TEST 7

Technology active end-of-year examination
Functions and graphs, algebra and calculus, probability and statistics
Section A: 22 marks
Section B: 58 marks
Suggested writing time: 120 minutes

Section A: Multiple-choice questions

Specific instructions to students

- A correct answer scores 1 mark, and an incorrect answer scores 0.
- Marks are not deducted for incorrect answers.
- No marks are given if more than one letter is shaded on the answer sheet.
- Choose the alternative that most correctly answers the question and mark your choice on the multiple-choice answer section at the bottom of each page, as shown in the example below.

1 A B C D — USE PENCIL ONLY

- Use pencil only.
- Working space is provided under those questions that require working out.

QUESTION 1

For what value(s) of m does the tangent to the curve $y = (m + 2)x^2 + 2mx + 4$ at the point $x = 2$ have a gradient of 6?

A $2, -\frac{1}{3}$

B $-\frac{1}{3}$

C 2

D $R\backslash\{2\}$

E $R\backslash\left\{-\frac{1}{3}, 2\right\}$

Tangent to $y = (m + 2)x^2 + 2mx + 4$, where $x = 2$, has a gradient of 6.

$y = (m + 2)x^2 + 2mx + 4$

$\frac{dy}{dx} = 2(m + 2)x + 2m$

When $x = 2$, $\frac{dy}{dx} = 4(m + 2) + 2m = 6m + 8$

Thus $6m + 8 = 6$

$\Rightarrow m = -\frac{1}{3}$

```
define f(x)=(m+2)x^2+2mx+4
                              done
solve(diff(f(x),x,1,2)=6,m)
                         {m=-1/3}
```

QUESTION 2

If $f(x) = 2x^3 + 7x^2 + \frac{x}{3} + \frac{1}{3}$, then $f'(x) =$

A $6x^2 + 14x - \frac{x}{3}$

B $6x^2 + 14x + 1$

C $\frac{18x^2 + 14x + 1}{3}$

D $\frac{18x^2 + 42x + 1}{3}$

E $6x^2 + 14x + \frac{x}{3} + \frac{1}{3}$

$f(x) = 2x^3 + 7x^2 + \frac{x}{3} + \frac{1}{3}$

$f'(x) = \frac{18x^2 + 42x + 1}{3}$

QUESTION 3

Emily is checking whether her CASICS professional netball shoes increase her chance of successfully intercepting a goal. She asks one sample of people to wear their old non-CASICS shoes, another to wear CASICS shoes for one game per season and a 3rd sample to wear CASICS for every game.
For each sample, the variable would be a random Bernoulli variable with success being 'successfully intercepting a goal'.
From 50 people who wore non-CASICS shoes, 10 successfully intercepted goals one season. The sample proportion of successfully intercepting goals is

A 0.2

B 0.1

C 0.5

D 1

E 5

Success is successfully intercepting goals.

Use the formula, **sample proportion**, $\hat{p} = \frac{x}{n} = \frac{10}{50} = \frac{1}{5}$

The sample proportion for those who successfully intercept goals is 0.2.

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QUESTION 4

The graph $f(x) = ax^3 + 6x^2 + bx - 2$ has a linear factor of $x + 1$, and when -2 is substituted for x, the answer is 3. The values of a and b, respectively, are

A $\frac{11}{6}, \frac{13}{6}$

B $-\frac{11}{6}, \frac{13}{6}$

C $\frac{9}{2}, -\frac{17}{2}$

D $-\frac{11}{6}, -\frac{13}{6}$

E $-\frac{9}{2}, \frac{17}{2}$

$f(x) = ax^3 + 6x^2 + bx - 2$ has $f(-1) = 0$ and $f(-2) = 3$.

Solving simultaneously,

```
define f(x)=ax^3+6x^2+bx-2
                                done
[f(-1)=0
[f(-2)=3 |a,b
                  {a=11/6, b=13/6}
```

$a = \frac{11}{6}, b = \frac{13}{6}$

QUESTION 5

If $g(x) = \cos^2 (f(x))$, then $g'(x) =$

A $2 \cos (f(x))$

B $-2 \sin (f(x))$

C $2 f'(x) \sin (f(x)) \cos (f(x))$

D $-2 f'(x) \sin (f(x)) \cos (f(x))$

E $-2 \sin (f(x)) \cos (f(x))$

$g(x) = \cos^2(f(x))$

Using the chain rule twice,

$g'(x) = 2 \times \cos (f(x)) \times [-\sin (f(x)) \times f'(x)]$

So $g'(x) = -2f'(x) \sin (f(x)) \cos (f(x))$

QUESTION 6

The graph $f(x) = ax^3 + x^2 + bx - c$ has a linear factor of $x + 1$ and a stationary point at (1, 4). The values of a, b and c, respectively, are

A 2, 4, 1

B 0, 2, –1

C 2, 4, –1

D –2, 4, –1

E no solution for a, b and c

$f(x) = ax^3 + x^2 + bx - c$ has $f(-1) = 0$ and $f'(1) = 4$.

Solving simultaneously,

```
define f(x)=ax^3+x^2+bx-c
                                done
define g(x)=d/dx(f(x))
                                done
[f(-1)=0
[f(1)=4
[g(1)=0 |a,b,c
                  {a=-2, b=4, c=-1}
```

gives $a = -2, b = 4, c = -1$

QUESTION 7

A continuous random variable is normally distributed with a mean of 4 and a standard deviation of 2.

$\Pr(X < 4 \mid X > 2)$ is closest to

A 0.4064

B 0.4058

C 0.8413

D 0.3413

E 0.4057

Normally distributed, mean = 4, standard deviation = 2.

$\Pr(X < 4 \mid X > 2) = \frac{\Pr(2 < X < 4)}{\Pr(X > 2)}$

```
normCDf(2,4,2,4)
----------------
normCDf(2,20,2,4)
                    0.4057132913
```

$\Pr(X < 4 \mid X > 2) = 0.4057$

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QUESTION 8

Consider $T: R^2 \to R^2$, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Applying this to the function $f(x) = \log_e(x + 1)$, the image function can be expressed as

A $y = \frac{1}{2}\log_e(x) - 1$

B $y = \frac{1}{2}\log_e(x) - 2$

C $y = 2\log_e(x) - 2$

D $y = 2\log_e(x) - 4$

E $y = \frac{1}{2}\log_e(x + 1) + 2$

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

So $\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

becomes $\begin{bmatrix} x+1 \\ \frac{1}{2}y - 2 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \begin{matrix} \Rightarrow x = x_1 - 1 \\ \Rightarrow y = 2(y_1 + 2) \end{matrix}$

In the equation $y = \log_e(x + 1)$, we get

$2(y_1 + 2) = \log_e(x_1 - 1 + 1)$

$(y_1 + 2) = 0.5\log_e(x_1)$

$y = 0.5\log_e(x) - 2$

The image function is $y = \frac{1}{2}\log_e(x) - 2$

QUESTION 9

A discrete random variable with a binomial distribution has 10 trials. It is known that $\Pr(X > 8) = 0.2$. The mean of the distribution is closest to

A 7.29

B 0.729

C 0.619

D 6.19

E 10

Binomial distribution, $n = 10$ and $\Pr(X > 8) = 0.2$.

$\Pr(X > 8) = \Pr(X = 9) + \Pr(X = 10)$

$\Pr(X > 8) = {}^{10}C_9\,(1 - p)^1\,p^9 + {}^{10}C_{10}\,(1 - p)^0\,p^{10}$

$\Rightarrow 10 \times (1 - p)^1\,p^9 + p^{10} = 0.2$

Solve for p.

```
solve(10·(1−p)·p^9+p^10=0.2, p)
                    {p=0.7290118496, p=1.101826388}
```

Select the value between 0 and 1.

$p = 0.7290$

Thus, mean $= np = 10 \times 0.7290 = 7.29$

QUESTION 10

The cost of a banana from a roadside stall, for traveller Tim, is \$1 if he has a discount card and \$2.50 if he doesn't have a discount card. A discrete random variable is displayed in the table below, where x is the number of discount cards that Tim has been able to collect at any one time.

x	0	1	2
$\Pr(X = x)$	0.4	0.5	0.1

The expected cost of a banana for Tim is

A 40c

B 60c

C \$1.60

D 70c

E \$1.70

$E(\$\text{ cost}) = (2.5 \times 0.4) + (1 \times 0.6) = 1.6$

The expected cost of a banana is \$1.60.

QUESTION 11

A probability density function is defined by

$$f(x) = \begin{cases} ax, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

The mean of the PDF is

A $\frac{2}{3}$

B $\frac{1}{2}$

C 2

D 1

E $\frac{\sqrt{2}}{2}$

Mean $= \int_0^1 (x \times ax)\,dx$, where $a = 2$

```
solve(∫[0,1] a·x dx=1, a)
                    {a=2}
∫[0,1] 2·x^2 dx
                    2/3
```

Mean $= \frac{2}{3}$

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QUESTION 12

A probability density function is defined by

$$f(x) = \begin{cases} ax, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

The median of the PDF is

A $\frac{2}{3}$

B $\frac{1}{2}$

C 2

D 1

E $\frac{\sqrt{2}}{2}$

Median = $\int_0^m (ax)\,dx = 0.5$, where $a = 2$ (refer to solution of Question 11)

solve$\left(\int_0^m 2\cdot x\,dx = 0.5, m\right)$

$\left\{m=\frac{-\sqrt{2}}{2}, m=\frac{\sqrt{2}}{2}\right\}$

In the domain, median = $\frac{\sqrt{2}}{2}$

QUESTION 13

The amount that parents will pay for a certain carnival ride is normally distributed with $\Pr(X > 10) = 0.15$, where X is the price in dollars for one ride. If we know that the standard deviation of the distribution is 0.3, the mean, in dollars, is closest to

A \$9.80

B \$10

C \$9.69

D \$9.36

E \$1.04

Normally distributed, $\Pr(X > 10) = 0.15$, standard deviation = 0.3.

Use inverse normal distribution to get $\Pr(Z > 1.0364) = 0.15$

Then use the formula

$z = \frac{x - \mu}{\sigma}$

and solve for μ in $1.0364 = \frac{10 - \mu}{0.3}$

invNormCDf("R", 0.15, 1, 0)

1.036433389

solve$\left(1.0364=\frac{10-u}{0.3}, u\right)$

{u=9.68908}

Mean = \$9.69

QUESTION 14

The maximal domain for the function

$f(x) = \frac{1}{4 - x^2} + \log_e(x)$ is

A $R\backslash\{2, -2\}$

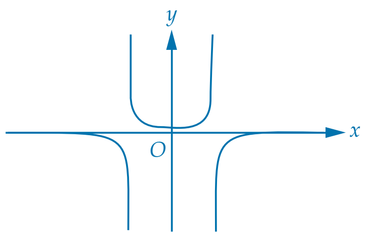
B $(-2, 2)$

C $(0, \infty)\backslash\{2, -2\}$

D $(0, \infty)\backslash\{2\}$

E $(0, \infty)$

Domain for $y = \frac{1}{4 - x^2}$ is $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$ or $R\backslash\{2, -2\}$.



Domain for $y = \log_e(x)$ is $(0, \infty)$.

The maximal domain for $f(x) = \frac{1}{4 - x^2} + \log_e(x)$ is the intersection of these domains.

Answer: $(0, \infty)\backslash\{2\}$

QUESTION 15

If $\int_0^1 f(x)dx = 5$, then $\int_1^0 (-f(x) + x)dx$ equals

A $5 + x$

B $-\frac{11}{2}$

C $\frac{11}{2}$

D $\frac{9}{2}$

E $-5 + \frac{x^2}{2}$

Given $\int_0^1 f(x)dx = 5$,

$$\int_1^0 (-f(x) + x)dx = -\int_1^0 f(x)dx + \int_1^0 x\,dx$$

$$= \int_0^1 f(x)dx + \left[\frac{x^2}{2}\right]_1^0 \quad \text{where } \int_0^1 f(x)dx = 5$$

$$= 5 - \frac{1}{2}$$

$$= \frac{9}{2}$$

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QUESTION 16

A cubic function is $y = x^3 - x^2 - 4x - 6$. The **incorrect** statement is

A there are 3 real linear factors

B there is only 1 linear factor

C a solution to the equation $y = 0$ is $x = 3$

D a quadratic factor is $x^2 + 2x + 2$

E the discriminant for $x^2 + 2x + 2$ is less than zero

$y = x^3 - x^2 - 4x - 6 = (x - 3)(x^2 + 2x + 2)$

It is true that the discriminant for $x^2 + 2x + 2 < 0$, so there are no further real linear factors.

The incorrect statement is that there are 3 real linear factors.

QUESTION 17

A continuous random variable, Y, is normally distributed with a mean of 164 and a variance of 4. If the random variable, Z, is the standard normal distribution, then $\Pr(Y > 160)$ is equal to

A $\Pr(Z > -1)$

B $1 - \Pr(Z < 1)$

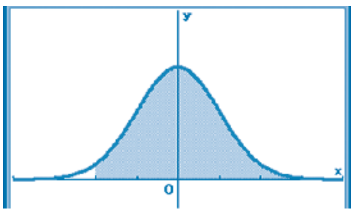
C $\Pr(Z > 3)$

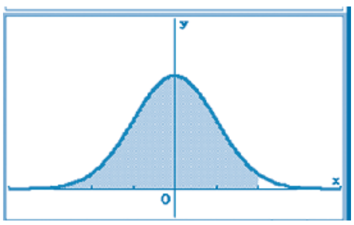
D $\Pr(Z < 2)$

E $1 - \Pr(Z > -2)$

Normally distributed, mean = 164

Variance of 4, s.d. = 2





$\Pr(Y > 160) = \Pr(Z > -2)$

This is equal to $\Pr(Z < 2)$.

So $\Pr(Y > 160)$ is equal to $\Pr(Z < 2)$.

QUESTION 18

Using a random sample of 100 adults, it is found that the sample proportion $\hat{p}$ is 0.3, where $\hat{p}$ is the sample proportion of those adults who jog for exercise. The approximate 95% CI for the proportion p of adults is

A (21, 39)

B (0.2102, 0.3898)

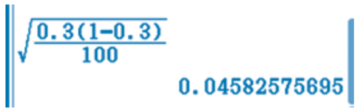
C (21.02, 38.98)

D (0.2, 0.4)

E 1.96

Since $\hat{p} = 0.3$ and $n = 100$

$$\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.0458$$

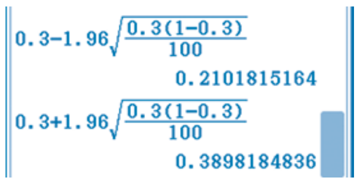


And using a 95% Confidence Interval

$$95\% \text{ CI} = \left(\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$$

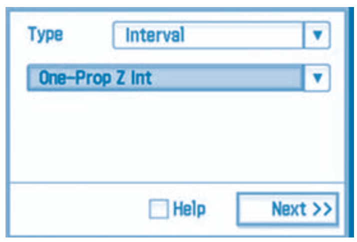
$$\text{gives} \left(0.3 - 1.96\sqrt{\frac{0.3(1-0.3)}{100}}, 0.3 + 1.96\sqrt{\frac{0.3(1-0.3)}{100}}\right)$$

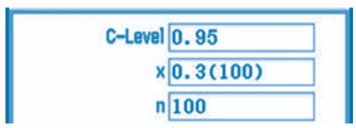
$= (0.3 - 1.96 \times 0.0458, 0.3 + 1.96 \times 0.0458)$

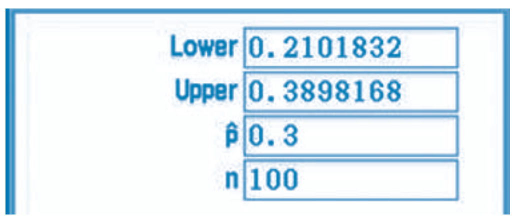


$= (0.2102, 0.3898)$

For a sample size of 100, 95% CI is (0.2102, 0.3898).







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QUESTION 19

The function $f(x) = ax^3 + 2ax^2 + 2x - 1$ has two turning points for

A $a \in R \setminus \left[0, \frac{3}{2}\right]$

B $a \in R \setminus \left(0, \frac{3}{2}\right)$

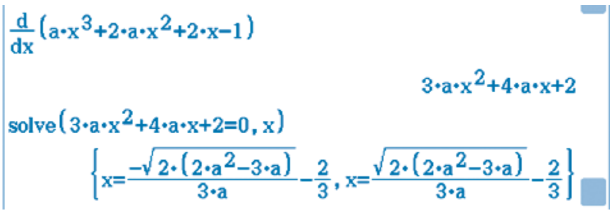
C $a \in R$

D $a \in \left(0, \frac{3}{2}\right)$

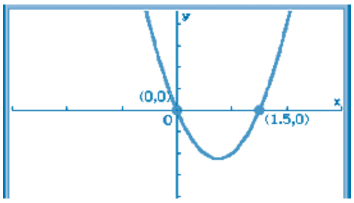
E $a \in \left[0, \frac{3}{2}\right]$

$f(x) = ax^3 + 2ax^2 + 2x - 1$

$f'(x) = 3ax^2 + 4ax + 2 = 0$ for stationary points



There are 2 solutions for the discriminant, $2(2a^2 - 3a) > 0$



Discriminant > 0 for $a < 0$, $a > 1.5$.

Therefore, the function $f(x)$ has two turning points for $a \in R \setminus \left[0, \frac{3}{2}\right]$.

QUESTION 20

The area between the curve $y = e^x$ and the line $y = e^2$ and the y-axis is

A 0

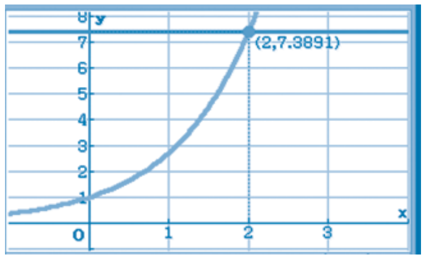
B e^2

C $e^2 - 1$

D $2e^2$

E $e^2 + 1$

To find the area between $y = e^x$, $y = e^2$ and the y-axis, first equate $e^x = e^2$ to find the point of intersection.



The point of intersection is at $x = 2$.

Area $= \int_0^2 e^2 - e^x \, dx = e^2 + 1$

Area $= e^2 + 1$

QUESTION 21

The value(s) of m for which the lines

$-y = (m + 2)x + 4$

and

$my + (m - 1)^2x = -m$

have infinite solutions are

A $m \in R \setminus \{-2, 1\}$

B $m \in R \setminus \left\{\frac{1}{4}\right\}$

C $m = \frac{1}{4}$

D $m \neq \frac{1}{4}$

E $m \notin R$

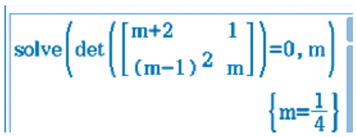
$-y = (m + 2)x + 4$ and $my + (m - 1)^2x = -m$

Rearranging the equations,

$(m + 2)x + y = -4$

$(m - 1)^2x + my = -m$

Find where det = 0



Alternative solution

Rearrange to find gradients and then equate.

We get $-(m+2) = -\frac{(m-1)^2}{m}$

solve to get $m = \frac{1}{4}$

$$\text{solve}\left(-(m+2)=-\frac{(m-1)^2}{m}, m\right)$$
$$\left\{m=\frac{1}{4}\right\}$$

Test $m = \frac{1}{4}$

$\left(\frac{1}{4}+2\right)x + y = -4$ and $\left(\frac{1}{4}-1\right)^2 x + \frac{1}{4}y = -\frac{1}{4}$

give

$\frac{9}{4}x + y = -4 \Rightarrow 9x + 4y = -16$

$\frac{9}{16}x + \frac{1}{4}y = -\frac{1}{4} \Rightarrow 9x + 4y = -4$

Parallel lines

No value of m gives infinite solutions, so $m \notin \boldsymbol{R}$.

QUESTION 22

If two events A and B in a given sample space are such that $\Pr(A \mid B) = 0.2$ and $\Pr(B' \cap A') = 0.1$, with $\Pr(B) = 0.6$, then $\Pr(B \mid A) =$

A 0.2

B 0.9

C $\frac{63}{250}$

D $\frac{2}{7}$

E $\frac{5}{7}$

We are given $\Pr(A \mid B) = 0.2$, $\Pr(B' \cap A') = 0.1$, $\Pr(B) = 0.6$.

$\Pr(A \mid B) = 0.2 = \frac{\Pr(A \cap B)}{\Pr(B)}$

So $\Pr(A \cap B) = 0.2 \times \Pr(B) = 0.2 \times 0.6 = 0.12$

	A	A'	
B	0.12	0.48	0.6
B'	0.3	0.1	0.4
	0.42	0.58	1

$\Pr(B \mid A) = \frac{\Pr(A \cap B)}{\Pr(A)} = \frac{0.12}{0.42} = \frac{6}{21} = \frac{2}{7}$

So $\Pr(B \mid A) = \frac{2}{7}$

ONE ANSWER PER LINE **USE PENCIL ONLY**

Question	Answer	Question	Answer
1	B	12	E
2	D	13	C
3	A	14	D
4	A	15	D
5	D	16	A
6	D	17	D
7	E	18	B
8	B	19	A
9	A	20	E
10	C	21	E
11	A	22	D

Section B: Extended-response questions

Specific instructions to students

- Answer all of the questions in the spaces provided.
- Show all workings in questions where more than one mark is available.
- An exact value must be provided in questions where a numerical answer is required, unless otherwise specified.

QUESTION 23

A binomial random variable distribution has probability, p, for success.

a **i** For 3 independent trials state an expression, in terms of p, that would find $\Pr(X > 2)$.

$\Pr(X > 2) = \Pr(X = 3) = {}^3C_3\,(1-p)^0 p^3 = p^3$

2 marks

ii For 3 independent trials state an expression, in terms of p, that would find $\Pr(X = 0)$.

$\Pr(X = 0) = {}^3C_0\,(1-p)^3 p^0 = (1-p)^3$

2 marks

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b i For what value of p is $\Pr(X > 2) = \Pr(X = 0)$?

Solve $\Pr(X > 2) = \Pr(X = 0)$.

$p^3 = (1 - p)^3$

```
solve(p^3=(1-p)^3, p)
                              {p=1/2}
```

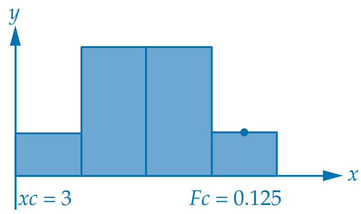
Thus, $p = \frac{1}{2}$

1 mark

ii Explain why you would expect this answer for p in this situation.

If $\Pr(X > 2) = \Pr(X = 0)$, then $\Pr(X = 3) = \Pr(X = 0)$.

The distribution is symmetric.



1 mark

For a 3-child family, the distribution table is shown below, where X = number of boys in a family and the probability of having a boy is 0.5.

c Complete the discrete distribution table below in terms of p.

x	0	1	2	3
$\Pr(X = x)$	$(1 - p)^3$	${}^3C_1\,(1-p)^2p^1 = 3(1-p)^2p^1$	${}^3C_2\,(1-p)^1p^2 = 3(1-p)^1p^2$	p^3

Using $\text{Bi}(n, p) = \text{Bi}(3, p)$.

4 marks

d What is the mean and median number of boys in a family for this situation?

Using $\text{Bi}(n, p) = \text{Bi}\left(3, \frac{1}{2}\right)$, the table now becomes

x	0	1	2	3
$\Pr(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$\text{Mean} = E(X) = \left(0 \times \frac{1}{8}\right) + \left(1 \times \frac{3}{8}\right) + \left(2 \times \frac{3}{8}\right) + \left(3 \times \frac{1}{8}\right) = \frac{12}{8}$$

$= 1.5$

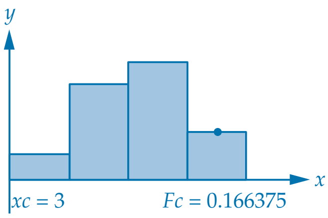
Median = 1.5

3 marks

The situation is changed and for a particular genetic community, a 3-child family where Y = number of boys in a family, the probability of having a boy is 0.55.

e Complete the discrete distribution table below, correct to 4 decimal places.

y	0	1	2	3
$\Pr(Y = y)$	0.0911	0.3341	0.4084	0.1664



2 marks

f i What is the mean and variance of boys in a family for this situation? Give your answers correct to 3 decimal places.

Mean = $E(X)$

$= (0 \times 0.0911) + (1 \times 0.3341) + (2 \times 0.4084) + (3 \times 0.1664)$

$= 1.650$

Variance = $E(X^2) - (E(X))^2$

$= (0^2 \times 0.0911) + (1^2 \times 0.3341) + (2^2 \times 0.4084) + (3^2 \times 0.1664) - (1.6501)^2$

$= 0.742$

3 marks

ii Find the probability that the distribution falls within 1 standard deviation of the mean.

Variance = 0.742 469 99 so s.d. = 0.8617

$\Pr(\mu - \sigma \le X \le \mu + \sigma) = \Pr(1.6501 - 0.8617 \le X \le 1.6501 + 0.8617)$

$= \Pr(0.7434 \le X \le 2.4668)$

For a discrete distribution,

$\Pr(\mu - \sigma \le X \le \mu + \sigma) = \Pr(1 \le X \le 2)$

$= 0.3341 + 0.4084$

$= 0.7425$

2 marks

iii In a family where the probability of a boy is 0.55, what is the probability, correct to 4 decimal places, that there will be at least 2 boys in the family?

$\Pr(X \ge 2) = \Pr(X = 2) + \Pr(X = 3)$

$= 0.4084 + 0.1664$

$= 0.5748$

```
binomialCDf(2,3,3,0.55)
                 0.57475
```

1 mark

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g How many trials will be needed for the probability of at least 2 boys in at least 2 families, where the probability of a boy is 0.55, to have the probability of at least 0.6?

$\text{Bi}(n, p)$ where $p = 0.5748$, $n = ?$

Solve $\Pr(X \geq 2) = 0.6$

$\Pr(X \geq 2) = 1 - (\Pr(X = 0) + \Pr(X = 1))$

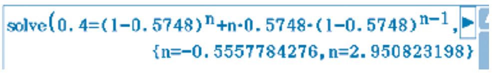
So $1 - (\Pr(X = 0) + \Pr(X = 1)) = 0.6$

$\Pr(X = 0) + \Pr(X = 1) = 0.4$

${}^nC_0(1 - 0.5748)^n(0.5748)^0 + {}^nC_1(1 - 0.5748)^{n-1}(0.5748)^1 = 0.4$

$\Rightarrow (1 - 0.5748)^n + n(1 - 0.5748)^{n-1}(0.5748)^1 = 0.4$

Solve for n.

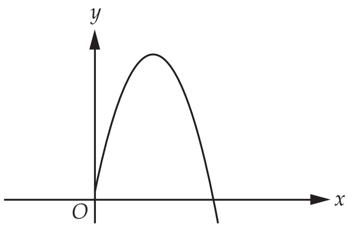


3 trials are needed.

3 marks
(Total: 24 marks)

QUESTION 24

Paddy inadvertently throws his cricket ball over the fence into the parkland behind his house. The ball follows the path of a parabola with the equation $h(x) = -x^2 + 10x + 1$, where $h(x)$ metres is the vertical height of the ball from the ground, at a horizontal distance, x metres, from Paddy's hand.
The graph of this function is shown below.



a At what height from the ground does the ball leave Paddy's hand?

$h(x) = -x^2 + 10x + 1$

$h(0) = 1$

The ball leaves Paddy's hand at a height of 1 metre.

1 mark

b What is the maximum height that the ball will reach?

$h(x) = -x^2 + 10x + 1$

To find the maximum, solve

$h'(x) = -2x + 10 = 0$

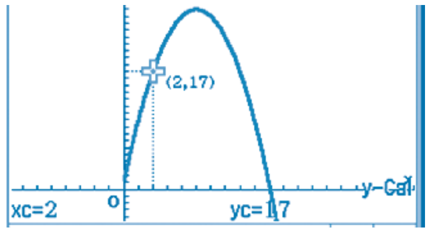
This gives $x = 5$.

$h(5) = 26$

$\therefore$ The maximum height = 26 m

1 mark

c The fence into the parkland has a height of 2 metres and is at a horizontal distance of 2 metres from Paddy. Will his ball hit the fence? Why/why not?



Path of the ball goes through (2, 17).

Fence has a height of 2 metres.

$17 > 2$

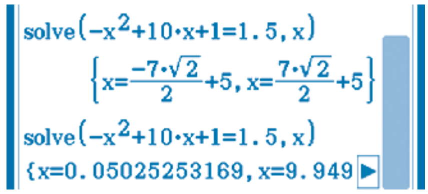
$\therefore$ The ball won't hit the fence.

2 marks

A jogging path has been built around the parkland and Imogen is jogging just as Paddy throws the ball. She sees the ball and stops jogging. Imogen is 1.5 metres tall and the ball actually hits her on her head.

d Where is Imogen standing, correct to 2 decimal places, when the ball hits her?

Solve $h(x) = -x^2 + 10x + 1 = 1.5$



Imogen gets hit after the ball goes over the fence, so take the larger x value.

$x = 9.95$

Imogen is 9.95 metres from Paddy.

1 mark

Imogen actually picks up the ball, runs away with it and puts it on the ground at the coordinates (30, 0).

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e What is the shortest distance from the ball on the ground to Paddy's hand, still at 1 metre vertically from the ground?

Distance from (30, 0) to (0, 1)
$= \sqrt{1^2 + 30^2} = \sqrt{901}$ m.

2 marks

f Paddy finds a way to crawl through the fence and runs to the ball. How far does Paddy run?

Horizontal distance to (30, 0) is 30 metres.

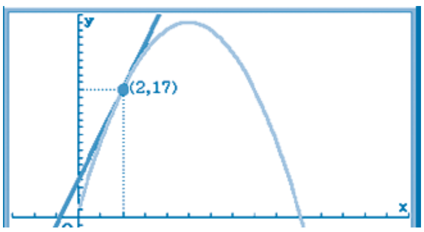
1 mark

g Find the equation of the tangent to the curve $h(x)$ at $x = 2$.

$h(x) = -x^2 + 10x + 1$
So $h(2) = 17$
$h'(x) = -2x + 10$
$h'(2) = 6$
The equation of the line is
$y - y_1 = m(x - x_1)$
$y - 17 = 6(x - 2)$
$y = 6x + 5$

2 marks

h **i** Hence state an integral to find the area between the tangent at $x = 2$, the curve $h(x)$ and the y-axis.



Area $= \int_0^2 (6x+5)-(-x^2+10x+1)\,dx$

2 marks

ii Evaluate this area.

Area $= \frac{8}{3}$ square units

1 mark
(Total: 13 marks)

QUESTION 25

The function f is defined by $f: [0, 2\pi] \rightarrow \boldsymbol{R}$, where $f(x) = 3e^{\frac{x}{10}}g(x)$, where $g(x) = \sin(2x)$.

a Show that the solutions to the equation $g(x) = 0$ for $x \in [0, 2\pi]$ are $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$.

$\sin(2x) = 0$ for $x \in [0, 2\pi]$
$2x = 0, \pi, 2\pi, 3\pi, 4\pi$
$x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$.

2 marks

b Hence state the solutions to the equation $f(x) = 0$.

$f(x) = 0$ for $f(x) = 3e^{\frac{x}{10}}g(x) = 0$
Solve $3e^{\frac{x}{10}}\sin(2x) = 0$
Either $3e^{\frac{x}{10}} = 0$ or $\sin(2x) = 0$
$3e^{\frac{x}{10}} = 0$ gives no solution
Answers: $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$.

1 mark

c Using the product rule, find $f'(x)$ and hence find the coordinates, correct to 2 decimal places, of the stationary points of the graph of $f(x)$.

$f(x) = 3e^{\frac{x}{10}}\sin(2x)$

To find the stationary points, solve
$f'(x) = \frac{3}{10}e^{\frac{x}{10}}\sin(2x) + 6e^{\frac{x}{10}}\cos(2x) = 0$
$e^{\frac{x}{10}}(0.3\sin(2x) + 6\cos(2x)) = 0$
$e^{\frac{x}{10}} = 0$ has no solution
$0.3\sin(2x) = -6\cos(2x)$
$\Rightarrow \tan(2x) = -20$
$\Rightarrow x = 0.81, x = 2.38, x = 3.95, x = 5.52$
Coordinates of the stationary points are (0.81, 3.25), (2.38, –3.80), (3.95, 4.45), (5.52, –5.21).

3 marks

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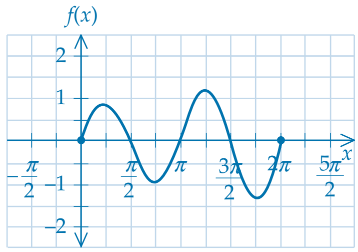
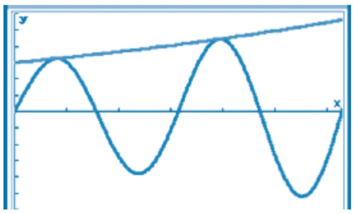
d Sketch the graph of $f(x)$, labelling axial intercepts and endpoints.

x-intercepts: $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$.



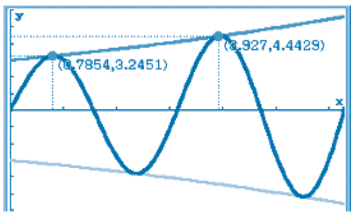
3 marks

e Let the points $A(a, f(a))$ and $B(b, f(b))$ be the first 2 points of intersection between the graphs of $f(x)$ and $y = 3e^{\frac{x}{10}}$. The line that joins A and B has a gradient of m. Show that $m = \frac{f(b) - f(a)}{\pi}$.



$A\ (a, f(a)) = \left(\frac{\pi}{4}, f\left(\frac{\pi}{4}\right)\right)$

$B\ (b, f(b)) = \left(\frac{5\pi}{4}, f\left(\frac{5\pi}{4}\right)\right)$

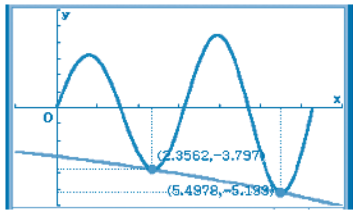


Gradient $= m = \frac{f(b) - f(a)}{b - a}$.

$$m = \frac{f\left(\frac{5\pi}{4}\right) - f\left(\frac{\pi}{4}\right)}{\frac{5\pi}{4} - \frac{\pi}{4}} = \frac{f\left(\frac{5\pi}{4}\right) - f\left(\frac{\pi}{4}\right)}{\pi}$$

2 marks

f Hence find an expression for the gradient, n, of points C and D, where C and D are the first 2 points of intersection between the graphs of $f(x)$ and $y = -3e^{\frac{x}{10}}$.



$$n = \frac{f\left(\frac{7\pi}{4}\right) - f\left(\frac{3\pi}{4}\right)}{\pi}$$

2 marks

(Total: 13 marks)

QUESTION 26

A section of a rollercoaster is modelled by a function with the equation $p(x) = a(bx^2 - cx)^2$.

a Show that the x-intercepts of the function are at $x = 0$ and $x = \frac{c}{b}$.

Factorise $p(x) = a(bx^2 - cx)^2$

$= a(bx^2 - cx)(bx^2 - cx)$

$= ax^2(bx - c)(bx - c)$

$= ax^2(bx - c)^2$

Let $ax^2(bx - c)^2 = 0$

$\Rightarrow x = 0, x = \frac{c}{b}$

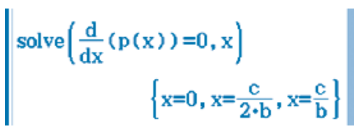
2 marks

b Show that the stationary points of the function are at $x = 0$, $x = \frac{c}{2b}$ and $x = \frac{c}{b}$.

$p(x) = ax^2(bx - c)^2$

$p'(x) = 2ax(bx - c)(2bx - c) = 0$ for stationary points.

This gives $x = 0$, $x = \frac{c}{2b}$ and $x = \frac{c}{b}$.



2 marks

This section of the rollercoaster has the domain $x \in [-1, 5]$, where x and y are respectively the horizontal and vertical displacement, in metres, from the point (0, 0). The point (1, 0) is where the safety officer sits.

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c We know that $p'\left(\frac{10}{3}\right) = 0$. Show that $\frac{10}{3} = \frac{c}{b}$.

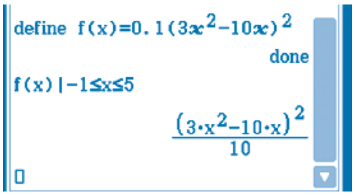
The stationary points are at $x = 0$, $x = \frac{c}{2b}$ and $x = \frac{c}{b}$.

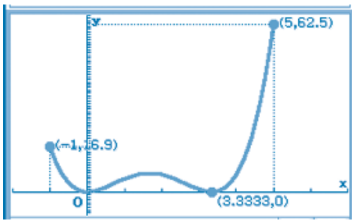
$p'\left(\frac{10}{3}\right) = 0$ gives the 2nd stationary point $x = \frac{c}{b}$.

So $p'\left(\frac{c}{b}\right) = 0$ and thus $\frac{10}{3} = \frac{c}{b}$.

1 mark

d It is also true that $p(5) = 62.5$ and $p'(0) = 0$. Find the values of a, b and c.





$a = 0.1$, $b = 3$, $c = 10$

1 mark

e The safety officer looks directly above him and decides that the rollercoaster is inoperable if the gradient is greater than 5. Does he shut down the rollercoaster?

The safety officer sits at (1, 0).

The rollercoaster goes through (1, 4.9).

At point (1, 4.9), the gradient is 5.6.

$5.6 > 5$

So the rollercoaster will be shut down.

2 marks

(Total: 8 marks)

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