

Year 2014
VCE
Mathematical Methods
Trial Examination 1
Solutions



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Question 1

a. $y = \frac{\cos(2x)}{3x}$ using the quotient rule

$$u = \cos(2x) \quad v = 3x$$

$$\frac{du}{dx} = -2\sin(2x) \quad \frac{dv}{dx} = 3$$

M1

$$\frac{dy}{dx} = \frac{-6x\sin(2x) - 3\cos(2x)}{(3x)^2}$$

$$\frac{dy}{dx} = \frac{-(2x\sin(2x) + \cos(2x))}{3x^2}$$

A1

b. $f(x) = \tan(\sqrt{x})$

$$y = \tan(\sqrt{x}) \text{ using the chain rule}$$

$$y = \tan(u) \text{ where } u = \sqrt{x} = x^{\frac{1}{2}}$$

$$\frac{dy}{du} = \frac{1}{\cos^2(u)} \quad \frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$f'(x) = \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \frac{1}{2\sqrt{x} \cos^2(\sqrt{x})}$$

M1

$$f'\left(\frac{\pi^2}{16}\right) = \frac{1}{2\sqrt{\frac{\pi^2}{16}} \cos^2\left(\sqrt{\frac{\pi^2}{16}}\right)} = \frac{2}{\pi \cos^2\left(\frac{\pi}{4}\right)} = \frac{2}{\pi \times \left(\frac{1}{2}\right)}$$

$$f'\left(\frac{\pi^2}{16}\right) = \frac{4}{\pi}$$

A1

Question 2

a. $y = \sqrt{x}$ into $y = 3 - \sqrt{2-x} = 3 - \sqrt{-(x-2)}$

$\frac{1}{2}$ mark for each correct transformation, the translations must come last.

reflect in the x -axis $y = -\sqrt{x}$

reflect in the y -axis $y = -\sqrt{-x}$

translate 2 units to the right parallel to the x -axis (or away from the y -axis) $y = -\sqrt{-(x-2)}$

translate 3 units up parallel to the y -axis (or away from the x -axis) $y = 3 - \sqrt{-(x-2)}$

b. $f: y = 3 - \sqrt{2-x}$ swap x and y
 $f^{-1} x = 3 - \sqrt{2-y}$
 $\sqrt{2-y} = 3-x$
 $2-y = (3-x)^2$ M1
 $y = 2 - (3-x)^2 = 2 - (9 - 6x + x^2)$
 $y = 6x - x^2 - 7$
 but $\text{dom } f = (-\infty, 2] = \text{ran } f^{-1}$ and $\text{ran } f = (-\infty, 3] = \text{dom } f^{-1}$
 To state the function, we must state its domain
 $f^{-1}: (-\infty, 3] \rightarrow \mathbb{R}, f^{-1}(x) = 6x - x^2 - 7$ A1

Question 3

$$3x - (k+2)y = k+1$$

$$kx - 5y = 4$$

$$\Delta = \begin{vmatrix} 3 & -(k+2) \\ k & -5 \end{vmatrix} = -15 + k(k+2) = k^2 + 2k - 15$$
 M1

$$\Delta = (k+5)(k-3)$$

i. There is a unique solution when $\Delta \neq 0$ that is $k \in \mathbb{R} \setminus \{-5, 3\}$ A1

When $k = -5$ the equations become

$$\begin{aligned} 3x + 3y &= -4 \\ -5x - 5y &= 4 \end{aligned}$$

these lines are parallel with different y -intercepts, therefore there is no solution when $k = -5$

ii. When $k = 3$ the equations become

$$\begin{aligned} 3x - 5y &= 4 \\ 3x - 5y &= 4 \end{aligned}$$

these lines are both the same line, therefore we have an infinite number of solutions when

$$k = 3$$
 A1

Question 4

$$\sqrt{3} \cos(2x) + \sin(2x) = 0$$

$$\sqrt{3} \cos(2x) = -\sin(2x)$$

$$\tan(2x) = -\sqrt{3} \quad \text{M1}$$

$$2x = n\pi + \tan^{-1}(-\sqrt{3}) = n\pi - \frac{\pi}{3}$$

$$2x = \frac{\pi}{3}(3n-1)$$

$$x = \frac{\pi}{6}(3n-1), \quad n \in \mathbb{Z} \quad \text{A1}$$

Question 5

a. $2\log_4(x-1) + \log_4(2) - \log_4(x) = \frac{3}{2}$

$$\log_4(x-1)^2 + \log_4(2) - \log_4(x) = \frac{3}{2}$$

$$\log_4 \left[\frac{2(x-1)^2}{x} \right] = \frac{3}{2} \quad \text{M1}$$

$$\frac{2(x-1)^2}{x} = 4^{\frac{3}{2}} = (\sqrt{4})^3 = 8$$

$$(x-1)^2 = 4x$$

$$x^2 - 2x + 1 = 4x$$

$$x^2 - 6x = -1$$

$$x^2 - 6x + 9 = -1 + 9 = 8$$

$$(x-3)^2 = 8 \quad \text{M1}$$

$$x-3 = \pm\sqrt{8}$$

$$x = 3 \pm 2\sqrt{2} \quad \text{but } x > 1$$

$$x = 3 + 2\sqrt{2} \quad \text{only} \quad \text{A1}$$

b. $8 \times 4^x + 16 \times 4^{-x} = 129$

let $u = 4^x$ then $4^{-x} = \frac{1}{4^x} = \frac{1}{u}$

$$8u + \frac{16}{u} = 129$$

$$8u^2 + 16 = 129u$$

$$8u^2 - 129u + 16 = 0$$

M1

$$(8u - 1)(u - 16)$$

$$u = 4^x = \frac{1}{8} \quad u = 4^x = 16$$

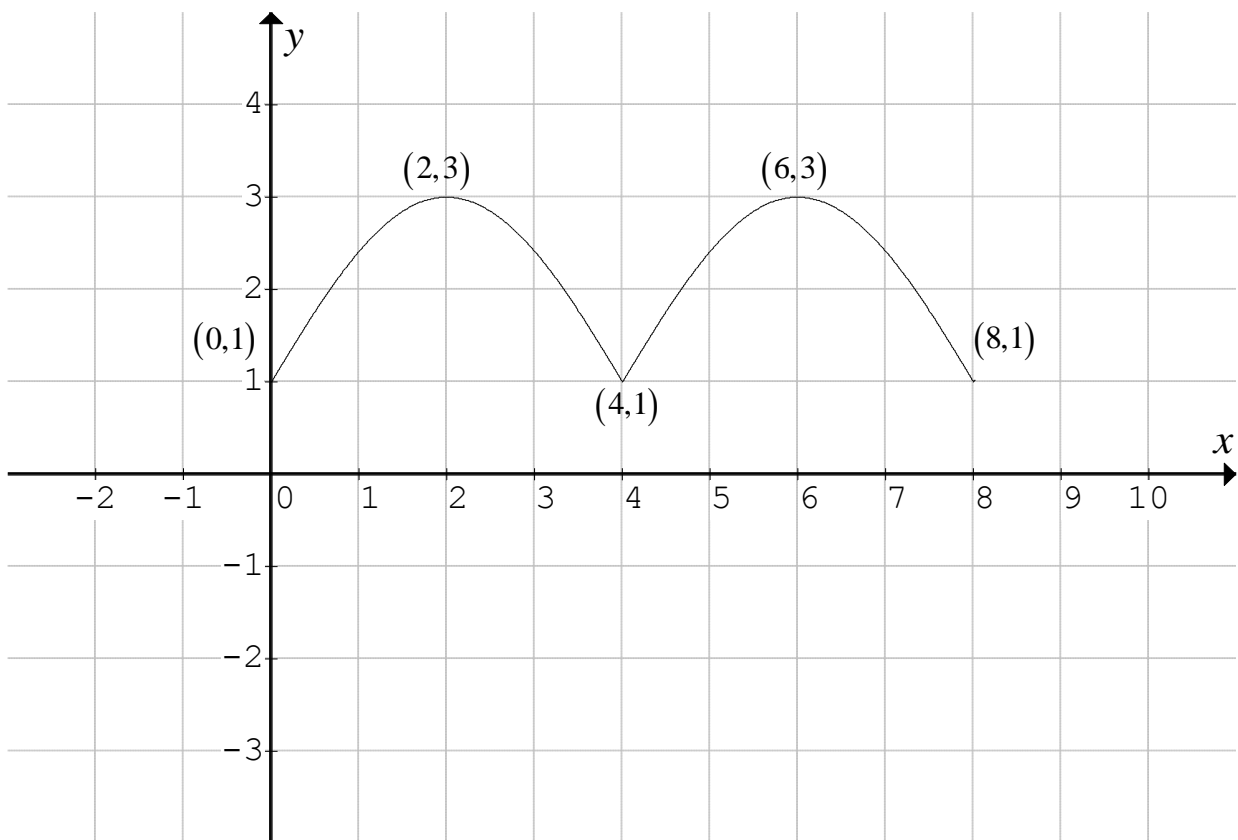
$$x = -\frac{3}{2} \quad \text{or} \quad x = 2$$

A1

Question 6

i. $f : [0, 8] \rightarrow \mathbb{R}, f(x) = \left| 2 \sin\left(\frac{\pi x}{4}\right) \right| + 1$

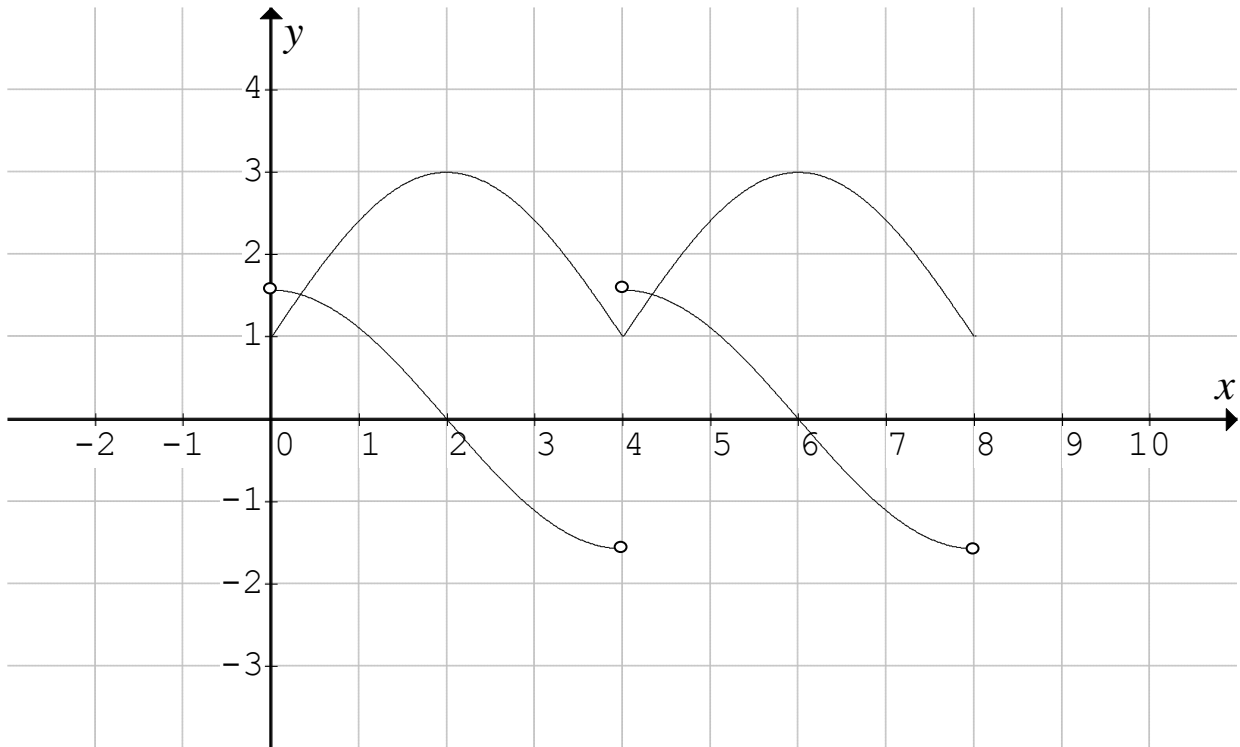
G2



ii. $f'(x) = \frac{\pi}{2} \cos\left(\frac{\pi x}{4}\right)$ for $0 < x < 4$ and $f'(x) = \frac{\pi}{2} \cos\left(\frac{\pi(x-4)}{4}\right)$ for $4 < x < 8$

Note $\frac{\pi}{2} \approx 1.57$ and that the gradient function crosses the x -axis at $x=2$ and $x=6$, the gradient function is not defined at $x=0, 4, 8$ must have open circles at these points.

G2



iii. $\bar{f} = \frac{1}{8} \int_0^8 \left(\left| 2 \sin\left(\frac{\pi x}{4}\right) \right| + 1 \right) dx = \frac{1}{4} \int_0^4 \left(2 \sin\left(\frac{\pi x}{4}\right) + 1 \right) dx$ by symmetry

$$\bar{f} = \frac{1}{4} \left[-\frac{8}{\pi} \cos\left(\frac{\pi x}{4}\right) + x \right]_0^4$$

$$\bar{f} = \frac{1}{4} \left[\left(-\frac{8}{\pi} \cos(\pi) + 4 \right) - \left(-\frac{8}{\pi} \cos(0) + 0 \right) \right] \quad \text{M1}$$

$$\bar{f} = \frac{1}{4} \left[\frac{16}{\pi} + 4 \right]$$

$$\bar{f} = \frac{4}{\pi} + 1 \quad \text{A1}$$

Question 7

a. Since it a probability density function $\int_0^4 \frac{k}{\sqrt{9-2x}} dx = 1$

$$k \int_0^4 (9-2x)^{-\frac{1}{2}} dx = 1$$

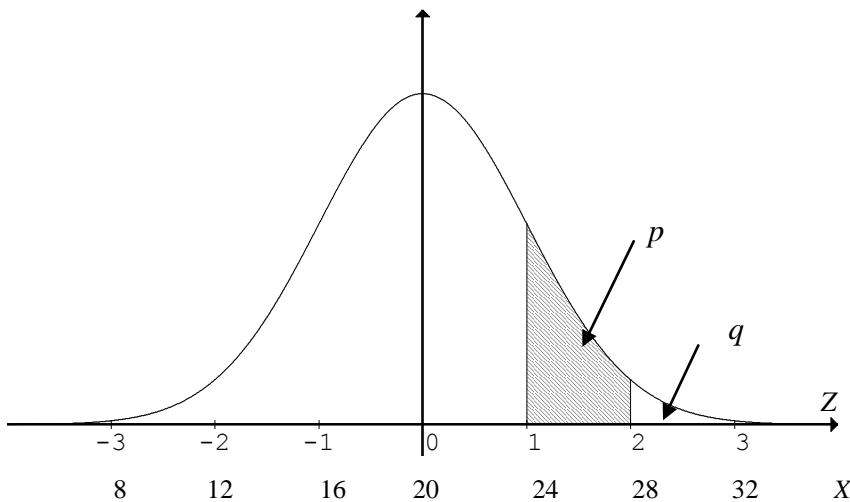
$$k \left[-\frac{2}{2} (9-2x)^{\frac{1}{2}} \right]_0^4 = k \left[-\sqrt{9-2x} \right]_0^4 \quad \text{M1}$$

$$= k \left[(-\sqrt{1}) - (-\sqrt{9}) \right] = 2k = 1$$

$$k = \frac{1}{2} \quad \text{A1}$$

b. Given that $\Pr(1 < Z < 2) = p$ and $\Pr(Z > 2) = q$

$$X \stackrel{d}{=} N(\mu = 20, \sigma^2 = 16) \quad \sigma = 4, \text{ and } Z = \frac{X - \mu}{\sigma}$$



$$\Pr(X < 24 | X > 12) = \frac{\Pr(12 < X < 24)}{\Pr(X > 12)} = \frac{\Pr(-2 < Z < 1)}{\Pr(Z > -2)} \quad \text{now using symmetry}$$

$$= \frac{\Pr(-1 < Z < 2)}{\Pr(Z > -2)} = \frac{2\Pr(0 < Z < 1) + \Pr(1 < Z < 2)}{\Pr(Z < 2)} = \frac{2\left(\frac{1}{2} - (p+q)\right) + p}{1-q} \quad \text{M1}$$

$$= \frac{1-p-2q}{1-q} \quad \text{A1}$$

Question 8

a. $X \stackrel{d}{=} Bi(n=3, p=?)$

$$\Pr(X=1) = \binom{3}{1} p^1 (1-p)^2 = \frac{p}{3}$$

$$3p(1-p^2) = \frac{p}{3} \text{ since } p \neq 0$$

$$(1-p)^2 = \frac{1}{9} \quad \text{M1}$$

$$1-p = \pm \frac{1}{3}$$

$$1-p = \frac{1}{3} \text{ since } 0 < p < 1$$

$$p = \frac{2}{3} \quad \text{A1}$$

b. Since the probabilities sum to one. $\sum \Pr(X=x) = 1$

$$k^2 + \frac{3k}{2} + \frac{k}{2} = 1$$

$$k^2 + 2k = 1$$

$$k^2 + 2k + 1 = 2$$

$$(k+1)^2 = 2 \quad \text{M1}$$

$$k+1 = \pm \sqrt{2}$$

$$k = -1 \pm \sqrt{2} \text{ but } 0 < k < 1$$

$$k = \sqrt{2} - 1$$

$$E(X) = \sum x \Pr(X=x) = 0 \times k^2 + 1 \times \frac{3k}{2} + 2 \times \frac{k}{2} = \frac{5k}{2}$$

$$E(X) = \frac{5}{2}(\sqrt{2} - 1) \quad \text{A1}$$

Question 9

a. $y = e^{-x}(2\cos(2x) + \sin(2x))$ using the product rule

$$u = e^{-x} \quad v = 2\cos(2x) + \sin(2x)$$

$$\frac{du}{dx} = -e^{-x} \quad \frac{dv}{dx} = -4\sin(2x) + 2\cos(2x) \quad \text{M1}$$

$$\frac{dy}{dx} = -e^{-x}(2\cos(2x) + \sin(2x)) + e^{-x}(-4\sin(2x) + 2\cos(2x))$$

$$\frac{dy}{dx} = e^{-x}(-2\cos(2x) - \sin(2x) - 4\sin(2x) + 2\cos(2x))$$

$$\frac{dy}{dx} = -5e^{-x}\sin(2x)$$

b. $f : R \rightarrow R, f(x) = e^{-x}\sin(2x)$

The graph crosses the x -axis when $\sin(2x) = 0$

$$2x = 0, \pi \text{ so that } x = 0, \frac{\pi}{2}$$

$$\text{The shaded area is } A = \int_0^{\frac{\pi}{2}} e^{-x}\sin(2x)dx \quad \text{A1}$$

Since $\frac{d}{dx}[e^{-x}(2\cos(2x) + \sin(2x))] = -5e^{-x}\sin(2x)$ it follows from **a.** that

$$\int e^{-x}\sin(2x)dx = -\frac{1}{5}e^{-x}(2\cos(2x) + \sin(2x))$$

$$A = -\frac{1}{5}\left[e^{-x}(2\cos(2x) + \sin(2x))\right]_0^{\frac{\pi}{2}}$$

$$A = -\frac{1}{5}\left[\left(e^{-\frac{\pi}{2}}(2\cos(\pi) + \sin(\pi))\right) - \left(e^0(2\cos(0) + \sin(0))\right)\right] \quad \text{M1}$$

$$A = -\frac{1}{5}\left[\left(-2e^{-\frac{\pi}{2}}\right) - 2\right]$$

$$A = \frac{2}{5}\left(1 + e^{-\frac{\pi}{2}}\right) \text{ units}^2 \quad \text{A1}$$

Question 10

$$f : [0, 9] \rightarrow R, f(x) = \pi x + 12 \cos\left(\frac{\pi x}{6}\right)$$

$$\text{for turning points } f'(x) = \pi - 2\pi \sin\left(\frac{\pi x}{6}\right) = 0$$

$$\sin\left(\frac{\pi x}{6}\right) = \frac{1}{2}$$

$$\frac{\pi x}{6} = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$x = 1, 5$$

A1

$$f(1) = \pi + 12 \cos\left(\frac{\pi}{6}\right) = \pi + 6\sqrt{3} \quad \text{and} \quad f(5) = 5\pi + 12 \cos\left(\frac{5\pi}{6}\right) = 5\pi - 6\sqrt{3} \quad \text{A1}$$

However since we have a restricted domain function, we must examine the endpoints.

$$f(0) = 12 \cos(0) = 12 \quad \text{and} \quad f(9) = 9\pi + 12 \cos\left(\frac{3\pi}{2}\right) = 9\pi$$

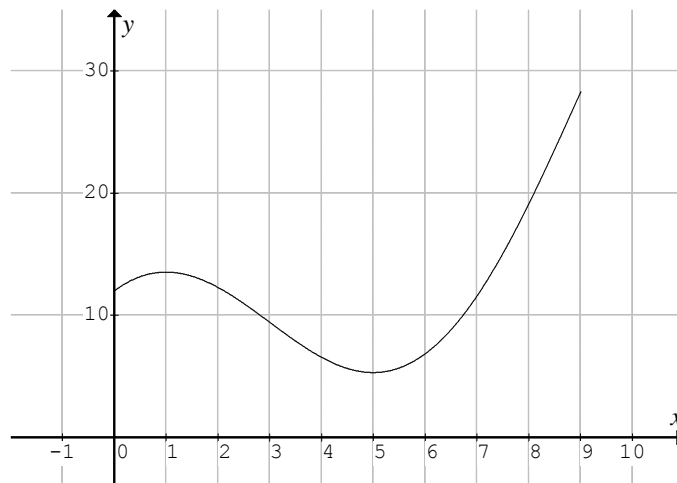
Now $9\pi > \pi + 6\sqrt{3} > 12 > 5\pi - 6\sqrt{3}$ so that

the maximum value is 9π and occurs when $x = 9$

A1

the minimum value is $5\pi - 6\sqrt{3}$ and occurs when $x = 5$

A1



END OF SUGGESTED SOLUTIONS