Year 2014

VCE

Mathematical Methods

Trial Examination 1

Solutions

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Question 1

a. $y = \dfrac{\cos(2x)}{3x}$ using the quotient rule

$u = \cos(2x) \qquad v = 3x$

$\dfrac{du}{dx} = -2\sin(2x) \qquad \dfrac{dv}{dx} = 3$ M1

$\dfrac{dy}{dx} = \dfrac{-6x\sin(2x) - 3\cos(2x)}{(3x)^2}$

$\dfrac{dy}{dx} = \dfrac{-(2x\sin(2x) + \cos(2x))}{3x^2}$ A1

b. $f(x) = \tan\left(\sqrt{x}\right)$

$y = \tan\left(\sqrt{x}\right)$ using the chain rule

$y = \tan(u)$ where $u = \sqrt{x} = x^{\frac{1}{2}}$

$\dfrac{dy}{du} = \dfrac{1}{\cos^2(u)} \qquad \dfrac{du}{dx} = \dfrac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$

$f'(x) = \dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \dfrac{1}{2\sqrt{x}\cos^2\left(\sqrt{x}\right)}$ M1

$f'\left(\dfrac{\pi^2}{16}\right) = \dfrac{1}{2\sqrt{\dfrac{\pi^2}{16}}\cos^2\left(\sqrt{\dfrac{\pi^2}{16}}\right)} = \dfrac{2}{\pi\cos^2\left(\dfrac{\pi}{4}\right)} = \dfrac{2}{\pi \times \left(\dfrac{1}{2}\right)}$

$f'\left(\dfrac{\pi^2}{16}\right) = \dfrac{4}{\pi}$ A1

Question 2

a. $y = \sqrt{x}$ into $y = 3 - \sqrt{2-x} = 3 - \sqrt{-(x-2)}$

$\dfrac{1}{2}$ mark for each correct transformation, the translations must come last.

reflect in the *x*-axis $y = -\sqrt{x}$

reflect in the *y*-axis $y = -\sqrt{-x}$

translate 2 units to the right parallel to the *x*-axis (or away from the *y*-axis) $y = -\sqrt{-(x-2)}$

translate 3 units up parallel to the *y*-axis (or away from the *x*-axis) $y = 3 - \sqrt{-(x-2)}$

b. $f: \quad y = 3 - \sqrt{2-x}$ swap x and y

$f^{-1} \quad x = 3 - \sqrt{2-y}$

$\sqrt{2-y} = 3 - x$

$2 - y = (3-x)^2$ M1

$y = 2 - (3-x)^2 = 2 - (9 - 6x + x^2)$

$y = 6x - x^2 - 7$

but dom $f = (-\infty, 2] = \text{ran } f^{-1}$ and ran $f = (-\infty, 3] = \text{dom } f^{-1}$

To state the function, we must state its domain

$f^{-1}: (-\infty, 3] \to R, \; f^{-1}(x) = 6x - x^2 - 7$ A1

Question 3

$3x - (k+2)y = k+1$

$kx - 5y = 4$

$\Delta = \begin{vmatrix} 3 & -(k+2) \\ k & -5 \end{vmatrix} = -15 + k(k+2) = k^2 + 2k - 15$ M1

$\Delta = (k+5)(k-3)$

i. There is a unique solution when $\Delta \neq 0$ that is $k \in R \setminus \{-5, 3\}$ A1

When $k = -5$ the equations become $\begin{aligned} 3x + 3y &= -4 \\ -5x - 5y &= 4 \end{aligned}$

these lines are parallel with different y-intercepts, therefore there is no solution when $k = -5$

ii. When $k = 3$ the equations become $\begin{aligned} 3x - 5y &= 4 \\ 3x - 5y &= 4 \end{aligned}$

these lines are both the same line, therefore we have an infinite number of solutions when

$k = 3$ A1

Question 4

$\sqrt{3}\cos(2x)+\sin(2x)=0$

$\sqrt{3}\cos(2x)=-\sin(2x)$

$\tan(2x)=-\sqrt{3}$ M1

$2x=n\pi+\tan^{-1}\left(-\sqrt{3}\right)=n\pi-\dfrac{\pi}{3}$

$2x=\dfrac{\pi}{3}(3n-1)$

$x=\dfrac{\pi}{6}(3n-1)\ ,\ n\in Z$ A1

Question 5

a. $2\log_4(x-1)+\log_4(2)-\log_4(x)=\dfrac{3}{2}$

$\log_4(x-1)^2+\log_4(2)-\log_4(x)=\dfrac{3}{2}$

$\log_4\left[\dfrac{2(x-1)^2}{x}\right]=\dfrac{3}{2}$ M1

$\dfrac{2(x-1)^2}{x}=4^{\frac{3}{2}}=\left(\sqrt{4}\right)^3=8$

$(x-1)^2=4x$

$x^2-2x+1=4x$

$x^2-6x=-1$

$x^2-6x+9=-1+9=8$ M1

$(x-3)^2=8$

$x-3=\pm\sqrt{8}$

$x=3\pm2\sqrt{2}$ but $x>1$

$x=3+2\sqrt{2}$ only A1

b. $8\times 4^{x}+16\times 4^{-x}=129$

let $u=4^{x}$ then $4^{-x}=\dfrac{1}{4^{x}}=\dfrac{1}{u}$

$8u+\dfrac{16}{u}=129$

$8u^{2}+16=129u$

$8u^{2}-129u+16=0$ M1

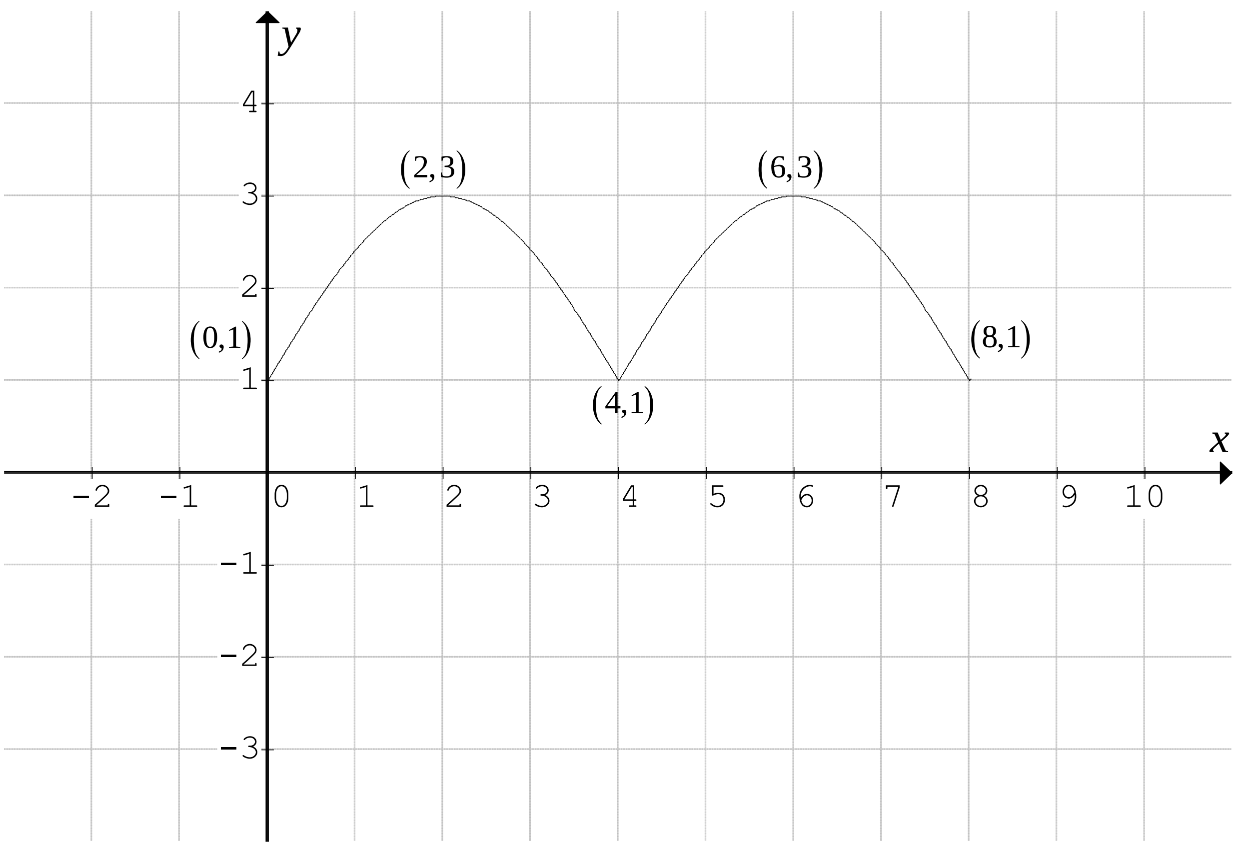
$(8u-1)(u-16)$

$u=4^{x}=\dfrac{1}{8}$ $\quad u=4^{x}=16$

$x=-\dfrac{3}{2}$ or $x=2$ A1

Question 6

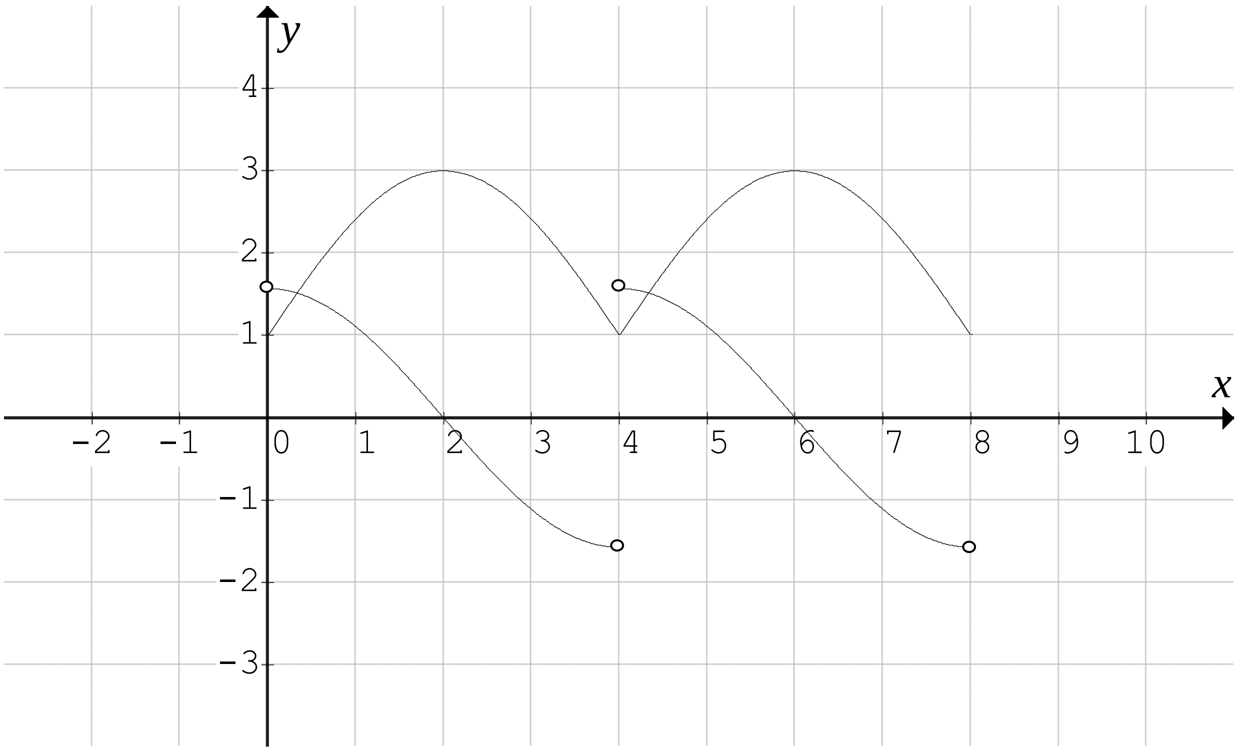
i. $f:[0,8]\to R\,,\ f(x)=\left|2\sin\left(\dfrac{\pi x}{4}\right)\right|+1$ G2



ii. $f'(x)=\dfrac{\pi}{2}\cos\left(\dfrac{\pi x}{4}\right)$ for $0<x<4$ and $f'(x)=\dfrac{\pi}{2}\cos\left(\dfrac{\pi(x-4)}{4}\right)$ for $4<x<8$

Note $\dfrac{\pi}{2}\approx 1.57$ and that the gradient function crosses the *x*-axis at $x=2$ and $x=6$, the gradient function is not defined at $x=0,4,8$ must have open circles at these points.

G2



iii. $\bar{f}=\dfrac{1}{8}\displaystyle\int_0^8\left(\left|2\sin\left(\frac{\pi x}{4}\right)\right|+1\right)dx=\frac{1}{4}\int_0^4\left(2\sin\left(\frac{\pi x}{4}\right)+1\right)dx$ by symmetry

$$\bar{f}=\frac{1}{4}\left[-\frac{8}{\pi}\cos\left(\frac{\pi x}{4}\right)+x\right]_0^4$$

$$\bar{f}=\frac{1}{4}\left[\left(-\frac{8}{\pi}\cos(\pi)+4\right)-\left(-\frac{8}{\pi}\cos(0)+0\right)\right]$$ M1

$$\bar{f}=\frac{1}{4}\left[\frac{16}{\pi}+4\right]$$

$$\bar{f}=\frac{4}{\pi}+1$$ A1

Question 7

a. Since it a probability density function $\int_0^4 \frac{k}{\sqrt{9-2x}}\,dx = 1$

$$k\int_0^4 (9-2x)^{-\frac{1}{2}}\,dx = 1$$

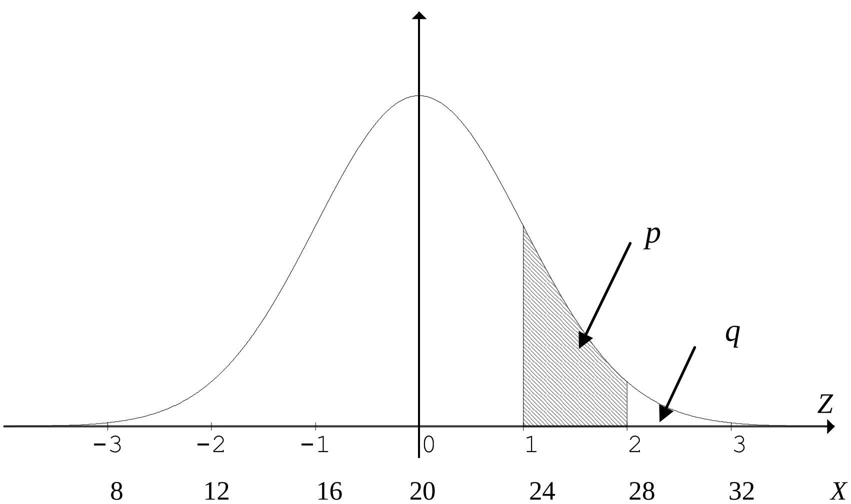
$$k\left[-\frac{2}{2}(9-2x)^{\frac{1}{2}}\right]_0^4 = k\left[-\sqrt{9-2x}\right]_0^4$$ M1

$$= k\left[\left(-\sqrt{1}\right)-\left(-\sqrt{9}\right)\right] = 2k = 1$$

$$k = \frac{1}{2}$$ A1

b. Given that $\Pr(1<Z<2) = p$ and $\Pr(Z>2) = q$

$X \stackrel{d}{=} N\left(\mu = 20, \sigma^2 = 16\right)$ $\sigma = 4$, and $Z = \frac{X-\mu}{\sigma}$



$$\Pr(X<24 \mid X>12) = \frac{\Pr(12<X<24)}{\Pr(X>12)} = \frac{\Pr(-2<Z<1)}{\Pr(Z>-2)}$$ now using symmetry

$$= \frac{\Pr(-1<Z<2)}{\Pr(Z>-2)} = \frac{2\Pr(0<Z<1)+\Pr(1<Z<2)}{\Pr(Z<2)} = \frac{2\left(\frac{1}{2}-(p+q)\right)+p}{1-q}$$ M1

$$= \frac{1-p-2q}{1-q}$$ A1

Question 8

a. $X \overset{d}{=} Bi\left(n=3, p=?\right)$

$\Pr\left(X=1\right)=\binom{3}{1}p^1\left(1-p\right)^2=\frac{p}{3}$

$3p\left(1-p^2\right)=\frac{p}{3}$ since $p \neq 0$

$\left(1-p\right)^2=\frac{1}{9}$ M1

$1-p=\pm\frac{1}{3}$

$1-p=\frac{1}{3}$ since $0<p<1$

$p=\frac{2}{3}$ A1

b. Since the probabilities sum to one. $\sum \Pr\left(X=x\right)=1$

$k^2+\frac{3k}{2}+\frac{k}{2}=1$

$k^2+2k=1$

$k^2+2k+1=2$

$\left(k+1\right)^2=2$

$k+1=\pm\sqrt{2}$ M1

$k=-1\pm\sqrt{2}$ but $0<k<1$

$k=\sqrt{2}-1$

$E\left(X\right)=\sum x\Pr\left(X=x\right)=0\times k^2+1\times\frac{3k}{2}+2\times\frac{k}{2}=\frac{5k}{2}$

$E\left(X\right)=\frac{5}{2}\left(\sqrt{2}-1\right)$ A1

Question 9

a. $y = e^{-x}\left(2\cos(2x)+\sin(2x)\right)$ using the product rule

$u = e^{-x} \qquad v = 2\cos(2x)+\sin(2x)$

$\frac{du}{dx} = -e^{-x} \qquad \frac{dv}{dx} = -4\sin(2x)+2\cos(2x)$ M1

$\frac{dy}{dx} = -e^{-x}\left(2\cos(2x)+\sin(2x)\right)+e^{-x}\left(-4\sin(2x)+2\cos(2x)\right)$

$\frac{dy}{dx} = e^{-x}\left(-2\cos(2x)-\sin(2x)-4\sin(2x)+2\cos(2x)\right)$

$\frac{dy}{dx} = -5e^{-x}\sin(2x)$

b. $f: R \to R\,,\ f(x) = e^{-x}\sin(2x)$

The graph crosses the x-axis when $\sin(2x)=0$

$2x = 0,\ \pi$ so that $x = 0\ ,\ \frac{\pi}{2}$

The shaded area is $A = \int_0^{\frac{\pi}{2}} e^{-x}\sin(2x)\,dx$ A1

Since $\frac{d}{dx}\left[e^{-x}\left(2\cos(2x)+\sin(2x)\right)\right] = -5e^{-x}\sin(2x)$ it follows from **a.** that

$\int e^{-x}\sin(2x)\,dx = -\frac{1}{5}e^{-x}\left(2\cos(2x)+\sin(2x)\right)$

$A = -\frac{1}{5}\left[e^{-x}\left(2\cos(2x)+\sin(2x)\right)\right]_0^{\frac{\pi}{2}}$

$A = -\frac{1}{5}\left[\left(e^{-\frac{\pi}{2}}\left(2\cos(\pi)+\sin(\pi)\right)\right)-\left(e^{0}\left(2\cos(0)+\sin(0)\right)\right)\right]$ M1

$A = -\frac{1}{5}\left[\left(-2e^{-\frac{\pi}{2}}\right)-2\right]$

$A = \frac{2}{5}\left(1+e^{-\frac{\pi}{2}}\right)$ units^2 A1

Question 10

$f:[0,9]\to R$, $f(x)=\pi x+12\cos\left(\frac{\pi x}{6}\right)$

for turning points $f'(x)=\pi-2\pi\sin\left(\frac{\pi x}{6}\right)=0$

$\sin\left(\frac{\pi x}{6}\right)=\frac{1}{2}$

$\frac{\pi x}{6}=\frac{\pi}{6},\frac{5\pi}{6}$

$x=1,5$ A1

$f(1)=\pi+12\cos\left(\frac{\pi}{6}\right)=\pi+6\sqrt{3}$ and $f(5)=5\pi+12\cos\left(\frac{5\pi}{6}\right)=5\pi-6\sqrt{3}$ A1

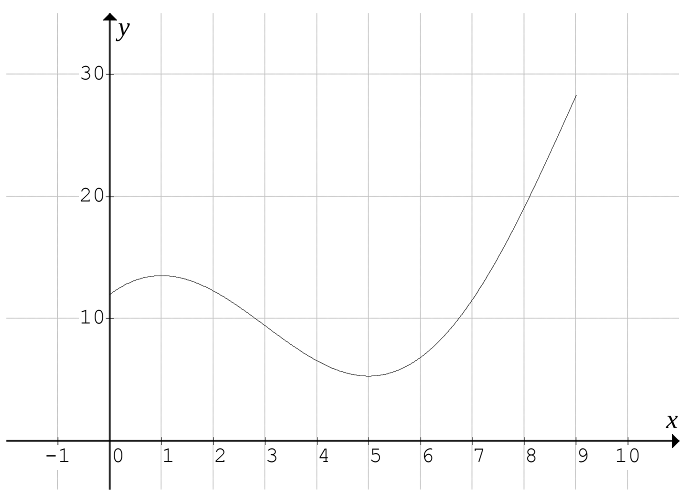
However since we have a restricted domain function, we must examine the endpoints.

$f(0)=12\cos(0)=12$ and $f(9)=9\pi+\cos\left(\frac{3\pi}{2}\right)=9\pi$

Now $9\pi>\pi+6\sqrt{3}>12>5\pi-6\sqrt{3}$ so that

the maximum value is 9π and occurs when $x=9$ A1

the minimum value is $5\pi-6\sqrt{3}$ and occurs when $x=5$ A1



END OF SUGGESTED SOLUTIONS