TEST 10

Functions, graphs, algebra, calculus, probability and statistics
Technology-free end-of-year examination
Total marks: 35
Suggested writing time: 55 minutes

Specific instructions to students

- Answer all of the questions in the spaces provided.
- Show all workings in questions where more than one mark is available.
- An exact value must be provided in questions where a numerical answer is required, unless otherwise specified.

QUESTION 1

Pr(A) and Pr(B) are independent events, where Pr(A) = 0.4 and Pr(B) = 0.2.

a Complete the Karnaugh map below.

	A	A'	
B	0.08	0.12	0.2
B'	0.32	0.48	0.8
	0.4	0.6	1

Independent events, so $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$

$= 0.4 \times 0.2 = 0.08$

2 marks

b Find $\Pr(A \mid B)$.

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$$= \frac{0.08}{0.2} = \frac{8}{20}$$

$$= 0.4$$

2 marks

c Find $\Pr(B \mid A)$.

$$\Pr(B \mid A) = \frac{\Pr(B \cap A)}{\Pr(A)}$$

$$= \frac{0.08}{0.4} = \frac{8}{40}$$

$$\Pr(B \mid A) = 0.2$$

1 mark
(Total: 5 marks)

QUESTION 2

Pr(A) and Pr(B) are mutually exclusive, where Pr(A) = 0.4 and Pr(B) = 0.5.

a Complete the Karnaugh map below.

	A	A'	
B	0	0.5	0.5
B'	0.4	0.1	0.5
	0.4	0.6	1

Mutually exclusive events, so $\Pr(A \cap B) = 0$

2 marks

b Find $\Pr(A \mid B)$.

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

$$= \frac{0}{0.5}$$

$$= 0$$

1 mark

c Find $\Pr(A \cup B)$.

$\Pr(A \cup B) = \Pr(A) + \Pr(B)$ (if mutually exclusive)

$\Pr(A \cup B) = 0.9$

2 marks
(Total: 5 marks)

QUESTION 3

I toss a biased coin three times. For this coin, Pr(H) = 0.6 and Pr(T) = 0.4.
What is the probability that I will get two heads and one tail?

Pr(HHT) or Pr(HTH) or Pr(THH)

$= 3 \times 0.6^2 \times 0.4$

$= 0.432$

2 marks

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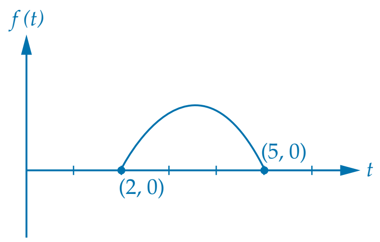
QUESTION 4

The contagious period for a certain virus is between 2 and 5 days after contact with the virus. The probability density function that describes the probability that symptoms will appear after t days is

$$f(t) = \begin{cases} \frac{2}{9}(t-5)(2-t), & 2 \le t \le 5 \\ 0, & \text{otherwise} \end{cases}$$

a Sketch a graph of the PDF.



2 marks

b Find the probability that symptoms will appear within 3 days after contact with the virus.

$$\Pr(T \le 3) = \int_2^3 \frac{2}{9}(t-5)(2-t)\,dt$$

$$= -\frac{2}{9}\int_2^3 (t^2 - 7t + 10)\,dt$$

$$= -\frac{2}{9}\left[\frac{t^3}{3} - \frac{7t^2}{2} + 10t\right]_2^3$$

$$= -\frac{2}{9}\left(9 - \frac{63}{2} + 30 - \frac{8}{3} + 14 - 20\right)$$

$$= -\frac{2}{9}\left(-\frac{7}{6}\right)$$

$$\Pr(T \le 3) = \frac{7}{27}$$

2 marks

c Find the modal time in which symptoms will appear after contact with the virus.

Mode = t-value where max exists

Mode = 3.5

2 marks
(Total: 6 marks)

QUESTION 5

a Find the anti-derivative with respect to x of e^{2x-1}.

$$\int e^{2x-1}dx = \frac{1}{2}e^{2x-1} + c$$

1 mark

b Find the anti-derivative with respect to x of $f'(x) = e^{2x-1}$ if $f(1) = 2$.

$$\int e^{2x-1}dx = \frac{1}{2}e^{2x-1} + c$$

Given $f(1) = 2$,

$$\Rightarrow 2 = \frac{1}{2}e^1 + c$$

$$\Rightarrow c = 2 - \frac{1}{2}e$$

So $\int e^{2x-1}dx = \frac{1}{2}e^{2x-1} + 2 - \frac{1}{2}e$

2 marks
(Total: 3 marks)

QUESTION 6

Consider the following functions.

$f: D \to \mathbf{R}, f(x) = \log_e(x-1)$ and

$g: \mathbf{R} \to \mathbf{R}, g(x) = x^2$

a For the maximal domain of $f(x)$, find D.

$f(x) = \log_e(x-1)$ exists for $(1, \infty)$.

$D = (1, \infty)$

1 mark

b For $f(g(x))$ to be defined, find the rule and domain of $f(g(x))$.

For $f(g(x))$ to exist, test ran $(g) \subseteq$ dom (f).

$[0, \infty) \not\subset (1, \infty)$

For $f(g(x))$ to exist, we need to restrict the range of g to $(1, \infty)$.

Domain $f(g(x))$ = dom $g(x) = (-\infty, -1) \cup (1, \infty)$

Rule: $f(g(x)) = \log_e(x^2 - 1)$

Domain $f(g(x)) = (-\infty, -1) \cup (1, \infty)$

3 marks
(Total: 4 marks)

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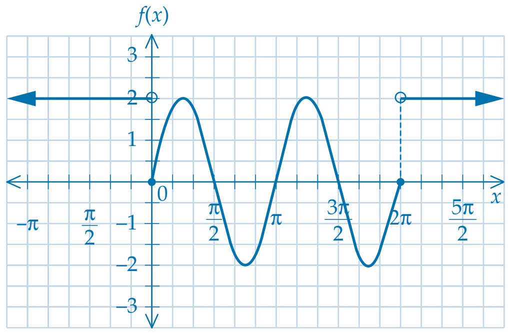
QUESTION 7

Given the hybrid function

$$f(x) = \begin{cases} -2\sin\left(2\left(x-\frac{\pi}{2}\right)\right), & 0 \le x \le 2\pi \\ 2, & x < 0 \cup x > 2\pi \end{cases}$$

a Sketch the graph of $f(x)$



2 marks

b State the domain for which $f'(x)$ is defined.

$f'(x)$ is defined for $\boldsymbol{R}\backslash\{0, 2\pi\}$.

1 mark

c With the help of your graph, solve the equation $f(x) = 2$.

$f(x) = 2$ for $x < 0$, $x > 2\pi$ and $x = \frac{\pi}{4}, \frac{5\pi}{4}$

2 marks
(Total: 5 marks)

QUESTION 8

Solve the equation $\log_e(2x) + \log_e(x) = 5$.

$\log_e(2x) + \log_e(x) = 5$

$\Rightarrow \log_e(2x^2) = 5$

$\Leftrightarrow e^5 = 2x^2$

$x^2 = \frac{e^5}{2}$

$\Rightarrow x = \pm\sqrt{\frac{e^5}{2}}$ reject negative solution

$x = \sqrt{\frac{e^5}{2}}$

2 marks

QUESTION 9

A discrete random variable is defined as follows.

x	0	1	2	3
$\Pr(X = x)$	0.2	$2a$	0.3	a

a Find a.

$0.2 + 2a + 0.3 + a = 1$

$3a + 0.5 = 1$

$a = \frac{1}{6}$

1 mark

b Hence find the mean of the distribution.

$E(X) = (0 \times 0.2) + (1 \times \frac{1}{3}) + (2 \times 0.3) + (3 \times \frac{1}{6})$

$= \frac{43}{30}$

2 marks
(Total: 3 marks)