Trial Examination 2014

# VCE Mathematical Methods (CAS) Units 3&4

Written Examination 2

## Suggested Solutions

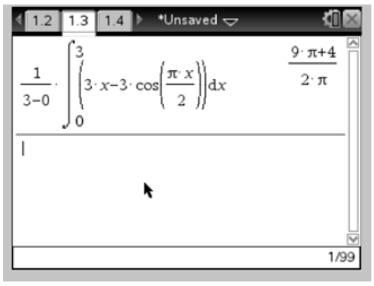
**SECTION 1**

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | A | B | **C** | D | E | 12 | A | B | C | **D** | E |
| 2 | A | B | C | D | **E** | 13 | A | B | C | **D** | E |
| 3 | A | B | C | D | **E** | 14 | A | B | C | D | **E** |
| 4 | A | B | C | D | **E** | 15 | A | B | C | D | **E** |
| 5 | A | B | C | **D** | E | 16 | **A** | B | C | D | E |
| 6 | A | B | **C** | D | E | 17 | A | B | **C** | D | E |
| 7 | A | **B** | C | D | E | 18 | A | B | C | D | **E** |
| 8 | **A** | B | C | D | E | 19 | A | **B** | C | D | E |
| 9 | A | B | C | **D** | E | 20 | A | B | C | **D** | E |
| 10 | A | B | **C** | D | E | 21 | A | B | **C** | D | E |
| 11 | A | B | C | **D** | E | 22 | A | B | C | D | **E** |

## SECTION 1

**Question 1** **C**

$$\text{average value} = \frac{1}{3}\int_0^3 3x - 3\cos\left(\frac{\pi x}{2}\right)dx$$
$$= \frac{9\pi + 4}{2\pi}$$
$$= \frac{9}{2} + \frac{2}{\pi}.$$



**Question 2** **E**

$$\text{average rate of change} = \frac{N(10\ 000) - N(0)}{10\ 000}$$
$$= -0.702$$

**Question 3** **E**

The transition matrix is $T = \begin{bmatrix} 0.45 & 0.8 \\ 0.55 & 0.2 \end{bmatrix}$, the initial state matrix is $S_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and the number of transitions is 20.

Therefore $S_{20} = T^{20}S_0$. CAS gives $\begin{bmatrix} 0.5926 \\ 0.4074 \end{bmatrix}$.

**Question 4** **E**

From the graph, the amplitude is 2 (all answers suitable); the vertical translation is 1 down (all answers suitable); the period is $2 \times \frac{4\pi}{3} = \frac{8\pi}{3}$ (so **D** is incorrect); and the horizontal translation will be $\frac{\pi}{3}$ to the right or $\frac{2\pi}{3}$ to the left (so **A** and **C** are incorrect).

With the translation $\frac{\pi}{3}$ to the right the solution will be cos with reflection in $x$-axis.

**Question 5 D**

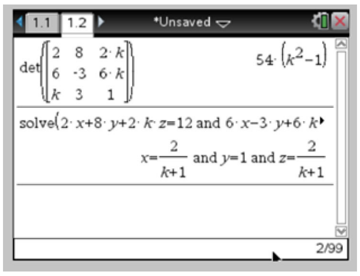
The matrix for this set of equations is $\begin{bmatrix} 2 & 8 & 2k \\ 6 & -3 & 6k \\ k & 3 & 1 \end{bmatrix}$.

Using CAS to find the determinant gives $\Delta = 54(k^2 - 1)$.

There is a unique solution when $k^2 \neq 1$.

$\therefore$ There is no unique solution when $k = \pm 1$.

When $k = 1$ there are infinitely many solutions, and when $k = -1$ there are no solutions.




**Question 6 C**

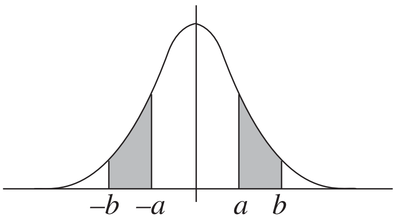
CAS gives $f'(x) = (4x - 3)\tan(2x^2 - 3x)$

$$= \frac{(3 - 4x)\sin(2x^2 - 3x)}{\cos(2x^2 - 3x)}$$

**Question 7 B**

$\Pr(-b < Z < -a) = \Pr(a < Z < b)$ by symmetry.

$= A - B$



**Question 8 A**

Construct a Karnaugh map.

| | Boys | Girls | |
|---|---|---|---|
| Not blonde | $\frac{13}{25}$ | $\frac{5}{25}$ | $\frac{18}{25}$ |
| Blonde | $\frac{2}{25}$ | $\frac{5}{25}$ | $\frac{7}{25}$ |
| | $\frac{15}{25}$ | $\frac{10}{25}$ | 1 |

$$\Pr(\text{blonde}|\text{boy}) = \frac{\Pr(\text{blonde} \cap \text{boy})}{\Pr(\text{boy})}$$

$$= \frac{\frac{2}{25}}{\frac{15}{25}}$$

$$= \frac{2}{15}$$

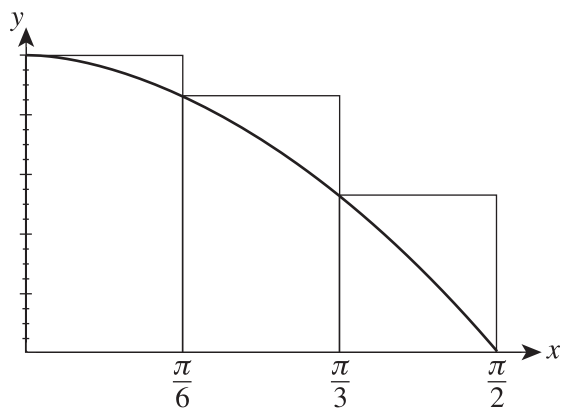
**Question 9 D**

Each rectangle has width $\frac{\pi}{6}$.

So area of three rectangles $= \frac{\pi}{6}\left(\cos(0) + \cos\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{3}\right)\right)$

$$= \frac{\pi}{6}\left(1 + \frac{\sqrt{3}}{2} + \frac{1}{2}\right)$$

$$= \frac{\pi(3 + \sqrt{3})}{12}$$



**Question 10** **C**

$\sin(2x) = \frac{-\sqrt{3}}{2}$

$2x = \frac{-\pi}{3} + 2\pi k$ or $2x = \frac{-2\pi}{3} + 2\pi k$

$x = \frac{-\pi}{6} + \pi k$ or $x = \frac{-\pi}{3} + \pi k$

$= \frac{\pi(6k-1)}{6}$ or $\frac{\pi(3k-1)}{3}$

**Question 11** **D**

$X \sim \text{Bi}(n, p)$

$$\begin{aligned}\Pr(X \geq 2) &= 1 - [\Pr(X = 0) + \Pr(X = 1)] \\ &= 1 - (p^0(1-p)^n) + (np^1(1-p)^{n-1}) \\ &= 1 - ((1-p)^n + np(1-p)^{n-1})\end{aligned}$$

Since $1 - (0.8^{12} + 12 \times 0.2 \times 0.8^{11})$ is required, $n = 12$ and $p = 0.2$.

**Question 12** **D**

Sketch graph on CAS and observe on which intervals the function is increasing. Endpoints are not included, as at the cusp points the derivative is undefined and at the maximums it equals 0.

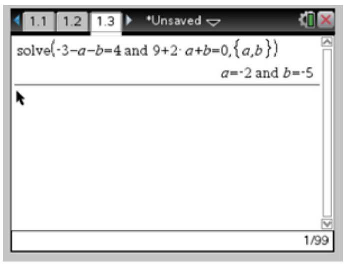
**Question 13** **D**

If $f(x) = 3x^3 - ax^2 + bx$ then $f'(x) = 9x^2 - 2ax + b$.

$f(-1) = 4$ yields $-3 - a - b = 4$

$f'(-1) = 0$ yields $9 + 2a + b = 0$

Solving simultaneously gives $a = -2$ and $b = -5$



**Question 14 E**

The probability of getting at least 2 triangles is equal to the sum of the probabilities of getting 2 triangles and 3 triangles.

$$\Pr(\Delta \geq 2) = 3 \times \frac{6 \times 7 \times 8}{13 \times 14 \times 15} + \frac{7 \times 6 \times 5}{13 \times 14 \times 15}$$

Probability of getting at least 1 triangle: $\Pr(\Delta \geq 1) = 1 - \Pr(\Delta = 0)$

$$= 1 - \frac{6 \times 7 \times 8}{13 \times 14 \times 15}$$

$$\Pr(\Delta \geq 2 | \Delta \geq 1) = \frac{\Pr(\Delta \geq 2)}{\Pr(\Delta \geq 1)}$$

$$= \frac{29}{57}$$

**Question 15 E**

The derivative function $f'(x)$ is a cubic function, therefore, the function $f(x)$ will be a quartic function. Only options **D** and **E** are graphs of quartics.

According to the graph, $f(x)$ will have zero gradient at $x = -1$ (point of inflection) and $x = 2$ (local maximum).

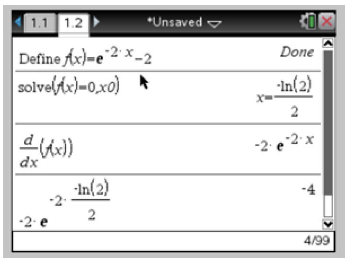
**Question 16 A**

Using CAS, $y = e^{-2x} - 2$ crosses the $x$-axis at $\left(\frac{-\log_e(2)}{2}, 0\right)$.

At that point, it has a gradient of –4.

$\therefore$ The tangent has a gradient of –4 and passes through the point $\left(\frac{-\log_e(2)}{2}, 0\right)$.

So the equation of the tangent is $y = -4\left(x + \frac{-\log_e 2}{2}\right)$.




**Question 17 C**

Sketch the graph of the given function on CAS, taking $a = 0$. Use the Analyse option to find coordinates of the point of inflexion, $\left(\frac{1}{2}, \frac{1}{48}\right)$. Thus $a = -1$.

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**Question 18 E**

Using the chain rule, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$, where $y = f(u)$ and $u = g(x) = \sin(2x)$:

$\frac{dy}{du} = f'(u)$ and $\frac{du}{dx} = 2\cos(2x)$

Therefore $\frac{dy}{dx} = f'(u) \times 2\cos(2x)$

$= 2f'(\sin(2x))\cos(2x)$

**Question 19 B**

$f(x^3) = 3\log_e(x)$

Substitution of answers into $f(y) - f(x)$ gives $3\log_e(x)$ for $y = x^4$.

**Question 20 D**

This is a related rates question where we are given $\frac{dV}{dt} = -200$ cm/minute.

Volume of a cone is given by $V = \frac{\pi}{3}r^2 h$. In this case, $h = d$.

Using similar triangles: $\frac{r}{d} = \frac{20}{50}$, giving $r = \frac{2d}{5}$

So $V = \frac{\pi}{3}\left(\frac{2d}{5}\right)^2 d$

$= \frac{4\pi}{75}d^3$

Therefore $\frac{dV}{dd} = \frac{4\pi}{25}d^2$.

Using $\frac{dV}{dt} = \frac{dV}{dd} \times \frac{dd}{dt}$ we require $\frac{dd}{dt} = \frac{dV}{dt} \times \frac{dd}{dV}$

$= -200 \times \frac{25}{4\pi d^2}$

$= -\frac{1250}{\pi d^2}$

**Question 21 C**

$$1 + \frac{\frac{1}{x^2}}{\frac{1}{y^2}} = 1 + \frac{y^2}{x^2} = \frac{x^2 + y^2}{x^2}$$

$$y^2\left(\frac{1}{x^2} + \frac{1}{y^2}\right) = \frac{x^2 + y^2}{x^2}$$

**Question 22** **E**

$y = 1 - \frac{1}{(2x-3)^2}$ can be arranged to give $-(y-1) = \frac{1}{(2x-3)^2}$

The transformations required use the following mappings:

$(x, y) \to (2x' - 3), -(y' - 1)$

$x = 2x' - 3 \Rightarrow x' = \frac{x+3}{2}$ and $y = -(y' - 1) \Rightarrow y' = y + 1$

$= \frac{x}{2} + \frac{3}{2}$

This is given by the matrices $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$.

## SECTION 2

### Question 1 (10 marks)

**a.** $g(x)=\dfrac{1}{(2(x+1))^3}-3$ A1

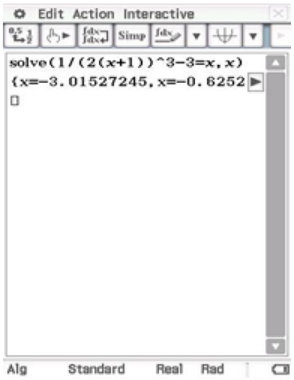
Rewrite this as $y=\dfrac{1}{(2(x+1))^3}-3$ and rearrange to make $x$ the subject, i.e. swap $x$ and $y$:

$g^{-1}(x)=\dfrac{1}{2\sqrt[3]{x+3}}-1$ M1

The domain is $R\setminus\{-3\}$ and range is $R\setminus\{-1\}$. A1

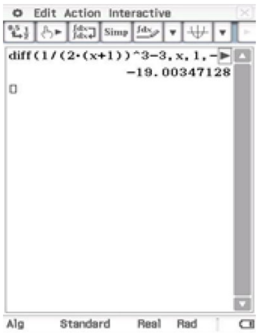
**b.** Find the right point of intersection of the graphs by solving $\dfrac{1}{(2(x+1))^3}-3=x$ (as the points of intersection must be on the line $y = x$).

So $x = -0.6252$ and $y = -0.6252$ A1




The gradient of the tangent to the graph is equal to the derivative at this point.

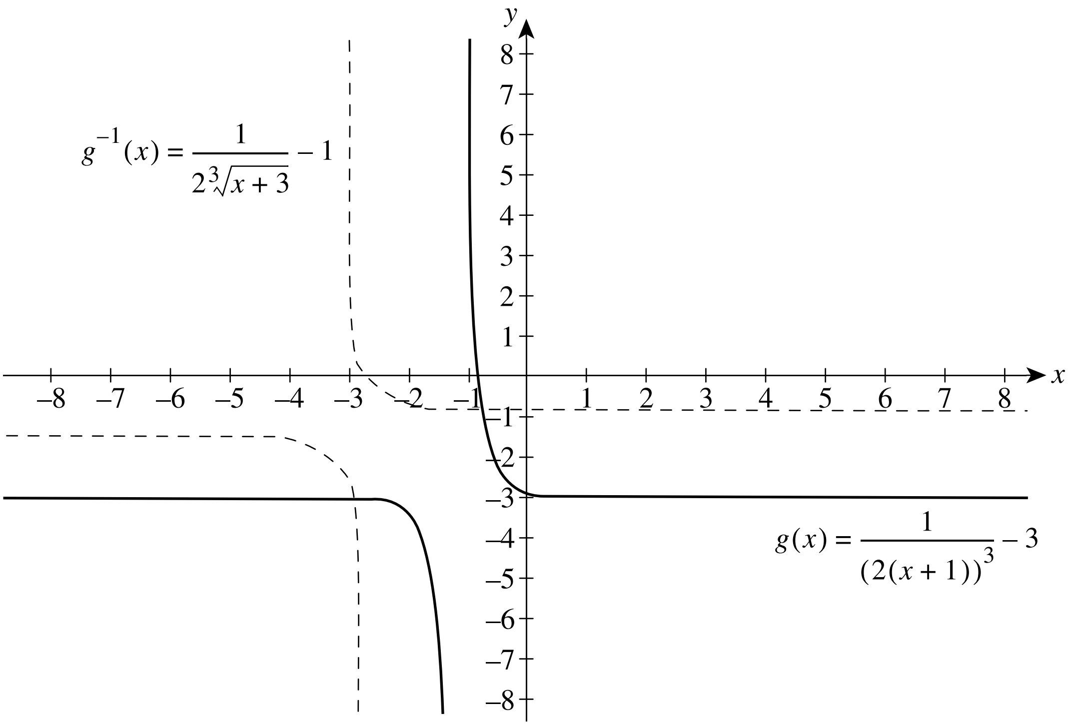
Therefore the gradient equals –19 and the gradient of the normal is $\dfrac{1}{19}$. M1




Substitute $x=-0.6252$ and $y=-0.6252$ into the equation: $y=\dfrac{1}{19}x+c$

Therefore $c=-0.5923$ and the equation of the normal is $y=\dfrac{1}{19}x-0.5923$. A1

**c.**

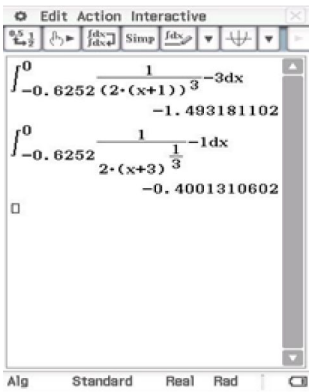



A1

Points of intersection of graphs are (–3.0153, –3.0153) and (–0.6252, –0.6252).

$g(x)$ has a $y$-intercept at –2.875 and an $x$-intercept at –0.6533; and vice versa for $g^{-1}(x)$. A1

**d.** Using CAS M1




area = 1.4932 – 0.4001 = 1.0931 square units A1

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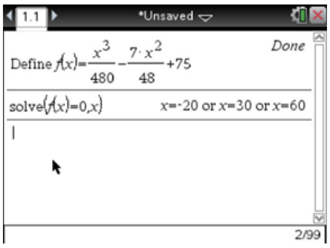
**Question 2 (18 marks)**

**a.** For $x$-intercepts: $f(x) = 0$ M1

$x = –20, 30, 60$




From the graph, $A$ corresponds to the lowest value of $x$ and $B$ corresponds to the highest.

$\therefore$ $A$ (–20, 0) and $B$ (60, 0) A1

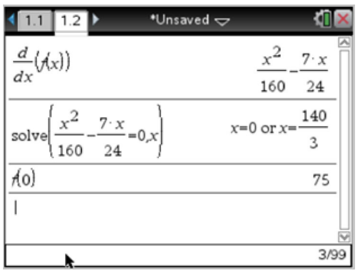
**b.** We require a maximum turning point of $f(x)$.

$f'(x) = 0$ M1

Turning points are at $x = 0$ and $\frac{140}{3}$

$T$ is at $x = 0$

$f(0) = 75$

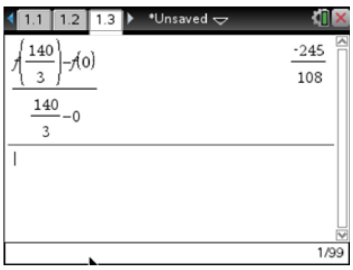



The height of $T$ above village is 75 m. A1

**c.** We require the average gradient from $T$ to $L$. The $x$-ordinate of $T$ is 0 and the $x$-ordinate of $L$, which is the minimum turning point of $f(x)$, is $\frac{140}{3}$ (from part **b.**). M1

$$\text{aveage gradient} = \frac{f(b) - f(a)}{b - a}$$ M1

$$= \frac{-245}{108}$$ A1



**d.** $V$ has coordinates (–60, 0) and $T$ has coordinates (0, 75).

gradient of line $VT = \frac{75-0}{0+60} = \frac{5}{4}$

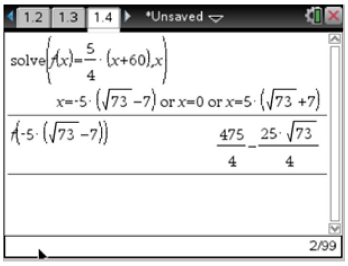
equation of line: $y = \frac{5}{4}(x+60)$ A1

**e.** We require the point of intersection of $f(x)$ and line $VT$.

$f(x) = \frac{5}{4}(x+60)$. M1

$x = 35 - 5\sqrt{73}$ at point $C$.

$f = (35 - 5\sqrt{73}) = \frac{25}{4}(19 - \sqrt{73})$




coordinates of $C$: $\left(35 - 5\sqrt{73}, \frac{25}{4}(19 - \sqrt{73})\right)$ A1

**f.** length $CT = \sqrt{((35 - 5\sqrt{73}) - 0)^2 + \left(\left(\frac{25}{4}(19 - \sqrt{73})\right) - 75\right)^2}$ M1

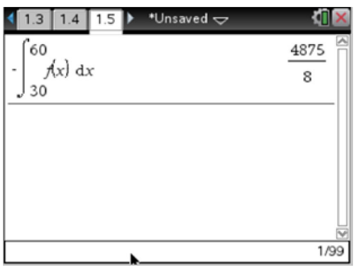
$= 12.358$

length of tunnel = 1236 cm A1

**g.** Point $P$ has coordinates (30, 0), found in part **a.** with $x$-intercepts.

area of pond $= -\int_{30}^{60} f(x)dx$ M1

$= \frac{4875}{8}$ m$^2$ A1



**h.** The deepest part of the pond is at point $L$, the local minimum.

$f\left(\frac{140}{3}\right) = -\frac{2500}{81}$ M1

The depth of the water at the deepest part of the pond is $\frac{2500}{81}$ m.

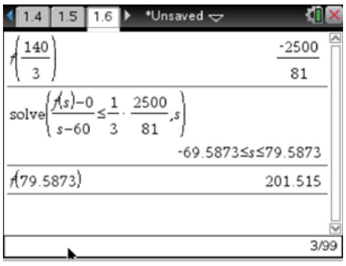
average gradient $\leq \frac{1}{3} \times \frac{2500}{81}$

average gradient between point $B$ (60, 0) and $S$ (s, $f(s)$) = $\frac{f(s) - 0}{s - 60}$

$\therefore \frac{f(s) - 0}{s - 60} \leq \frac{1}{3} \times \frac{2500}{81}$ M1

$s \leq 79.5873$ A1

$f(79.5873) = 201.515$




The coordinates of $S$, to the nearest centimetre: (7959, 20152) A1

**Question 3 (18 marks)**

**a.** $\frac{2\pi}{\sqrt{\frac{k}{0.16}}} = 2$

$k = 0.16\pi^2$

$k = 1.58$ A1

**b.** $0 = -v_m \sin(wt)$

$wt = \pi$

$t = \frac{\pi}{w}$

$t = 1$ s A1

**c.** $x(t) = \int (-v_m \sin(wt))dt$

$= \frac{v_m}{w}\cos(wt) + C$

$= 10\cos(\pi t) + C$ M1

$t = 0$, $x(0) = 10$, so $C = 0$

$x(t) = 10\cos(\pi t)$ A1

**d.** $a(t) = \frac{dv}{dt}$

$= -10\pi^2 \cos(\pi t)$ M1

Velocity is equal to 0 at $t = 0$, so $a(t) = -10\pi^2$ A1

**e.** To find the area we need to find the $t$-intercepts first.

$t = n$, where $n \in N$ (natural numbers) M1

We also need to take definite integrals of regions above the $t$-axis as '+' and regions below the $t$-axis as '–'.

Also, we know that the antiderivative of $v(t)$ is $x(t)$, so:

area $= [10\cos(\pi t)]_{1.5}^{2} - [10\cos(\pi t)]_{2}^{3} + [10\cos(\pi t)]_{3}^{4} - [10\cos(\pi t)]_{4}^{4.75}$ A1

distance = area = 10 + 20 + 20 + 17.07 = 67.07 A1

**f.** As period of $u(t)$ is $\pi$, the third maximum will be at $\frac{11\pi}{4}$. M1

$0.0266 = 2e^{\frac{-11\pi}{4}r}$ M1

Solve with CAS: $d = 0.50$ A1

**g.** Displacement is the antiderivative of velocity, and finding greatest displacement both in negative and positive directions means that we are looking for stationary points of displacement, i.e. the points where velocity (as derivative of displacement) is equal to 0.

$u(t) = -2e^{-dt}\sin(2t) = 0$ M1 A1

$\sin(2t) = 0$

$t = \frac{\pi}{2}n$, where $n \in N$ (natural number) A1

**h.** displacement $= \int_0^{10} -2e^{-rt}\sin(2t)dt$ M1 A1

Solve with CAS: displacement = –0.48 A1

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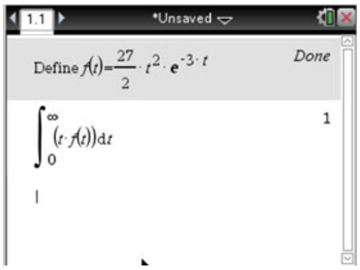
**Question 4 (12 marks)**

**a.** **i.** $E(t) = \int_{-\infty}^{\infty} tf(t)dt$

$= \int_{0}^{\infty} t\left(\frac{27}{2}t^2e^{-3t}\right)dt$ M1

$= \frac{27}{2}\int_{0}^{\infty} t^3e^{-3t}dt$

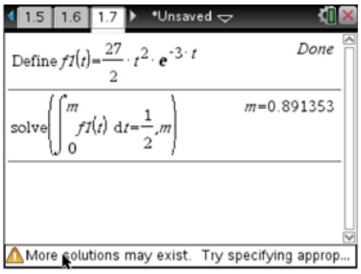
$= \frac{2}{27}$ A1




**ii.** $\int_{0}^{m} f(t)dt = \frac{1}{2}$ M1

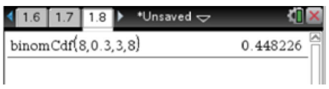
$m = 0.8913$

median = 0.89 hours A1




**b.** We require $\Pr(X \geq 3)$, where $X \sim \text{Bi}\,(8,\ 0.3)$. M1

$\Pr(X \geq 3) = 0.4482$ A1



**c.** $S \sim N(\mu, \sigma^2)$

$\Pr(S > 1.5) = 0.55$

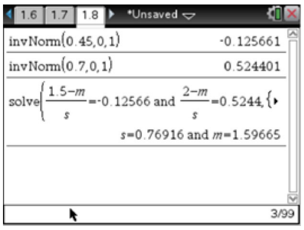
$\therefore \Pr(S < 1.5) = 1 - 0.55 = 0.45.\ z = -0.12566$

$\Pr(S < 2) = 0.7.\ z = 0.5244$ M1

$\dfrac{1.5 - \mu}{\sigma} = -0.12566$ and $\dfrac{2 - \mu}{\sigma} = 0.5244$ M1

Solve the simultaneous equations using CAS.

$\mu = 1.60$ and $\sigma = 0.77$ A1




**d.** Conditional probability: M1

$$\Pr(Y > 1.5 \mid Y > 1) = \frac{\Pr(Y > 1.5 \cap Y > 1)}{\Pr(Y > 1)}$$

$$= \frac{\Pr(Y > 1.5)}{\Pr(Y > 1)}$$ M1

Using CAS:

$$= \frac{0.549998}{0.781042} = 0.7042$$ A1