STUDENT NUMBER | LETTER

Figures									
Words									

MATHEMATICAL METHODS

Written examination 2

Reading time: 15 minutes

Writing time: 2 hours

QUESTION AND ANSWER BOOK

Section	Number of questions	Number of questions to be answered	Marks
1	22	22	22
2	4	4	58

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved CAS calculator (memory DOES NOT need to be cleared) and, if desired, one scientific calculator. For approved computer-based CAS, their full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper, white out liquid/tape or a calculator of any type.

Materials supplied
- Question and answer book of 25 pages, with a detachable sheet of miscellaneous formulas.
- Working space is provided throughout the book.

Instructions
- Detach the formula sheet from the back of this book during reading time.
- Write your **student number** in the space provided above on this page.
- All written responses must be in English.
- Unless otherwise indicated, the diagrams in this book are not drawn to scale.

At the end of the examination
- Place the answer sheet for multiple-choice questions inside the front cover of this book.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

ISBN 9780170354141

MATHS METHODS

Written Examination 2
Multiple-Choice Answer Sheet

STUDENT NAME:	*Your name will be printed here*

STUDENT NUMBER

9	9	1	2	3	4	5	6	A
0	0	0	0	0	0	0	0	A
1								E
2								F
3								G
4	4	4	4	4	4	4	4	J
5	5	5	5	5	5	5	5	L
6	6	6	6	6	6	6	6	R
7	7	7	7	7	7	7	7	T
8	8	8	8	8	8	8	8	W
9	9	9	9	9	9	9	9	X

Your student number will be recorded here for you to check

INSTRUCTIONS: USE PENCIL ONLY

SIGN BELOW IF YOUR NAME AND NUMBER ARE PRINTED CORRECTLY

SIGNATURE:

If your name or number on this sheet is incorrect, notify the Supervisor.
Use a PENCIL for ALL entries. For each question, shade the box which indicates your answer. All answers must be completed like THIS example: A B C D
Marks will NOT be deducted for incorrect answers.
NO MARK will be given if more than ONE answer is completed for any question.
If you make a mistake, ERASE the incorrect answer — DO NOT cross it out.

SUPERVISOR USE ONLY

USE PENCIL ONLY

Shade the "**ABSENT**" box if the student was absent from the examination.

ABSENT

SUPERVISOR'S INITIALS

ONE ANSWER PER LINE

1	■	B	C	D	E
2	A	B	C	■	E
3	A	B	■	D	E
4	A	■	C	D	E
5	A	B	C	■	E
6	A	■	C	D	E
7	A	B	■	D	E
8	A	B	C	D	■
9	A	B	C	■	E
10	A	B	C	■	E
11	A	B	C	■	E

ONE ANSWER PER LINE

12	A	B	■	D	E
13	A	B	C	■	E
14	A	■	C	D	E
15	■	B	C	D	E
16	A	B	C	■	E
17	A	■	C	D	E
18	A	B	■	D	E
19	A	■	C	D	E
20	A	B	C	■	E
21	A	B	C	D	■
22	A	B	■	D	E

Please DO NOT fold, bend or staple this form.

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Section A: Multiple-choice questions

Specific instructions to students

- Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
- A correct answer scores 1 mark, and an incorrect answer scores 0.
- Marks are not deducted for incorrect answers.
- No marks are given if more than one answer is completed for any question.

QUESTION 1

The area of the region enclosed by the graph of $f(x) = (x + 2)(x - 1)(x - 4)$ and the x-axis is

A $\dfrac{81}{2}$

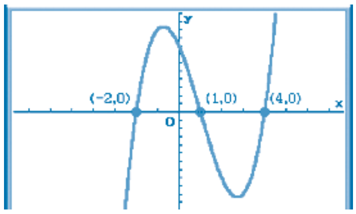
B $\dfrac{81}{4}$

C 81

D 0

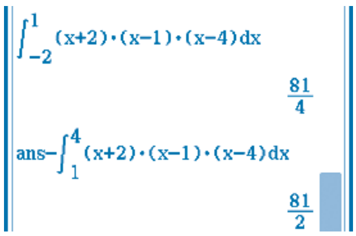
E 48

Area between $f(x) = (x + 2)(x - 1)(x - 4)$ and the x-axis.



Graph shows the area above and below the x-axis.

Area = $\int_{-2}^{1} f(x)dx + \left|\int_{1}^{4} f(x)dx\right|$ or $\int_{-2}^{1} f(x)dx - \int_{1}^{4} f(x)dx$



Area = $\dfrac{81}{2}$

QUESTION 2

If $f(x) = \log_e(x + 2) + \dfrac{x}{3} + \dfrac{1}{3}$, then $f'(x) =$

A $\dfrac{1}{x+2} - \dfrac{x}{3}$

B $2x \log_e(x + 2) + \dfrac{1}{3}$

C $\dfrac{x+5}{3}$

D $\dfrac{x+5}{3(x+2)}$

E $\dfrac{1}{x+2} + \dfrac{x}{3}$

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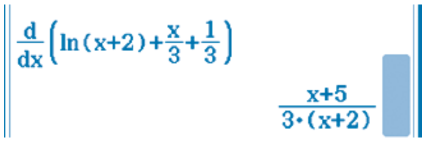
$f(x) = \log_e(x+2) + \frac{x}{3} + \frac{1}{3}$



$f'(x) = \dfrac{x+5}{3(x+2)}$

QUESTION 3

The point $P(6, -2)$ is translated by 2 units in the positive direction of the x-axis, then reflected over the y-axis. The coordinates of the final image of P are

A (6, 2)

B 2

C (–8, –2)

D (8, 2)

E (–4, –2)

$P(6, -2)$

Translated by 2 units in the positive direction of the x-axis, it becomes (8, –2).

Then reflected over the y-axis, it becomes (–8, –2).

So the image of P is (–8, –2).

QUESTION 4

A function is defined by $f: D \to \boldsymbol{R}, f(x) = 3 - 2x$ and has a range of [–1, 5). The domain, D, is

A [–1, 2]

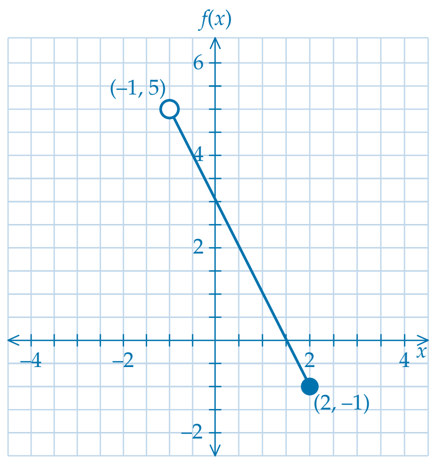
B (–1, 2]

C [2, –1)

D $\boldsymbol{R}$

E [5, –1)

$f: D \to \boldsymbol{R}, f(x) = 3 - 2x$ and has a range of [–1, 5).



The graph's endpoints are (–1, 5) and (2, –1).

Domain, D, is (–1, 2].

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QUESTION 5

If $g(x) = e^{f(x)}$, then $g'(x) =$

A $e^{f(x)}$

B $f(x)\, e^{f(x)}$

C $f(x)\, e^{f(x)-1}$

D $f'(x)\, e^{f(x)}$

E $g'(x)\, e^{f(x)}$

> $g(x) = e^{f(x)}$
>
> Using the chain rule,
>
> $g'(x) = f'(x)\, e^{f(x)}$

QUESTION 6

The graph of $f(x) = ax^4 - c$ has an x-intercept of $x = 2$ and the equation of the tangent at $x = 2$ is $y = -4x + 8$. The values of a and c, respectively, are

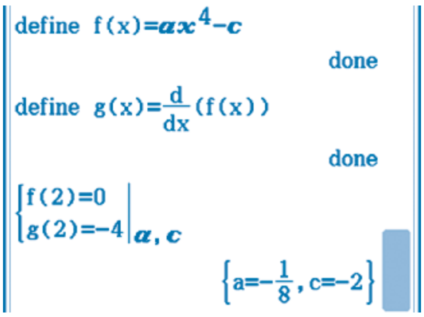
A $-\frac{1}{8}, 2$

B $-\frac{1}{8}, -2$

C $-\frac{1}{128}, -\frac{1}{8}$

D $-2, -\frac{1}{8}$

E $-8, -2$

> $f(x) = ax^4 - c$
>
> x-intercept at $x = 2$, the equation of the tangent at $x = 2$ is $y = -4x + 8$.
>
> We need $f(2) = 0$ and $f'(2) = -4$
>
> Solving simultaneously,
>
> 
>
>
> we get $a = -\frac{1}{8}$ and $c = -2$.

QUESTION 7

A continuous random variable, X, is normally distributed with a mean of 10 and a standard deviation of 0.7. $\Pr(X < 9.3)$ is equal to

A $\Pr(Z > 0.5)$

B $\Pr(Z > -1)$

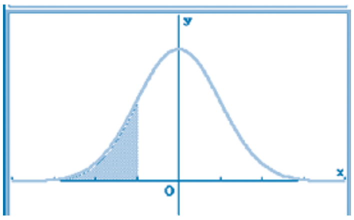
C $\Pr(Z > 1)$

D $\Pr(Z < 9.3)$

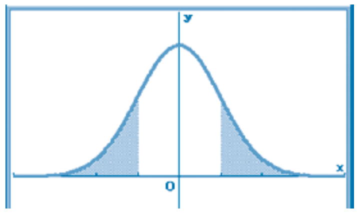
E $\Pr(Z < 1)$

Normally distributed, mean = 10, standard deviation = 0.7

1 s.d. lower than 10 is 9.3.



is the same as



$\Pr(X < 9.3)$ is equal to $\Pr(Z > 1)$.

QUESTION 8

The transformation $T: R^2 \to R^2$, with the rule $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ maps the line $y + 2x = 6$ onto the line with equation

A $y = \frac{3}{4}x + 26$

B $y = \frac{4}{3}x - 13$

C $3y + 4x = 13$

D $4x - 3y = 52$

E $4x - 3y = 26$

$\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ becomes $\begin{bmatrix} 3x - 1 \\ -2y + 2 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$

$3x - 1 = x_1 \Rightarrow x = \frac{1}{3}(x_1 + 1)$

$-2y + 2 = y_1 \Rightarrow y = -\frac{1}{2}(y_1 - 2)$

Applied to equation $y + 2x = 6$, we get

$-\frac{1}{2}(y_1 - 2) + \frac{2}{3}(x_1 + 1) = 6$

$-\frac{1}{2}y_1 + 1 + \frac{2}{3}x_1 + \frac{2}{3} = 6$

$-3y_1 + 4x_1 = 26$

$4x - 3y = 26$

QUESTION 9

Let f be the function with domain R such that $f'(3) = 0$ and $f'(x) > 0$ when $x \neq 3$. At $x = 3$, the graph of f has a

A local minimum

B local maximum

C gradient of 3

D stationary point of inflection

E non-stationary point of inflection

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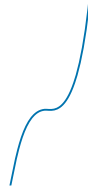
$f'(3) = 0$ and $f'(x) > 0$ when $x \neq 3$ give the general shape of



At $x = 3$, f has a stationary point of inflection.

QUESTION 10

A function defined by $f: D \to \mathbf{R}$, where $f(x) = x^4 + \frac{20}{3}x^3 - 16x^2 - 48x + 2$ will **NOT** have an inverse function if D equals

A $[-1, 2]$

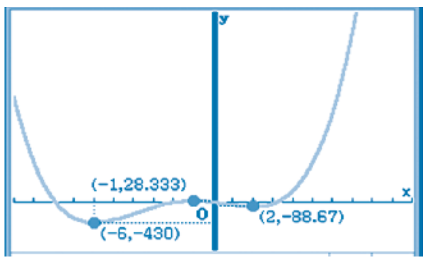
B $(-1, 2)$

C $[-6, -1)$

D $[-6, 2]$

E $[3, \infty)$

$f: D \to \mathbf{R}$, where $f(x) = x^4 + \frac{20}{3}x^3 - 16x^2 - 48x + 2$



For the inverse, we need to have a one-to-one section.

We will **NOT** have an inverse function if D equals $[-6, 2]$.

QUESTION 11

A probability density function defined by

$$f(x) = \begin{cases} k\sin(x), & 0 \le x \le \pi, \text{ where } k \text{ is a real constant} \\ 0, & \text{otherwise} \end{cases}$$

has a mean of $\frac{\pi}{2}$. The variance of f is

A $\frac{\pi}{2}$

B $\frac{\pi^2}{4}$

C $\frac{\pi^2}{2} - 2$

D $\frac{\pi^2 - 8}{4}$

E $\pi\left(\frac{\pi - 2}{4}\right)$

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$f(x) = \begin{cases} k \sin(x), & 0 \le x \le \pi \\ 0, & \text{otherwise} \end{cases}$ has a mean of $\frac{\pi}{2}$.

solving integral = 1 gives $k = \frac{1}{2}$

Variance $= E(X^2) - (E(X))^2$

$$= \frac{\pi^2}{2} - 2 - \left(\frac{\pi}{2}\right)^2$$

$$= \frac{\pi^2}{2} - 2 - \frac{\pi^2}{4}$$

$$= \frac{\pi^2 - 8}{4}$$

The variance of f is $\frac{\pi^2 - 8}{4}$.

$\text{solve}\left(\int_0^{\pi} k \cdot \sin(x)\,dx = 1, k\right)$

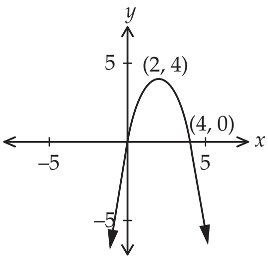
$\left\{k = \frac{1}{2}\right\}$

$\int_0^{\pi} \frac{1}{2} x^2 \sin(x)\,dx - \left(\frac{\pi}{2}\right)^2$

$\frac{\pi^2}{4} - 2$

QUESTION 12

The rule of the function whose graph is shown below is



A $y = (x - 2)^2 + 4$

B $y = -2(x - 2)^2 + 4$

C $y = -(x - 2)^2 + 4$

D $y = (x - 4)^2 + 2$

E $y = -(x - 4)^2 + 2$

The rule is of the form $y = a(x - 2)^2 + 4$

Substitute (4, 0) into the equation.

We get $0 = 4a + 4$

So $a = -1$

Thus, the rule is $y = -(x - 2)^2 + 4$

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QUESTION 13

The functional equation $\frac{1}{4}f(f'(x)) = x^2$ is satisfied by the rule

A $f(x) = 2x$

B $f(x) = \log_e(x)$

C $f(x) = e^x$

D $f(x) = x^2$

E $f(x) = 2 - x$

Functional equation: $\frac{1}{4}f(f'(x)) = x^2$

Test Option D: $f(x) = x^2$

$f'(x) = 2x$ so

$\text{LHS} = \frac{1}{4}f(f'(x)) = \frac{1}{4}f(2x) = \frac{1}{4}4x^2 = x^2$

$\text{RHS} = x^2 = \text{LHS}$

Hence, the functional equation is satisfied by the rule $f(x) = x^2$.

QUESTION 14

The inverse of the function $f: (2, \infty) \rightarrow \boldsymbol{R}, f(x) = \frac{1}{\sqrt{x-2}}$ is

A $f^{-1}: \boldsymbol{R} \rightarrow (2, \infty), f^{-1}(x) = \frac{1}{\sqrt{x-2}}$

B $f^{-1}: (0, \infty) \rightarrow \boldsymbol{R}, f^{-1}(x) = \frac{1}{x^2} + 2$

C $f^{-1}: (2, \infty) \rightarrow \boldsymbol{R}, f^{-1}(x) = \sqrt{x-2}$

D $f^{-1}: \boldsymbol{R}\backslash\{0\} \rightarrow \boldsymbol{R}, f^{-1}(x) = \frac{1}{x^2} + 2$

E $f^{-1}: (0, \infty) \rightarrow \boldsymbol{R}, f^{-1}(x) = \sqrt{x-2}$

$f:(2, \infty) \rightarrow \boldsymbol{R}, f(x) = \frac{1}{\sqrt{x-2}}$

Swap x and y and rearrange to find the inverse.

$x = \frac{1}{\sqrt{y-2}}$

$\Rightarrow y - 2 = \frac{1}{x^2}$

$\Rightarrow y = \frac{1}{x^2} + 2$

Domain of f^{-1} = range of $f = (0, \infty)$

The inverse of the function is $f^{-1}: (0, \infty) \rightarrow \boldsymbol{R}, f^{-1}(x) = \frac{1}{x^2} + 2$

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QUESTION 15

If $\int_1^5 f(x)\,dx = 3$, then $\int_1^5 (3-2f(x))\,dx$ is equal to

A 6

B −6

C 12

D 9

E 3

$$\begin{aligned}\int_1^5 (3-2f(x))\,dx &= \int_1^5 3\,dx - \int_1^5 2f(x)\,dx\\ &= [3x]_1^5 - 2\int_1^5 f(x)\,dx \qquad \text{where } \int_1^5 f(x)\,dx = 3\\ &= 12 - 2\times 3\\ &= 6\end{aligned}$$

QUESTION 16

A magic bag has coloured cubes in it. There are 3 green cubes and 6 red cubes. I select a cube, look at it, and put it back. I then select another cube. What is the probability that I select two cubes of different colours?

A $\frac{1}{9}$

B $\frac{2}{9}$

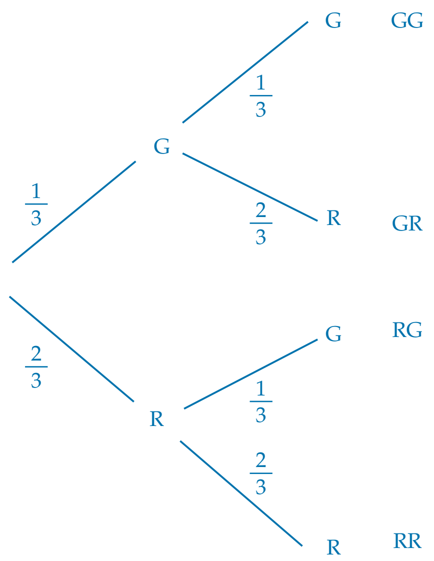
C $\frac{1}{3}$

D $\frac{4}{9}$

E $\frac{2}{3}$

There are 3 green cubes and 6 red cubes, a total of 9 cubes.

The tree diagram for selecting 2 with replacement is below.



$$\Pr(\text{GR or RG}) = \frac{1}{3}\times\frac{2}{3}+\frac{2}{3}\times\frac{1}{3} = \frac{4}{9}$$

The probability that I select two cubes of different colours $= \frac{4}{9}$

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QUESTION 17

The domain of the composite function $f(x) = \log_e(\cos(x))$, where $\cos(x)$ is defined for $x \in [0, 2\pi]$, is

A $[0, \frac{\pi}{2})$

B $[0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi]$

C $(\frac{3\pi}{2}, 2\pi]$

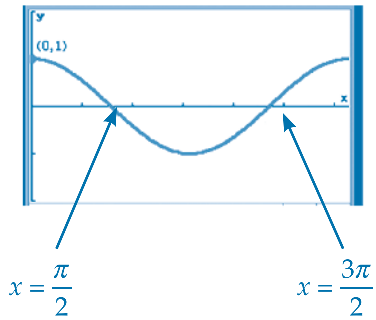
D $[0, 2\pi]$

E $(0, \infty)$

$f(x) = \log_e(\cos(x))$, where $\cos(x)$ is defined for $x \in [0, 2\pi]$.

Test ran (inner) $\subseteq$ dom (outer)

$[-1, 1] \not\subset (0, \infty)$, so restrict $[-1, 1]$ to $(0, 1]$.



This gives dom of $f(x) = (\log_e(\cos(x)) =$ dom of $\cos(x)$.

Domain of composite function $= [0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi]$

QUESTION 18

If X is a random variable such that $\Pr(X < 3) = a$ and $\Pr(X < 7) = b$, then $\Pr(X > 7 \mid X > 3)$ equals

A $\frac{b}{a} - 1$

B $1 - \frac{b}{a}$

C $\frac{1-b}{1-a}$

D $\frac{a}{b}$

E $1 - \frac{a}{b}$

For $\Pr(X < 3) = a$ and $\Pr(X < 7) = b$,

$$\begin{aligned}\Pr(X > 7 \mid X > 3) &= \frac{\Pr(X > 7 \cap X > 3)}{\Pr(X > 3)} \\ &= \frac{\Pr(X > 7)}{\Pr(X > 3)} \\ &= \frac{1-b}{1-a}\end{aligned}$$

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QUESTION 19

The simultaneous equations $kx - 2y = 2$ and $-3x + ky = 2k$, where k is a real constant, have a unique solution for

A $k \in \boldsymbol{R} \backslash \left[-\sqrt{6}, \sqrt{6}\right]$

B $k \in \boldsymbol{R} \backslash \left\{-\sqrt{6}, \sqrt{6}\right\}$

C $k \in \left\{-\sqrt{6}, \sqrt{6}\right\}$

D $k = \sqrt{6}$

E $k = \pm 6$

Solve for k in $kx - 2y = 2$ and $-3x + ky = 2k$.

$$\text{solve}\left(\det\left(\begin{bmatrix} k & -2 \\ -3 & k \end{bmatrix}\right)=0, k\right)$$

$$\{k=-\sqrt{6}, k=\sqrt{6}\}$$

$k \in \left\{-\sqrt{6}, \sqrt{6}\right\}$ is where there is no unique solution.

So we have a unique solution for $k \in \boldsymbol{R} \backslash \left\{-\sqrt{6}, \sqrt{6}\right\}$.

Alternative Solution

Rearrange the equations

$kx - 2y = 2$ and $-3x + ky = 2k$

to get

$y = \frac{kx-2}{2}$ and $y = \frac{3x+2k}{k}$

Equate the gradients to find where there is **no** unique solution.

$\Rightarrow \frac{k}{2} = \frac{3}{k}$

$\Rightarrow k^2 = 6$

So $k = \pm\sqrt{6}$

So we have a unique solution for $k \in \boldsymbol{R} \backslash \left\{-\sqrt{6}, \sqrt{6}\right\}$.

QUESTION 20

The graphs of $f(x) = x^3 + 2x^2 - x - 6$ and $f(x) = x^3 + x^2 - 2x + k$ intersect only once for

A $k = \boldsymbol{R} \backslash \left\{\frac{5}{2}\right\}$

B $k > \frac{5}{2}$

C $0 < k < \frac{25}{4}$

D $k = -\frac{25}{4}$

E $k > \frac{25}{4}$

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Solve for k in $x^3 + 2x^2 - x - 6 = x^3 + x^2 - 2x + k$.

```
define f(x)=x^3+2x^2-x-6
                                done
define g(x)=x^3+x^2-2x+k
                                done
f(x)-g(x)=0
                        x^2+x-k-6=0
```

Discriminant $= 1 + 4(k + 6) = 4k + 25$

$4k + 25 = 0$ for one solution.

So $k = -\frac{25}{4}$

The graphs intersect only once for $k = -\frac{25}{4}$.

QUESTION 21

Andy and Ben are playing basketball hoops. The probability that they score a goal is independent of each other. The probability that Andy scores a goal is 0.6 and the probability that Ben scores a goal is 0.7. In a game of hoops where Andy and Ben play 10 games each, taking turns shooting hoops, how many games do they each have to play so that the probability that Andy and Ben score an equal number of goals is the same?

A 2

B 3

C 5

D 6

E 7

$\Pr(A) = 0.6$, $\text{Bi}(n, 0.6)$ where $\Pr(X = n) = {}^{10}C_n\, 0.4^{10-n}\, 0.6^n$

$\Pr(B) = 0.7$, $\text{Bi}(n, 0.7)$ where $\Pr(Y = n) = {}^{10}C_n\, 0.3^{10-n}\, 0.7^n$

Solve ${}^{10}C_n\, 0.4^{10-n}\, 0.6^n = {}^{10}C_n 0.3^{10-n}\, 0.7^n$

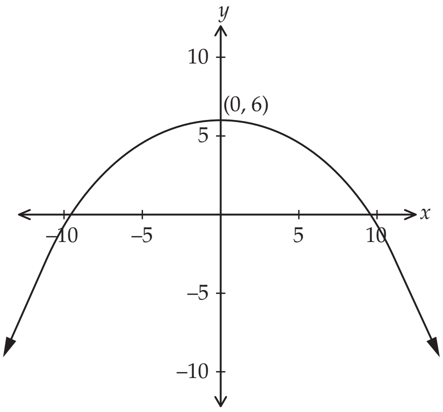
```
solve(0.4^(10-x)·0.6^x=0.3^(10-x)·0.7^x,x)
                         {x=6.511107902}
```

$n = 7$

7 games must be played so that the probability that Andy and Ben score an equal number of goals is the same.

QUESTION 22

The graph shown below is of the form $y = ax^2 + b$. The turning point is at (0, 6) and the gradient of the graph at $x = 1$ is –4. The average value of the graph from $x = -2$ to $x = 2$ is



A 0

B 6

C $\dfrac{10}{3}$

D $\dfrac{5}{3}$

E $\dfrac{40}{3}$

The general form of the graph shown is $y = ax^2 + b$. The turning point is at (0, 6), the gradient at $x = 1$ is –4.

$$\text{define } g(x)=ax^2+b \qquad \text{done}$$

$$\text{define } h(x)=\frac{d}{dx}(g(x)) \qquad \text{done}$$

$$\begin{cases} g(0)=6 \\ h(1)=-4 \end{cases}\Bigg|_{a,\,b} \qquad \{a=-2,\ b=6\}$$

The equation is $y = -2x^2 + 6$.

$$\text{Average value} = \frac{1}{2-(-2)}\int_{-2}^{2}\left(-2x^2+6\right)dx$$

$$= \frac{1}{4}\int_{-2}^{2}\left(-2x^2+6\right)dx$$

$$\text{define } f(x)=-2x^2+6 \qquad \text{done}$$

$$\int_{-2}^{2}\frac{1}{4}f(x)\,dx \qquad \frac{10}{3}$$

Average value $= \dfrac{10}{3}$

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Section 2: Extended-response questions

Specific instructions to students

- Answer **all** of the questions in the spaces provided.
- Show all workings in questions where more than one mark is available.
- An exact value must be provided in questions where a numerical answer is required, unless otherwise specified.
- Unless otherwise indicated, the diagrams in this book are not drawn to scale.

QUESTION 23

Consider the function $f: [-1, 3] \to \boldsymbol{R}, f(x) = 3x^2 - x^3$

a Find the coordinates of the stationary points of f.

$f(x) = 3x^2 - x^3$

$f'(x) = 6x - 3x^2$

For stationary points, solve for $f'(x) = 0$.

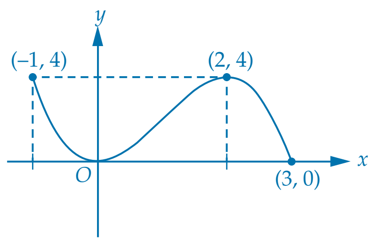
$6x - 3x^2 = 3x(2 - x) = 0$

This gives $x = 0, x = 2$.

The coordinates are $(0, 0)$ and $(2, 4)$.

2 marks

b Sketch the graph of f, labelling endpoints and stationary points with their coordinates.



3 marks

c What is the maximum value of f and for what x value(s) does the maximum exist?

Max = 4 at $x = -1$ and $x = 2$.

2 marks

d What is the minimum value of f and for what x-value(s) does the minimum exist?

Min = 0 at $x = 0$ and $x = 3$.

2 marks

e For what x-values is the graph of f strictly increasing?

Strictly increasing for $x \in [0, 2]$.

1 mark

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f A tangent is drawn to the curve at $x = \frac{7}{4}$. What is the equation of this tangent?

Use $y - y_1 = m(x - x_1)$

$f'(x) = 6x - 3x^2$

$f'\left(\frac{7}{4}\right) = \frac{21}{16}$

Also, $f\left(\frac{7}{4}\right) = \frac{245}{64}$

so $y - \frac{245}{64} = \frac{21}{16}\left(x - \frac{7}{4}\right)$

define $f(x)=3x^2-x^3 \mid -1 \le x \le 3$

done

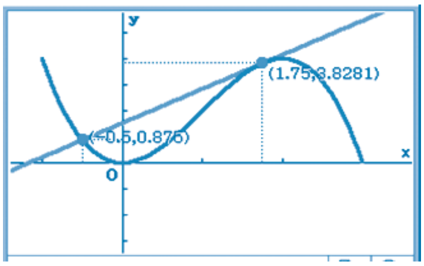
$\text{tanLine}\left(f(x), x, \frac{7}{4}\right)$

$\frac{21 \cdot x}{16} + \frac{49}{32}$

The equation of this tangent is $y = \frac{21}{16}x + \frac{49}{32}$

2 marks

g **i** An area is formed that is enclosed by the curve f and the tangent. Write down the integral for finding this area.



The tangent hits the curve again at point $\left(-\frac{1}{2}, \frac{7}{8}\right)$.

Area = $\int_{-\frac{1}{2}}^{\frac{7}{4}} \left(\frac{21x}{16} + \frac{49}{32}\right) - \left(3x^2 - x^3\right)dx$

3 marks

ii Hence evaluate this area.

Area = $\int_{-\frac{1}{2}}^{\frac{7}{4}} \left(\frac{21x}{16} + \frac{49}{32}\right) - \left(3x^2 - x^3\right)dx$

$\int_{-\frac{1}{2}}^{\frac{7}{4}} \frac{21 \cdot x}{16} + \frac{49}{32} - f(x)dx$

$\frac{2187}{1024}$

Area = $\frac{2187}{1024}$ square units

1 mark

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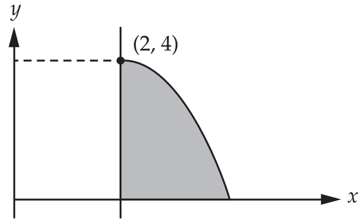
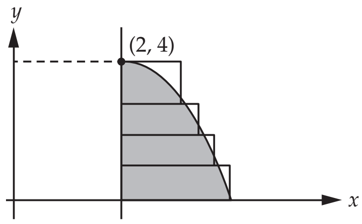
A different area is formed when a line is drawn parallel to the y-axis, starting at the point (2, 0) and ending at (2, 4), as shown in the diagram below.



h An approximation to the area between the line $x = 2$ and the curve to the right side of the line is found by a series of rectangles of width 1 unit, as shown below.



Find an approximation to the area using the four rectangles. Give your answer correct to 2 decimal places.

$$\begin{aligned}\text{Area} &= 1 \times \left(\sum f(y)\right)\\ &= 1(f(0) + f(1) + f(2) + f(3))\\ &= 1(1 + 0.879 + 0.732 + 0.532)\\ &= 3.14 \text{ square units}\end{aligned}$$

3 marks

i Find the ratio between the shaded area found by integration and your approximated area from part **h**.

$$\text{Area by integration} = \int_2^3 \left(3x^2 - x^3\right) dx = 2.75 \text{ square units}$$

$$\begin{aligned}\text{Ratio} &= 2.75 : 3.14\\ &= 275 : 314\end{aligned}$$

2 marks

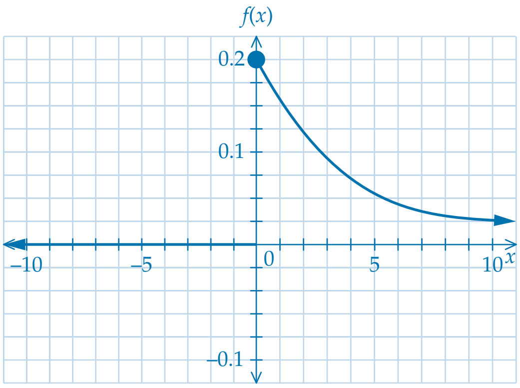
(Total: 21 marks)

QUESTION 24

The sales in a crockery company, x in \$1000s, follow a probability density function $f(x)$ where

$$f(x) = \begin{cases} \frac{1}{5}e^{-\frac{x}{5}}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

a Sketch a graph of $f(x)$.



2 marks

b State the mean sales of the crockery company.

Mean = $\int_0^\infty x \times \frac{1}{5} e^{-\frac{x}{5}} dx$

$$\text{define } f(x)=\frac{1}{5}\cdot e^{-\frac{x}{5}} \mid x\geq 0 \quad \text{done}$$

$$\int_0^\infty x\cdot f(x)\,dx \quad 5$$

Mean = 5

Mean sales are \$5000.

1 mark

c The median of X is m. Determine the value of m.

To find the median, solve $\int_0^m \frac{1}{5} e^{-\frac{x}{5}} dx = 0.5$ for m.

$$\int_0^\infty \frac{1}{5}\cdot e^{-\frac{x}{5}} dx \quad 1$$

$$\text{solve}\left(\int_0^m \frac{1}{5}\cdot e^{-\frac{x}{5}} dx=0.5, m\right) \quad \{m=5\cdot\ln(2)\}$$

Median = $5 \log_e (2)$

2 marks

d Find the probability, correct to 4 decimal places, that the sales are at least \$5000 per month.

Find $\Pr(X > 5)$.

$$\int_5^\infty \frac{1}{5}\cdot e^{-\frac{x}{5}} dx \quad e^{-1}$$

$\Pr(X > 5) = e^{-1} \approx 0.3679$ (to 4 decimal places)

2 marks

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e Find $\Pr(X < 1 \mid X \leq \mu)$, where μ is the mean correct to 3 decimal places.

$$\Pr(X < 1 \mid X \leq \mu) = \frac{\Pr(X < 1 \cap X < \mu)}{\Pr(X < \mu)}$$

From part **d**, $\Pr(X > 5) = \Pr(X > \mu) = e^{-1}$

So $\Pr(X < \mu) = 1 - e^{-1}$

$$\begin{aligned}\frac{\Pr(X < 1 \cap X < \mu)}{\Pr(X < \mu)} &= \frac{\Pr(X < 1)}{\Pr(X < 5)} \\ &= \frac{\Pr(0 < X < 1)}{\Pr(0 < X < 5)} \\ &= \frac{\int_0^1 \frac{1}{5} e^{-\frac{x}{5}} dx}{1 - e^{-1}} \\ &\approx \frac{0.18127}{0.63212} \\ &\approx 0.287\end{aligned}$$

2 marks

The company is concerned about the decreasing sales. After a certain time, they find that the sales start to follow a normal distribution with a mean of \$5000. It is known that $\Pr(X > 1200) = 0.9$.

f Determine the standard deviation of the distribution, rounded to the nearest dollar.

$\Pr(X > 1200) = 0.9$

```
invNormCDf("R",0.9,1,0)
                                   -1.281551566
solve(-1.28155=(1200-5000)/s,s)
                                  {s=2965.159377}
```

Using $z = \dfrac{x - \mu}{\sigma}$, gives s.d. = \$2965

2 marks

Claire, who owns the crockery company, is very worried when the company receives the lowest 20% of sales. She is overjoyed when the company receives the highest 30% of sales.

g Find the maximum sales amount, correct to the nearest dollar, for which Claire will still be worried.

$N(\mu, \sigma^2) = N(5000, 2965^2)$

$\Pr(X < ?) = 0.2$

```
invNormCDf("L",0.2,2965,5000)
                                    2504.593042
```

$\Rightarrow \Pr(X < 2505) = 0.2$

Claire will be worried about sales amounts of anything less than \$2505.

2 marks

h Find the minimum sales amount, correct to the nearest dollar, for which Claire will be overjoyed.

$N(\mu, \sigma^2) = N(5000, 2965^2)$

$\Pr(X > ?) = 0.3$

```
invNormCDf("R",0.3,2965,5000)
                                    6554.84752
```

$\Rightarrow \Pr(X > 6555) = 0.3$

Claire will be overjoyed about sales amounts of anything greater than \$6555.

1 mark

Another section of Claire's company makes antique replica teapots. Claire makes one teapot per week. Claire demands these teapots are perfect before selling. Only teapots that are perfect are adequate for the postal sales. Over a period of time, it is found that if a teapot has a perfection probability of p, it is adequate for the postal sales. The probability p stays fixed for a week.

i Find the probability, in terms of p, that in the third week the antique replica teapots are adequate for the postal sales.

$$\Pr(\text{3rd week adequate}) = ppp + (1-p)pp + p(1-p)p + (1-p)(1-p)p$$
$$= p$$

1 mark

j In one particular year, the probability that in ten weeks at least 8 antique replica tea pots are adequate for postal sales is equal to 0.8. Find p for this instance.

$\text{Bi}(n, p) = \text{Bi}(10, p)$

$\Pr(X > 8) = 0.8$

$\Pr(X = 9) + \Pr(X = 10) = 0.8$

$\Rightarrow {}^{10}C_9(1-p)^1p^9 + {}^{10}C_{10}(1-p)^0p^{10} = 0.8$

$\Rightarrow 10(1-p)^1p^9 + p^{10} = 0.8$

Solve for p.

```
solve(10·(1-p)·p^9+p^10=0.8, p)
                    {p=0.9167400634, p=1.0572562}
```

$p = 0.9167$ ($p = 1.057$ not possible)

2 marks

(Total: 17 marks)

QUESTION 25

The population of the common loon in one particular lake in the great lakes region of North America varies according to the rule $p(t) = 10\,000 - 5000\cos\left(\frac{\pi t}{6}\right)$, where p is the population of the common loon and t is the number of months after 1 January 2014 with $t \in [0, 12]$.

Lucas is the conservation officer in charge of maintaining the population health of the loon birds.

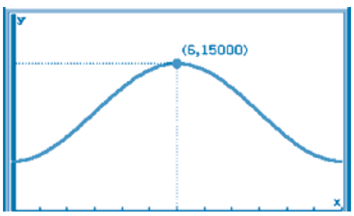
a State the period and amplitude of the function p.

$\text{Period} = \dfrac{2\pi}{\frac{\pi}{6}} = 12$

$\text{Amplitude} = 5000$

2 marks

b Find the maximum number of common loons in this lake and state when this maximum occurs.



Maximum = 15 000, at $t = 6$

Maximum occurs in June.

2 marks

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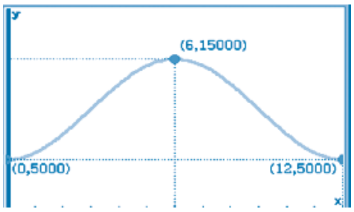
c Find the minimum number of common loons in this lake and state when this minimum occurs.



Minimum = 5 000, at $t = 0$ and $t = 12$

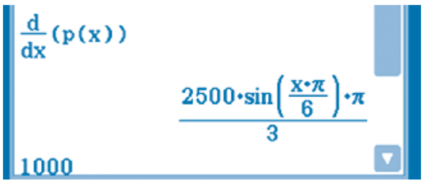
Minimum occurs in January and December.

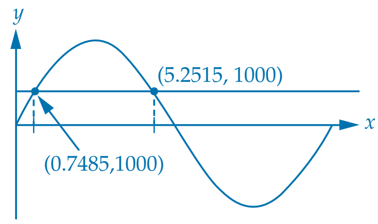
1 mark

In his first year of work, starting on 1 January 2014, Lucas reports that the loon population is healthy when the rate of population change is greater than 1000 birds per month.

d Find the fraction of time, over the first 12 months of his job, when Lucas decides that the population is healthy. Give your answer as an interval for t, correct to 2 decimal places.

Sketch the derivative graph $p'(t)$.





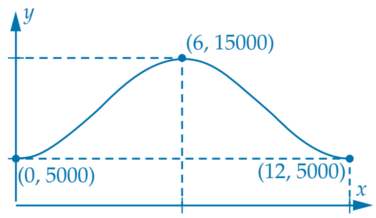
Solve for $p'(t) > 1000$.

$p'(t) > 1000$ for $0.7485 < t < 5.2515$

$\Rightarrow t \in (0.75, 5.25)$

2 marks

e Sketch the graph of $p(t)$, labelling axial intercepts and endpoints.



2 marks

Lucas moves to another lake for the second year of his job, starting on 1 January 2015. This time, the population of the yellow-billed loon varies according to the rule $y(t) = 5000 \sin\left(\frac{\pi t}{4}\right) + 80\,000$, where y is the population of the yellow-billed loon and t is the number of months after 1 January 2015. The standard for a healthy loon population stays the same.

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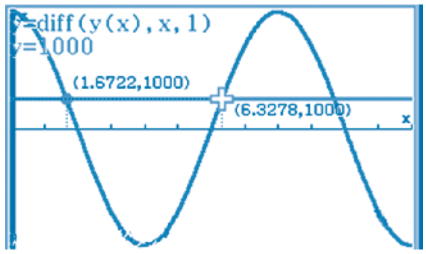
f Find the ratio of time when the yellow-billed loon population is healthy compared to when the common loon population is healthy.

$y(t) = 5000 \sin\left(\frac{\pi t}{4}\right) + 80\,000$

$y'(t) = 1250\pi \cos\left(\frac{\pi t}{4}\right)$

Sketch the derivative graph $y'(t)$.



Solve for $y'(t) > 1000$

$y'(t) > 1000$ for $0 < t < 1.6722,\ 6.3278 < t < 9.6722$

$\Rightarrow t \in (0, 1.67) \cup (6.33, 9.67)$

Ratio comparison

$p'(t) > 1000 \Rightarrow t \in (0.75, 5.25)$

$y'(t) > 1000 \Rightarrow t \in (0, 1.67) \cup (6.33, 9.67)$

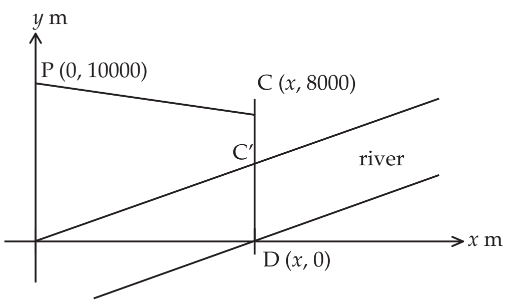
$\text{Ratio} = \frac{p'(t)}{y'(t)} = \frac{5.25 - 0.75}{1.67 + 9.67 - 6.33} = \frac{150}{167}$

Ratio of yellow-billed loon : common loon = 167 : 150

3 marks
(Total: 12 marks)

QUESTION 26

Toby Jones is hiking through the woods towards a river and is deciding whether he will swim or walk to his destination. He is able to walk at a rate of 6 metres per second and swim at a rate of k metres per second, where k is a constant. Toby has a 'mud map' that he is following that is sketched below.



In the diagram above, P is Toby's starting point, C is the coffee shop and D is his destination on the opposite bank of the river. P has coordinates (0, 10 000). D has coordinates $(x, 0)$. C has coordinates $(x, 8000)$. C′ is where Toby hits the river. Let x be the horizontal distance between the origin and D. Let $(1000 - \sqrt{x})$ metres be the distance between C and C′.

a State an expression for $T(x)$, the time in seconds that Toby walks and swims for if he goes from P to C and then from C to D.

$$\text{Speed} = \frac{\text{distance}}{\text{time}}, \text{ so time} = \frac{\text{distance}}{\text{speed}}$$

$$T(x) = \frac{PC}{6} + \frac{CC'}{6} + \frac{C'D}{k}$$

$$T(x) = \frac{\sqrt{x^2 + 2000^2}}{6} + \frac{1000 - \sqrt{x}}{6} + \frac{8000 - (1000 - \sqrt{x})}{k}$$

$$T(x) = \frac{\sqrt{x^2 + 2000^2}}{6} + \frac{1000 - \sqrt{x}}{6} + \frac{7000 + \sqrt{x}}{k}$$

2 marks

b Find $T'(x)$ and hence find the minimum time that Toby walks if $k = 10$. Give your answer to the nearest second.

$$T(x) = \frac{\sqrt{x^2 + 2000^2}}{6} + \frac{1000 - \sqrt{x}}{6} + \frac{7000 + \sqrt{x}}{k}$$

$$\text{define } t(x) = \frac{\sqrt{x^2+2000^2}}{6} + \frac{1000-\sqrt{x}}{6} + \frac{7000+\sqrt{x}}{k} \qquad \text{done}$$

$$\frac{d}{dx}(t(x))$$

$$\frac{2\cdot k\cdot x^{\frac{3}{2}} - k\cdot\sqrt{x^2+4000000} + 6\cdot\sqrt{x^2+4000000}}{12\cdot k\cdot\sqrt{x}\cdot(x^2+4000000)}$$

Let $k = 10$.

$$10 \Rightarrow k \qquad 10$$

$$\frac{d}{dx}(t(x))$$

$$\frac{5\cdot x^{\frac{3}{2}} - \sqrt{x^2+4000000}}{30\cdot\sqrt{x}\cdot(x^2+4000000)}$$

So at $k = 10$.

$$T'(x) = \frac{5x^{\frac{3}{2}} - \sqrt{x^2 + 2000^2}}{30\sqrt{x(x^2 + 2000^2)}}$$

Solve $\dfrac{5x^{\frac{3}{2}} - \sqrt{x^2 + 2000^2}}{30\sqrt{x(x^2 + 2000^2)}} = 0$ for the stationary point.

$$\text{solve}\left(\frac{d}{dx}(t(x)) = 0, x\right)$$

$$\{x = 54.30168894\}$$

The solution for $\dfrac{5x^{\frac{3}{2}} - \sqrt{x^2 + 2000^2}}{30\sqrt{x(x^2 + 2000^2)}} = 0$ is $x = 54.3017$

Test using a variety of methods if this answer is max or min. The graph shows it is a local min.

So the minimum time $= T(54.3017) = 1199.63$

To nearest second

Answer = 1200 seconds

2 marks

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c What is the restriction on x in this derivative?

x must not be zero so $x \in (0, \infty)$

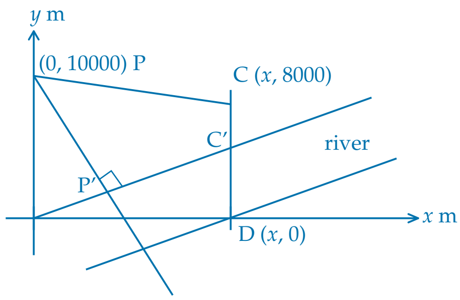
and $1000 - \sqrt{x} > 0$

Answer: $x \in (0, 1000^2)$

1 mark

d Find the time taken, to the nearest second, if Toby decides to walk in a line directly from P to the nearest bank of the river. Assume that this direct line is at right angles to the bank of the river.

A diagram will help.



Let the point on the river bank be P′

Speed $= \dfrac{\text{dist}}{\text{time}}$, so time $= \dfrac{\text{dist}}{\text{speed}}$

$T(x) = \dfrac{\text{PP}'}{6}$

The line from the origin to C′ has the equation $y = mx$,

where $m = \dfrac{8000 - (1000 - \sqrt{x})}{x} = \dfrac{7000 + \sqrt{x}}{x}$

So the line from the origin to C′ has the equation $y = \dfrac{7000 + \sqrt{x}}{x}x$

Perpendicular gradient $= \dfrac{-x}{7000 + \sqrt{x}}$

To find equation of line PP′,

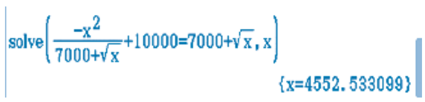
$y = mx + c$ where $c = 10\,000$.

Equation of line PP′ is $y = \dfrac{-x}{7000 + \sqrt{x}}x + 10\,000$

Now equate to find the coordinates of P′.

$\dfrac{-x}{7000 + \sqrt{x}}x + 10\,000 = \dfrac{7000 + \sqrt{x}}{x}x$

Solve for x: $\dfrac{-x}{7000 + \sqrt{x}}x + 10\,000 = 7000 + \sqrt{x}$



$x = 4552.533$

Coordinates of P′ = (4552.533, 7067.472)

Distance PP′ $= \sqrt{4552.533^2 + (10000 - 7067.472)^2}$

$= 5415.282$ metres.

$T(x) = \dfrac{5415.282}{6} = 902.54697$

Time from P to P′ = 903 seconds

3 marks
(Total: 8 marks)

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Mathematical Methods Formulas

Mensuration

area of trapezium:	$\frac{1}{2}(a+b)h$
curved surface area of a cylinder:	$2prh$
volume of a cylinder:	pr^2h
volume of a cone:	$\frac{1}{3}\pi r^2 h$
volume of a pyramid:	$\frac{1}{3}Ah$
volume of a sphere:	$\frac{4}{3}\pi r^3$
area of a triangle:	$\frac{1}{2}bc\sin A$

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$

$\int x^n \, dx = \frac{1}{n+1}x^{n+1} + c, n \neq -1$

$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$

$\int e^{ax} \, dx = \frac{1}{a}e^{ax} + c$

$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$

$\int \frac{1}{x} dx = \log_e |x| + c$

$\frac{d}{dx}\left(\sin(ax)\right) = a\cos(ax)$

$\int \sin(ax)\, dx = -\frac{1}{a}\cos(ax) + c$

$\frac{d}{dx}\left(\cos(ax)\right) = -a\sin(ax)$

$\int \cos(ax)\, dx = \frac{1}{a}\sin(ax) + c$

$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\sec^2(ax)$

product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

chain rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

approximation: $f(x+h) \approx f(x) + hf'(x)$

Probability

$\Pr(A) = 1 - \Pr(A')$

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$

variance: $Var(X) = \sigma^2 = E\left((X-\mu)^2\right) = E(X^2) - \mu^2$

mean: $\mu = E(X)$

Probability distribution		Mean	Variance
discrete	$\Pr(X = x) = p(x)$	$\mu = \sum x\, p(x)$	$\sigma^2 = \sum (x-\mu)^2 p(x)$
continuous	$\Pr(a < X < b) = \int_a^b f(x)\, dx$	$\mu = \int_{-\infty}^{\infty} x\, f(x)\, dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 \, f(x)\, dx$

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