TEST 11

Functions, graphs, algebra, calculus, probability and statistics
Technology-free end-of-year examination
Total marks: 40
Suggested writing time: 60 minutes

Specific instructions to students

- Answer all of the questions in the spaces provided.
- Show all workings in questions where more than one mark is available.
- An exact value must be provided in questions where a numerical answer is required, unless otherwise specified.

QUESTION 1

Let $f(x) = \cos^2(x)\log_e(2x+1)$. Find $f'(\pi)$.

$f(x) = \cos^2(x)\log_e(2x+1)$

$= (\cos(x))^2 \times \log_e(2x+1)$

Using the product and chain rules,

$f'(x) = [\log_e(2x+1) \times 2\cos(x) \times [-\sin(x)]]$

$+\left[\cos^2(x)\times\left(\frac{2}{2x+1}\right)\right]$

$= -2\cos(x)\sin(x)[\log_e(2x+1)] + \frac{2\cos^2(x)}{2x+1}$

$f'(\pi) = -2\cos(\pi)\sin(\pi)[\log_e(2\pi+1)] + \frac{2\cos^2(\pi)}{2\pi+1}$

$= \frac{2}{2\pi+1}$

3 marks

QUESTION 2

If $\Pr(A) = 0.25$, $\Pr(B) = 0.15$ and $\Pr(A \mid B) = 0.2$, find $\Pr(A \cup B)$.

$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} = 0.2$

Thus, $\frac{\Pr(A \cap B)}{0.15} = 0.2$

$\Rightarrow \Pr(A \cap B) = 0.2 \times 0.15 = 0.03$

Use addition formula:

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$\Pr(A \cup B) = 0.25 + 0.15 - 0.03$

$= 0.37$

2 marks

QUESTION 3

a I throw a fair die twice and note the product of the two uppermost numbers. Set up a discrete random variable distribution where x is the event of the product producing an even number or an odd number.

Two dice are rolled.

Draw a grid representing the 36 possible outcomes.

11	12	13	14	15	16
21	22	23	24	25	26
31	32	33	34	35	36
41	42	43	44	45	46
51	52	53	54	55	56
61	62	63	64	65	66

Shade the product that is odd. There are 9 cases where the product is odd.

So there are 27 cases where the product is even.

x	even	odd
$\Pr(X = x)$	$\frac{27}{36} = \frac{3}{4}$	$\frac{9}{36} = \frac{1}{4}$

2 marks

b A game is played where it costs \$2 to play the game. I win \$10 if the product is odd and lose \$1 if the product is even. What is my expected profit?

	even	odd
x	–3	8
$\Pr(X = x)$	$\frac{27}{36} = \frac{3}{4}$	$\frac{9}{36} = \frac{1}{4}$

$E(X) = -3 \times 0.75 + 8 \times 0.25 = -0.25$

Expected profit is a loss of 25 cents.

2 marks

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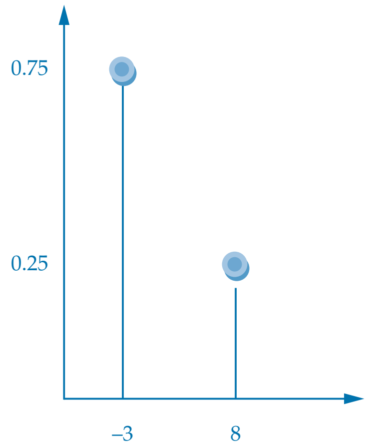
c Sketch a graph of this distribution described in part **b**.



1 mark
(Total: 5 marks)

QUESTION 4

a Find $\frac{d}{dt}(t\cos(2t))$.

$$\frac{d}{dt}(t\cos(2t)) = \cos(2t) \times 1 - t \times 2\sin(2t)$$
$$= \cos(2t) - 2t\sin(2t)$$

The probability density function below describes the probability that a certain drug relieves symptoms after t days.

$$f(t) = \begin{cases} kt\sin(2t), & 0 \le t \le \frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases}$$

2 marks

b Hence find k.

$$\int_0^{\frac{\pi}{2}} kt\sin(2t)\,dt = 1$$

We have

$$\frac{d}{dt}(t\cos(2t)) = \cos(2t) - 2t\sin(2t)$$

$$\Rightarrow \int \cos(2t) - 2t\sin(2t)\,dt = t\cos(2t)$$

$$\Rightarrow \int 2t\sin(2t)\,dt = -t\cos(2t) + \int \cos(2t)\,dt$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} kt\sin(2t)\,dt = -\frac{k}{2}[t\cos(2t)]_0^{\frac{\pi}{2}} + \frac{k}{2}\int_0^{\frac{\pi}{2}} \cos(2t)\,dt$$

Now $\int_0^{\frac{\pi}{2}} kt\sin(2t)\,dt = 1$

$$-\frac{k}{2}[t\cos(2t)]_0^{\frac{\pi}{2}} + \frac{k}{2}\int_0^{\frac{\pi}{2}} \cos(2t)\,dt = 1$$

$$-[t\cos(2t)]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos(2t)\,dt = \frac{2}{k}$$

$$-\frac{\pi}{2}\cos(\pi) + \frac{1}{2}[\sin(2t)]_0^{\frac{\pi}{2}} = \frac{2}{k}$$

$$\frac{\pi}{2} = \frac{2}{k}$$

$$k = \frac{4}{\pi}$$

4 marks
(Total: 6 marks)

QUESTION 5

a Find $\int \frac{-1}{\sqrt{1-2x}}\,dx$.

$$\int \frac{-1}{\sqrt{1-2x}}\,dx = -\int (1-2x)^{-\frac{1}{2}}dx$$
$$= \frac{-\sqrt{1-2x}}{-2 \times \frac{1}{2}} + c$$
$$= \frac{-2}{-2}(1-2x)^{\frac{1}{2}} + c$$
$$= \sqrt{1-2x} + c$$

2 marks

b Hence find the anti-derivative with respect to x of $f'(x) = \dfrac{-1}{\sqrt{1-2x}}$ if $f(0) = 2$.

$$\int \frac{-1}{\sqrt{1-2x}}\,dx = \sqrt{1-2x} + c$$

$f(x) = \sqrt{1-2x} + c$

Since $f(0) = 2$,

$2 = 1 + c$, so $c = 1$

$f(x) = \sqrt{1-2x} + 1$

2 marks

(Total: 4 marks)

QUESTION 6

A tangent to the graph of $y = \log_e(x-1)$ at $x = 3$ has the equation $y = \dfrac{x}{2} + q$. Find q.

$y = \log_e(x-1)$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{x-1}$

At $x = 3$, $\dfrac{dy}{dx} = \dfrac{1}{2}$

Use the equation of a line $y - y_1 = m(x - x_1)$

where m = gradient of curve.

Using the point $(3, \log_e(2))$,

$y - \log_e(2) = \dfrac{1}{2}(x-3)$

The equation of the tangent is

$y = \dfrac{1}{2}x - \dfrac{3}{2} + \log_e(2)$

$\Rightarrow q = \log_e(2) - \dfrac{3}{2}$

3 marks

QUESTION 7

Consider the graph of $y = -2\cos\left(2\left(x - \dfrac{\pi}{4}\right)\right)$

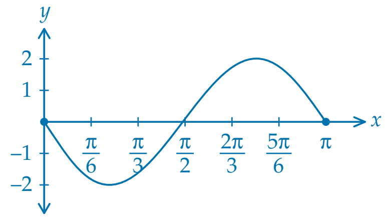
a State the amplitude and period.

Amplitude = 2

Period = π

2 marks

b Hence, sketch the graph of $y = -2\cos\left(2\left(x - \dfrac{\pi}{4}\right)\right)$ for $x \in [0, \pi]$.



2 marks

(Total: 4 marks)

QUESTION 8

Solve the equation $\tan(2x) = -\sqrt{3}$ for $x \in [0, \pi]$.

$\tan(2x) = -\sqrt{3}$ for $x \in [0, \pi]$.

Reference angle = $\dfrac{\pi}{3}$

$2x = \pi - \dfrac{\pi}{3}, 2\pi - \dfrac{\pi}{3}$

$= \dfrac{2\pi}{3}, \dfrac{5\pi}{3}$

$x = \dfrac{\pi}{3}, \dfrac{5\pi}{6}$

2 marks

QUESTION 9

A function has the equation $f(x) = ax^3 + 3x^2 + 4ax + 6$. For what value(s) of a will the graph of $f(x)$ have exactly 2 stationary points?

$f(x) = ax^3 + 3x^2 + 4ax + 6$

$f'(x) = 3ax^2 + 6x + 4a$

$\Delta = 6^2 - 4 \times 3a \times 4a$

$= 36 - 48a^2$

We need $\Delta > 0$ for 2 stationary points.

Solve $36 - 48a^2 > 0$

$-\dfrac{\sqrt{3}}{2} < a < \dfrac{\sqrt{3}}{2}$

3 marks

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QUESTION 10

a A normal distribution has a mean of 10 and a standard deviation of 2. Find $\Pr(X < 12)$.

$\mu = 10, \sigma = 2$

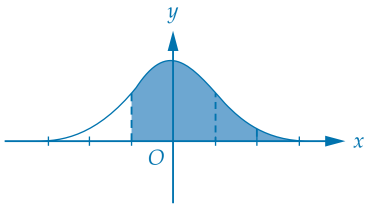
We know that $\Pr(X < 10) = 0.5$ and $\Pr(8 < X < 12) = 0.68$

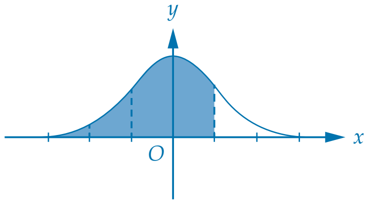
$\Pr(X < 12) = 0.5 + 0.34 = 0.84$

2 marks

b A normal distribution has a mean of 10 and a standard deviation of 2. Find a if $\Pr(X < a) = \Pr(X > -1) = 0.84$.

$\mu = 10, \sigma = 2$





$\Pr(X > -1) = 0.84$ is equal to $\Pr(X < 1) = 0.84$.

Use $z = \dfrac{x-\mu}{\sigma}$

$1 = \dfrac{x-10}{2}$

$x = 12$

So $a = 12$

2 marks
(Total: 4 marks)

QUESTION 11

A fair coin is tossed 20 times and the number of heads is noted. This experiment is repeated many times. What is the expected value and standard deviation of the sample proportion of heads?

Parameters of $\hat{p}$ are $p = 0.5$, $q = 0.5$, $n = 20$

$E(\hat{p}) = p$ and $\mathrm{sd}(\hat{p}) = \sqrt{\dfrac{p(1-p)}{n}}$

So $E(\hat{p}) = 0.5$ and $\mathrm{sd}(\hat{p}) = \sqrt{\dfrac{0.5 \times 0.5}{20}} = \sqrt{\dfrac{1}{80}}$

Mean proportion of heads = 0.5

With a standard deviation of $\dfrac{\sqrt{5}}{20}$

3 marks

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