TEST 6

Technology active end-of-year examination
Functions and graphs, algebra and calculus, probability and statistics
Section A: 20 marks
Section B: 40 marks
Suggested writing time: 90 minutes

Section A: Multiple-choice questions

Specific instructions to students

- A correct answer scores 1 mark, and an incorrect answer scores 0.
- Marks are not deducted for incorrect answers.
- No marks are given if more than one letter is shaded on the answer sheet.
- Choose the alternative that most correctly answers the question and mark your choice on the multiple-choice answer section at the bottom of each page, as shown in the example below.

1 A B C D **USE PENCIL ONLY**

- Use pencil only.
- Working space is provided under those questions that require working out.

QUESTION 1

The value(s) of m for which the pair of equations
$(m + 2)x + 3y = m + 3$
and
$4x + (2m - 1)y = 5$
have infinite solutions are

A $2, -\frac{7}{2}$

B $-\frac{7}{2}$

C 2

D $R\backslash\{2\}$

E $R\backslash\left\{-\frac{7}{2}, 2\right\}$

$(m + 2)x + 3y = m + 3$ and $4x + (2m - 1)y = 5$

Coefficient matrix $= \begin{bmatrix} m+2 & 3 \\ 4 & 2m-1 \end{bmatrix}$

$\text{Det}\begin{bmatrix} m+2 & 3 \\ 4 & 2m-1 \end{bmatrix} = (m + 2)(2m - 1) - 12 = 0$ for a singular matrix.

This gives $m = 2, m = -\frac{7}{2}$

$\det\left(\begin{bmatrix} m+2 & 3 \\ 4 & 2m-1 \end{bmatrix}\right)$
$2\cdot m^2+3\cdot m-14$
$\text{solve}(2\cdot m^2+3\cdot m-14=0, m)$
$\left\{m=2, m=-\frac{7}{2}\right\}$

Alternative solution

Find gradient of each line:

$\frac{-(m+2)}{3}$ and $\frac{-4}{(2m-1)}$

Then equate gradients to find $m = 2$ and $m = -\frac{7}{2}$.

$\text{solve}\left(\frac{-(m+2)}{3}=\frac{-4}{2m-1}, m\right)$
$\left\{m=2, m=-\frac{7}{2}\right\}$

Test $m = 2$	$m = -\frac{7}{2}$
$4x + 3y = 5$	$-\frac{3}{2}x + 3y = -\frac{1}{2} \Rightarrow -3x + 6y = -1$
$4x + 3y = 5$	$4x - 8y = 5$
Infinite solutions	No solutions

For infinite solutions, $m = 2$

QUESTION 2

If $2x + a$ is a factor of the polynomial $P(x) = 2x^3 + 7x^2 + ax$, where $a \in R\backslash\{0\}$, the value of a is

A -1

B 0

C 1

D 3

E 5

$P(x) = 2x^3 + 7x^2 + ax$

$2x + a$ is a factor, so we need $P\left(-\frac{a}{2}\right) = 0$

$\text{define } f(x)=2x^3+7x^2+ax$
done
$\text{solve}\left(f\left(-\frac{a}{2}\right)=0, a\right)$
$\{a=0, a=5\}$

Because $a \in R\backslash\{0\}$, we have $a = 5$.

QUESTION 3

The line that is a tangent to the graph $f(x) = 2x^3 + 5x^2 + 3x - 2$ at $x = 1$ is

A $y = -x - 3$

B $y = 47x - 54$

C $y = 19x$

D $y = 22x + 1$

E $y = 19x - 11$

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$f(x) = 2x^3 + 5x^2 + 3x - 2$

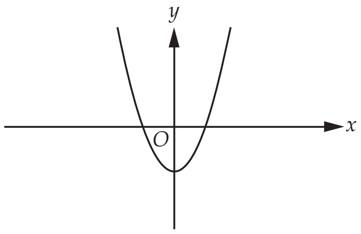
Tangent at $x = 1$

```
tanLine(2·x^3+5·x^2+3·x−2, x, 1)
                         19·x−11
```

$y = 19x - 11$

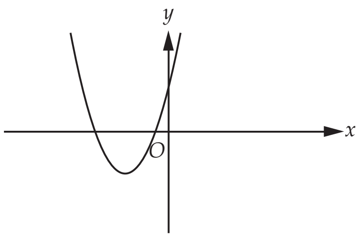
QUESTION 4

If $f(x)$ is represented by the following graph

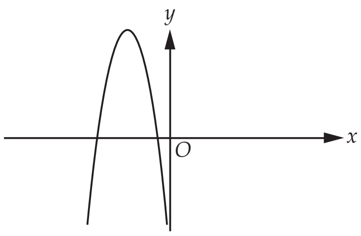


and $g(x) = f(x + 2)$, then the graph of $y = -3f(g(x))$ looks similar to

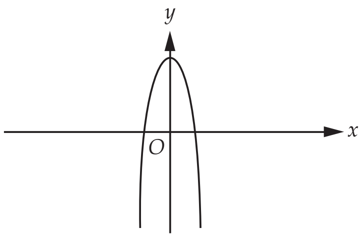
A



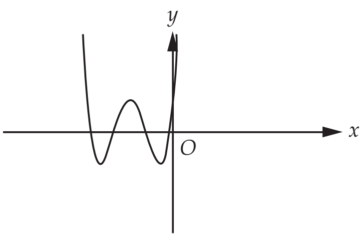
B



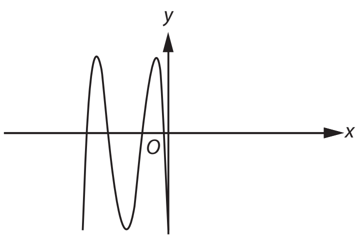
C



D



E



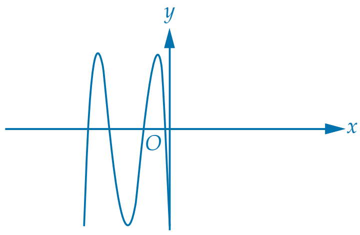
$f(x)$ is a +ve parabola that is symmetric around the y-axis.

$g(x) = f(x + 2)$, that is, $f(x)$ moved 2 units to the left, and then becomes the inner of composite function $f(g(x))$, which becomes a quartic function. You could assume what the equation of the original graph may have been.

```
define f(x)=x^2−2
                         done
define g(x)=f(x+2)
                         done
f(g(x))
               ((x+2)^2−2)^2−2
```

$y = -3f(g(x))$ then dilates the quartic by 3 from the x-axis and is then reflected over the x-axis.

$y = -3f(g(x))$ looks similar to



QUESTION 5

If $g(x) = 3(f(x))^2 + 2x$, then $g'(x) =$

A $6f(x)$

B $6f(x) + 2$

C $6f(x)f'(x) + 2$

D $(6f(x) + 2)f'(x)$

E $6f(x) + 2f'(x)$

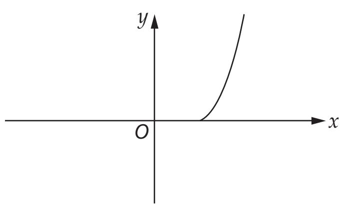
$g(x) = 3(f(x))^2 + 2x$

$g'(x) = 6f(x)f'(x) + 2$

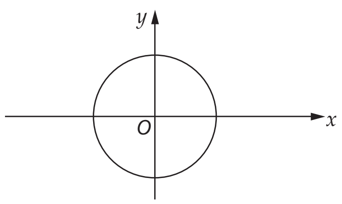
QUESTION 6

Each of the following displays a relationship between x and y. The graph that does not have an inverse is

A



B



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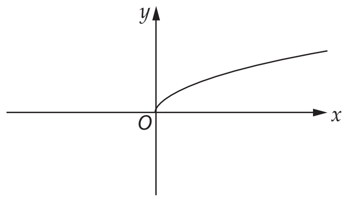
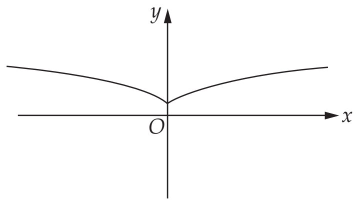
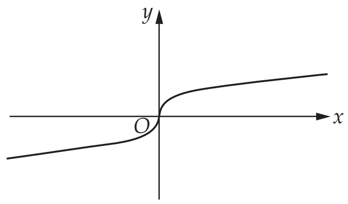
C



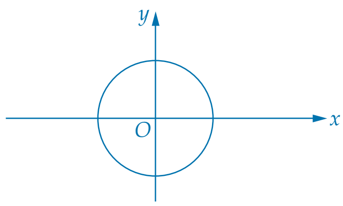
D



E



The graph that does not have an inverse is the circle



QUESTION 7

A continuous random variable is normally distributed with a mean of 3 and a variance of 16. $\Pr(X < 7)$ is closest to

A 0.5

B 0.68

C 0.84

D 0.52

E 0.17

Normal distribution. Mean = 3 and variance = 16, so s.d. = 4.

$\Pr(X < 3) = 0.5$

Also $3 - 4 < X < 3 + 4 = 0.68$ so $3 < X < 7 = 0.34$

So $\Pr(X < 7) = 0.84$

QUESTION 8

A graph is dilated by 2 units from the y-axis, then dilated by 2 units parallel to the y-axis. The transformation matrix, T, which describes this is

A $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}$

B $\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

C $\begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

D $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

E $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Dilated by 2 units from the y-axis, then dilated by 2 units parallel to the y-axis.

T will be $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}$.

QUESTION 9

A discrete random variable follows the binomial distribution with a mean of 4 and a variance of 3.2. The number of trials is

A $\sqrt{3.2}$

B 4

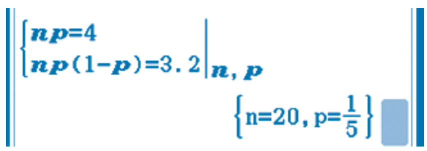
C $2\sqrt{5}$

D 20

E 0.2

Binomial mean = 4, variance = 3.2.

Use mean = np and variance = $np(1 - p)$, and solve simultaneously



$n = 20$

QUESTION 10

A discrete random variable is displayed in the table below.

x	0	1	2	3
$\Pr(X = x)$	0.4	0.1	0.2	0.3

The mean and the mode, respectively, are

A 1.8 and 1

B 1.4 and 2

C 1.4 and 3

D 1.4 and 0

E 1.8 and 0

Mean = $(0 \times 0.4) + (1 \times 0.1) + (2 \times 0.2) + (3 \times 0.3) = 1.4$

Mode = 0

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QUESTION 11

A probability density function is defined by

$$f(x) = \begin{cases} kx^2, & 0 \le x \le 10 \\ 0, & \text{otherwise} \end{cases}$$

The value of k is

A 0.003

B 0.0003

C 3

D 1

E 0

$$f(x) = \begin{cases} kx^2, & 0 \le x \le 10 \\ 0, & \text{otherwise} \end{cases}$$

For a PDF, $\int_0^{10} kx^2 \, dx = 1$

```
solve(∫_0^10 k·x^2 dx=1, k)
                    {k=3/1000}
```

$k = 0.003$

QUESTION 12

Amy throws a netball at a goal ring in practice. She successfully scores a goal half of the time. The probability that in 10 trials Amy scores at least 6 goals is closest to

A 0.633

B 0.623

C 0.172

D 0.377

E 0.6

$\text{Bi}(n, p) = \text{Bi}(10, 0.5)$

$\Pr(X \ge 6) = 0.377$

```
binomialCDf(6,10,10,0.5)
                0.376953125
```

QUESTION 13

One white ball, two red balls and three black balls are placed in a bag. If a ball is drawn from the bag, noted, returned to the bag and then a 2nd ball is drawn, the probability that no white ball is drawn is

A $\frac{1}{36}$

B $\frac{25}{36}$

C $\frac{11}{36}$

D $\frac{35}{36}$

E $\frac{1}{12}$

One white ball, two red balls, three black balls.

We are sampling with replacement, and we need to find Pr(0W).

$$\begin{aligned} \Pr(0W) &= 1 - (\text{at least } 1W) \\ &= 1 - (1W, 1 \text{ other} + 1 \text{ other}, 1W + 2W) \\ &= 1 - \left(2 \times \frac{1}{6} \times \frac{5}{6} + \frac{1}{6} \times \frac{1}{6}\right) \\ &= 1 - \left(\frac{10}{36} + \frac{1}{36}\right) \\ &= 1 - \frac{11}{36} \\ &= \frac{25}{36} \end{aligned}$$

QUESTION 14

The maximal domain for the function

$f(x) = \frac{1}{\sqrt{9 - x^2}} + \log_e(x + 1)$ is

A $R \backslash \{3, -3\}$

B $(-1, 3)$

C $[-1, 3]$

D $[-3, 3]$

E $(-1, \infty)$

Domain of $y = \frac{1}{\sqrt{9 - x^2}}$ is $(-3, 3)$.

Domain of $y = \log_e(x + 1)$ is $(-1, \infty)$.

So, the domain of $f(x) = \frac{1}{\sqrt{9 - x^2}} + \log_e(x + 1)$, using the intersection of domains, is $(-1, 3)$.

QUESTION 15

If $\int_0^3 f(x)\, dx = 5$, then $-\int_0^3 (3f(x) + 1)\, dx$ equals

A 16

B −5

C −12

D 18

E −18

We are given $\int_0^3 f(x)\, dx = 5$

$$\begin{aligned} \text{So } -\int_0^3 (3f(x)+1)\,dx &= -3\int_0^3 f(x)\,dx - \int_0^3 1\,dx \\ &= -3 \times 5 - \int_0^3 1\,dx \\ &= -15 - [x]_0^3 \\ &= -15 - 3 \\ &= -18 \end{aligned}$$

So $-\int_0^3 (3f(x)+1)\, dx = -18$

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QUESTION 16

The amplitude and period respectively of

$y = -3\sin\left(\frac{\pi x}{6}\right) + 1$ are

A $-3, 2\pi$

B $-3, 12$

C $3, 12$

D $3, 6\pi$

E $3, 12\pi$

> $y = -3\sin\left(\frac{\pi x}{6}\right) + 1$
>
> Amplitude = 3
>
> Period $= \dfrac{2\pi}{\frac{\pi}{6}} = 12$

QUESTION 17

The discrete random variable X has the following probability distribution.

x	-1	0	1
$\Pr(X = x)$	a	0.5	b

The **incorrect** statement is

A $a + b = 0.5$

B $E(X) = -a + b$

C $E(X^2) = a + b$

D $Var(X) = 2a$

E median = 0

> We know that $a + b + 0.5 = 1$
>
> and that $E(X) = -a + b$
>
> $Var(X) = E(X^2) - (-a + b)^2$
>
> $= a + b - (b - a)^2$
>
> So $Var(X) = 2a$ is wrong.

QUESTION 18

Consider the function $y = 3 - \frac{2}{x+3}$. The **incorrect** statement is

A its graph does not have an inverse

B its graph has a horizontal asymptote at $y = 3$

C its graph has a vertical asymptote at $x = -3$

D its graph is in the 2nd and 4th quadrants

E its graph goes through the point $(-1, 2)$

> $y = 3 - \frac{2}{x+3}$ has an inverse.
>
> The incorrect statement is 'its graph does not have an inverse'.

QUESTION 19

The temperature of coffee, in °C, just after it is made is modelled by the function $f(t)$, where t is the time in minutes after the coffee is made. The average temperature of the coffee over the first 10 minutes is expressed by

A $\int_0^{10} f(t)\,dt$

B $10\int_0^{10} f(t)\,dt$

C $\frac{1}{2}\int_0^{10} f(t)\,dt$

D $\frac{1}{10-0}\int_0^{10} f(t)\,dt$

E $\frac{1}{600}\int_0^{600} f(t)\,dt$

> Average value $= \frac{1}{10-0}\int_0^{10} f(t)\,dt$

QUESTION 20

The temperature of coffee, in °C, just after it is made is modelled by the function $f(t) = 80e^{-\frac{t}{5}} + 20$, where t is the time in minutes after the coffee is made. The rate of temperature change in the coffee after 10 minutes, in °C/min, is

A -6.917

B 30.83

C $-16e^{-2}$

D $80(e^{-2} - 1)$

E $\frac{16}{5}e^{-2}$

> $f(t) = 80e^{-\frac{t}{5}} + 20$
>
> $f'(t) = -0.2 \times 80e^{-\frac{t}{5}} = -16e$
>
> $f'(10) = -16e^{-2}$

```
define f(x)=80e^(-x/5)+20
                                done
diff(f(x),x,1,10)
                              -16·e^-2
```

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ONE ANSWER PER LINE						USE PENCIL ONLY					
1	A	B	■	D	E	11	■	B	C	D	E
2	A	B	C	D	■	12	A	B	C	■	E
3	A	B	C	D	■	13	A	■	C	D	E
4	A	B	C	D	■	14	A	■	C	D	E
5	A	B	■	D	E	15	A	B	C	D	■
6	A	■	C	D	E	16	A	B	■	D	E
7	A	B	■	D	E	17	A	B	C	■	E
8	■	B	C	D	E	18	■	B	C	D	E
9	A	B	C	■	E	19	A	B	C	■	E
10	A	B	C	■	E	20	A	B	■	D	E

Section B: Extended-response questions

Specific instructions to students

- Answer all of the questions in the spaces provided.
- Show all workings in questions where more than one mark is available.
- An exact value must be provided in questions where a numerical answer is required, unless otherwise specified.

QUESTION 21

A drink outlet, Michigan Milkshakes, has a game for customers to play when they buy a milkshake. Whenever the outlet sells one milkshake, the customer is given a 6-sided die to roll. If the number on the uppermost face of the die is a prime number, then the customer gets a free scoop of ice-cream.

a What is the probability that a randomly-selected customer gets a free scoop of ice-cream?

$\Pr(\text{prime}) = \Pr(2, 3, 5) = \frac{3}{6} = \frac{1}{2}$

2 marks

b What is the probability that 2 randomly-selected customers get a free scoop of ice-cream each?

$\Pr(\text{2prime}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

1 mark

c There are now 10 customers. What is the probability, correct to 4 decimal places, that not more than 2 customers get a free scoop of ice-cream?

$\text{Bi}(n, p) = \text{Bi}\left(10, \frac{1}{2}\right)$

$\Pr(X \le 2) = 0.0547$

$\text{binomialCDF}\left(0, 2, 10, \frac{1}{2}\right)$ 0.0546875

2 marks

The game is changed and the customer is given two dice to roll. If the numbers uppermost on the dice add to 7, then the customer gets hot fudge on the top of their milkshake.

d What is the probability that 2 randomly-selected customers get hot fudge on the top of their milkshakes?

$\Pr(\text{sum to } 7) = \Pr(1{,}6 + 6{,}1 + 2{,}5 + 5{,}2 + 4{,}3 + 3{,}4)$

$= \frac{6}{36} = \frac{1}{6}$

$\Pr(\text{2 sum to } 7) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

2 marks

The two dice actually have the Michigan Milkshakes logo in place of the 1 on their dice, so the 1 no longer counts. When the 2 dice are rolled, a sum of 7 earns hot fudge and if either uppermost number is a prime, the customer earns 1 free scoop of ice-cream.

e What is the probability that Manuel gets a free scoop of ice-cream **and** hot fudge on top of his milkshake?

$\Pr(\text{prime} \cap \text{sum to } 7)$

= prime 1st and sum to 7, or prime 2nd and sum to 7, or prime both and sum to 7

$= \Pr(3, 4) + \Pr(4, 3) + \Pr(2, 5) + \Pr(5, 2)$

$= 4 \times \frac{1}{36} = \frac{1}{9}$

3 marks

After milkshakes, Manuel leaves Michigan Milkshakes and goes to a show called Michigan Mimes. The show is priced in an interesting way. The ticket costs \$9 plus the number, in dollars, on top of a 6-sided die when rolled at the ticket counter.

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f What is the probability that Manuel pays an even number of dollars for his ticket to Michigan Mimes?

Pr(even) = \$9 and 1, or \$9 and 3, or \$9 and 5

$= Pr(1) + Pr(3) + Pr(5)$

$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$

$= \frac{1}{2}$

2 marks

If the patrons going to Michigan Mimes tweeted about the show in advance and show their tweet, they get to play for their ticket price with a 6-sided die that has the 6 blanked out.

g In this case, what is the probability that a randomly-selected customer pays an odd number of dollars for their ticket to Michigan Mimes?

Pr(odd) = Pr(2) + Pr(4) + 6 missing

$= \frac{1}{6} + \frac{1}{6}$

$= \frac{1}{3}$

2 marks

If the patrons booked their ticket online, they pay \$20 in advance and then get to roll a 7-sided die, with the numbers 1 to 7, and then subtract that amount in dollars from the \$20 that they already paid.

h Manuel paid for his ticket in this way. What is the probability that Manuel pays an odd number of dollars for his ticket to Michigan Mimes?

Pr(odd) = \$20 less 1, or \$20 less 3, or \$20 less 5, or \$20 less 7

$= Pr(1) + Pr(3) + Pr(5) + Pr(7)$

$= \frac{1}{7} + \frac{1}{7} + \frac{1}{7} + \frac{1}{7}$

$= \frac{4}{7}$

2 marks

(Total: 16 marks)

QUESTION 22

Consider the function $f(x) = (x - 1)(x^2 - k)$, where k is a real constant.

a Find the rule for $g(x)$ if $g(x) = -f(2x - 1) + 3$. Give your answer in the form of $ax^3 + bx^2 + cx + d$.

$f(x) = (x - 1)(x^2 - k)$

$f(2x - 1) = (2x - 1 - 1)[(2x - 1)^2 - k]$

$-f(2x - 1) = -(2x - 2)[(2x - 1)^2 - k]$

$-f(2x - 1) + 3 = -(2x - 2)[(2x - 1)^2 - k] + 3$

```
expand(-(2·x-2)·((2·x-1)^2-k)+3)
                    -8·x^3+16·x^2+2·k·x-10·x-2·k+5
```

$g(x) = -f(2x - 1) + 3 = -8x^3 + 16x^2 + x(2k - 10) + 5 - 2k$

2 marks

b Find the equation of the tangent line to the curve $g(x)$, when $x = 1$, in terms of k.

$y = -8x^3 + 16x^2 + x(2k - 10) + 5 - 2k$

$\frac{dy}{dx} = -24x^2 + 32x + (2k - 10)$

When $x = 1$, $\frac{dy}{dx} = -24 + 32 + (2k - 10)$

Gradient $= 2k - 2$

when $x = 1$, $y = 3$

$y - y_1 = m(x - x_1)$

$y - 3 = (2k - 2)(x - 1)$

gives

$y = (2k - 2)x + 5 - 2k$

```
define g(x)=-8x^3+16x^2+x(2k-10)+5-2k
                                   done
diff(g(x),x,1,1)
                                   2·k-2
tanLine(g(x),x,1)
                          x·(2·k-2)-2·k+5
```

3 marks

c Find the equation of the line that is perpendicular to the tangent found in part **b**, also going through the point when $x = 1$, in terms of k.

When $x = 1$, perpendicular gradient $= -\frac{1}{2k - 2}$

$y - y_1 = m(x - x_1)$

$y - 3 = -\frac{1}{2k - 2}(x - 1)$

gives $y = -\frac{x}{2k - 2} + \frac{1}{2k - 2} + 3$

```
normal(g(x),x,1)
                 -x/(2·k-2) + 1/(2·k-2) + 3
```

3 marks

d Find the value(s) of k, if it is possible, such that the tangent found in part **b** is also a tangent to $f(x) = (x - 1)(x^2 - k)$ at $x = 1$.

The equation of the tangent at $x = 1$ to $f(x) = (x - 1)(x^2 - k)$ is

$y = -x(k - 1) + k + 1$

```
define f(x)=(x−1)(x^2−k)
                                done
tanLine(f(x),x,1)
                     −x·(k−1)+k−1
solve(−x·(k−1)+k−1=(2·k−2)·x+5−2·k,k)
                         {k=(x−2)/(x−1)}
```

Equate the 2 tangents and solve for k.

$(2k - 2)x + 5 - 2k = -x(k - 1) + k - 1$

$(2k - 2)x + 5 - 2k = -xk + x + k - 1$

$2kx + xk - 3k = 2x - 5 + x - 1$

$3k(x - 1) = 3x - 6$

$k = \dfrac{3(x-2)}{3(x-1)}$

$k = \dfrac{x-2}{x-1}$

For $x = 1$ this is not possible.

3 marks

(Total: 11 marks)

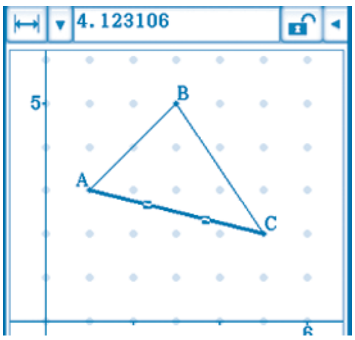
QUESTION 23

William likes playing with secret stones. He digs a hole in which to bury the stones and sets up the digging site with sticks at the points $A(1, 3)$, $B(3, 5)$ and $C(5, 2)$, where the distances are in metres.

a Find the distance AC.

$AC = \sqrt{(2-3)^2 + (5-1)^2} = \sqrt{1^2 + 4^2}$

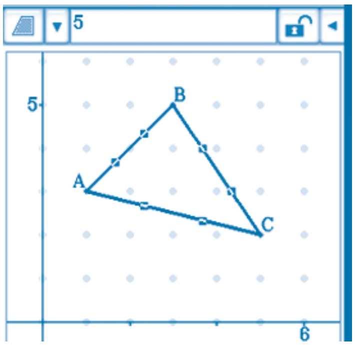
$AC = \sqrt{17}$ metres



Length AC shown here is 4.123 m.

1 mark

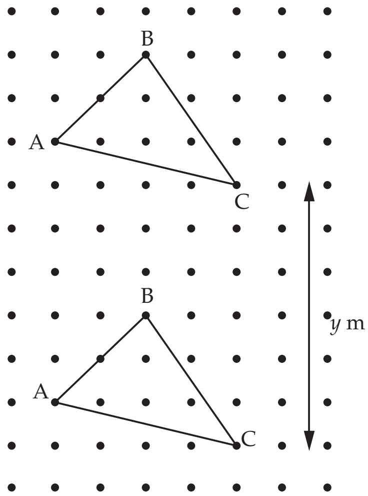
b Find the area of the triangle ABC.



Area of triangle = 5 m^2

3 marks

William plans this triangle to form a triangular prism with a depth of y metres.



c What is the volume of the prism in terms of y?

Vol = area of base × height

Vol = $5y$ m^3

1 mark

William drops the stone into the hole with the velocity given by $v(t) = -2e^t + b$, where t is the time in seconds after the stone has been dropped and b is a constant.

d Find, in terms of b, when the velocity equals zero.

Solve $v(t) = 0$.

$t = \log_e\left(\dfrac{b}{2}\right)$

```
define v(t)=−2e^t+b
                                done
solve(v(t)=0,t)
                     {t=ln(b)−ln(2)}
simplify({t=ln(b)−ln(2)})
                         {t=ln(b/2)}
```

1 mark

ISBN 9780170354141

e Define a function $x(t)$ that describes the displacement of the stone after t seconds, if the origin is at the point of throwing.

$x(t) = \int(-2e^t + b)\,dt$

$\int v(t)\,dt \quad -2\cdot e^t + b\cdot t$

$x(t) = -2e^t + bt + c$

$0 = -2 + c$ so $c = 2$

$x(t) = -2e^t + bt + 2$

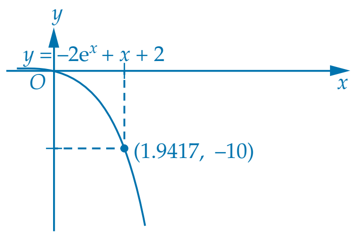
2 marks

f Sketch a graph of $x(t)$, letting $b = 1$.

$x(t) = -2e^t + bt + 2$

When $b = 1$

$x(t) = -2e^t + t + 2$



3 marks

g At what velocity does the stone hit the bottom of the hole if $y = 10$ and $b = 1$? (Answer to 3 decimal places.)

At height y, displacement $= -y$

Solve $y = -2e^t + bt + 2$, where $b = 1$ and $y = 10$.

$-10 = -2e^t + t + 2$

$\text{solve}(y=-2\cdot e^t+t+2, t) \quad \{-2\cdot e^t-y+t+2=0\}$

$\text{solve}(-10=-2\cdot e^t+t+2, t, 0, -\infty, \infty) \quad \{t=-11.99998771, t=1.941740041\}$

$t = 1.9417$

$v(t) = -2e^t + 1$

$v(1.9417) = -2e^{1.9417} + 1$

$= -12.941$ m/s

2 marks

(Total: 13 marks)