2014 Mathematical Methods (CAS) Trial Exam 1 Solutions

Q1a $y = mx + c$ and $y_1 = -4x - x^2$:

$mx + c = -4x - x^2$, $x^2 + mx + 4x + c = 0$, $x^2 + (m+4)x + c = 0$

One solution only for $y = mx + c$ to be a tangent, .: $\Delta = 0$

.: $(m+4)^2 - 4c = 0$, $m^2 + 8m + 4(4-c) = 0$

$y = mx + c$ and $y_2 = (x-2)^2 - 2$:

$(x-2)^2 - 2 = mx + c$, $x^2 - 4x + 2 - mx - c = 0$,

$x^2 - (m+4)x + (2-c) = 0$, $\Delta = (m+4)^2 - 4(2-c) = 0$

.: $m^2 + 8m + 4(2+c) = 0$

Q1b $m^2 + 8m + 4(4-c) = 0$ .... (1)

$m^2 + 8m + 4(2+c) = 0$ .... (2)

(2) – (1): $c = 1$

Substitute in (2), $m^2 + 8m + 12 = 0$, $(m+6)(m+2) = 0$

.: $m = -6, -2$

The common tangents are: $y = -6x + 1$, $y = -2x + 1$

Q2 $f(x+k) = 2 - f(x)$, $a\cos\left(\frac{x+k}{3}\right) + 1 = 2 - \left(a\cos\left(\frac{x}{3}\right) + 1\right)$

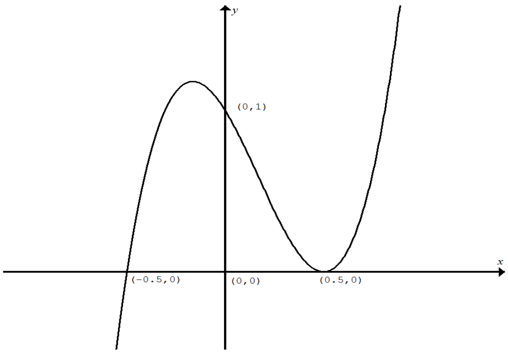
.: $\cos\left(\frac{x+k}{3}\right) = -\cos\left(\frac{x}{3}\right) = \cos\left(\frac{x}{3} + n\pi\right)$

.: $\frac{k}{3} = n\pi$ where n is an integer

Since $-3\pi \le k \le 3\pi$, .: $k = \pm 3\pi$

Q3a $g(x) = 8x^3 - 4x^2 - 2x + 1 = 4x^2(2x-1) - (2x-1)$

$= (4x^2 - 1)(2x-1) = (2x+1)(2x-1)^2$



Q3b $g(x) = f(1-2x)$, $g\left(\frac{x}{2}\right) = f(1-x)$, $g\left(\frac{-x}{2}\right) = f(1+x)$

$g\left(\frac{-(x-1)}{2}\right) = f(x)$, .: $f(x) = g\left(\frac{1-x}{2}\right)$

$= \left(2\left(\frac{1-x}{2}\right) + 1\right)\left(2\left(\frac{1-x}{2}\right) - 1\right)^2 = (2-x)(-x)^2 = -x^3 + 2x^2$

.: $a = -1$, $b = 2$, $c = 0$ and $d = 0$

Q4a $\Pr(2 \text{ cups of coffee}) = \Pr(\text{cctt}) + \Pr(\text{ctct}) + \Pr(\text{cttc})$

$= 1 \times \frac{3}{5} \times \frac{2}{5} \times \frac{4}{5} + 1 \times \frac{2}{5} \times \frac{1}{5} \times \frac{2}{5} + 1 \times \frac{2}{5} \times \frac{4}{5} \times \frac{1}{5} = \frac{36}{125}$

Q4b $\Pr(\text{at least 2 cups of coffee}) = 1 - \Pr(1 \text{ cup of coffee})$

$= 1 - \Pr(\text{cttt}) = 1 - 1 \times \frac{2}{5} \times \frac{4}{5} \times \frac{4}{5} = \frac{93}{125}$

Q4c $\Pr(3 \text{ cups of coffee}) = \Pr(\text{ccct}) + \Pr(\text{cctc}) + \Pr(\text{ctcc})$

$= 1 \times \frac{3}{5} \times \frac{3}{5} \times \frac{2}{5} + 1 \times \frac{3}{5} \times \frac{2}{5} \times \frac{1}{5} + 1 \times \frac{2}{5} \times \frac{1}{5} \times \frac{3}{5} = \frac{30}{125}$

$\Pr(4 \text{ cups of coffee}) = \Pr(\text{cccc}) = 1 \times \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} = \frac{27}{125}$

x cups of coffee	1	2	3	4
$\Pr(X = x)$	$\frac{32}{125}$	$\frac{36}{125}$	$\frac{30}{125}$	$\frac{27}{125}$

$E(X) = 1 \times \frac{32}{125} + 2 \times \frac{36}{125} + 3 \times \frac{30}{125} + 4 \times \frac{27}{125} = \frac{302}{125}$

Q5a $f(x) = \log_{10}(9 - x^2)$, $9 - x^2 > 0$, .: $-3 < x < 3$

The domain is $(-3, 3)$.

Q5b $f(x) = \log_{10}(9 - x^2) = \frac{\log_e(9 - x^2)}{\log_e 10}$

$f'(x) = \frac{-2x}{(\log_e 10)(9 - x^2)}$

$f'(-1) = \frac{-2(-1)}{(\log_e 10)(9 - (-1)^2)} = \frac{1}{4\log_e 10}$

Q5c Area $\approx 1 \times \log_{10} 5 + 1 \times \log_{10} 8 + 1 \times \log_{10} 9 + 1 \times \log_{10} 8$

$= \log_{10}(5 \times 8 \times 9 \times 8) = \log_{10} 2880$ or $1 + \log_{10} 288$

Q6 $-12\sin\frac{\pi x}{3}$ completes 2 cycles and $\frac{1}{12}\cos\frac{\pi x}{6}$ completes a cycle in $[-6, 6]$, .: the average value of each one in the interval is zero.

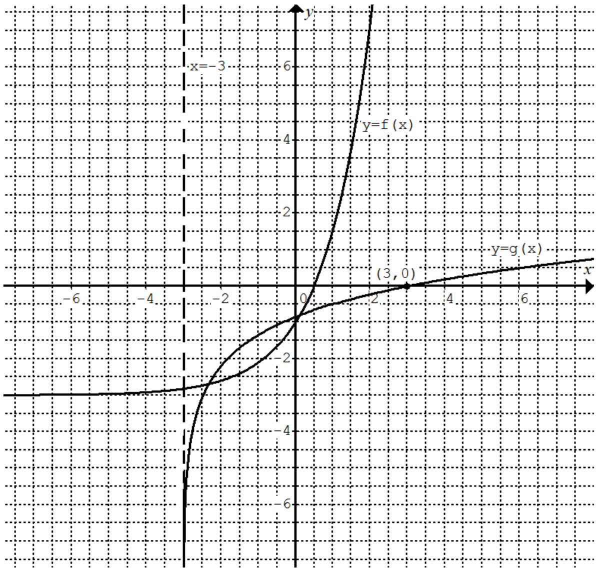
Average value of $f(x)$

$= \frac{\int_{-6}^{6} f(x)\,dx}{6 - (-6)} = \frac{\int_{-6}^{6}\left(1 - 12\sin\frac{\pi x}{3} + \frac{1}{12}\cos\frac{\pi x}{6}\right)dx}{12}$

$= \frac{\int_{-6}^{6} 1\,dx}{12} + \frac{\int_{-6}^{6} -\sin\frac{\pi x}{3}\,dx}{12} + \frac{\int_{-6}^{6} \cos\frac{\pi x}{6}\,dx}{12}$

$= \frac{\int_{-6}^{6} 1\,dx}{12} = 1$

Q7a



Q7b Sequence of transformations of $f(x)$:

$f(x) \to f^{-1}(x) \to f^{-1}(x-2) \to f^{-1}\left(\frac{x}{3}-2\right)$

$\therefore\ g(x) = f^{-1}\left(\frac{x}{3}-2\right)$

Q8a $y = |x|e^{-|x|} = \begin{cases} x e^{-x}, & x \ge 0 \\ -x e^{x}, & x \le 0 \end{cases}$

Note: The two sections in the hybrid function are reflections of each other in the *y*-axis, i.e. the *y*-axis is the axis of symmetry of the function.

$\frac{dy}{dx} = \begin{cases} e^{-x} - x e^{-x}, & x > 0 \\ -e^{x} - x e^{x}, & x < 0 \end{cases}$

Note: $\frac{dy}{dx}$ is undefined at $x = 0$

Q8b From Q8a, $\frac{dy}{dx} = e^{-x} - x e^{-x}$ for $x > 0$

$\therefore \int_a^{\log_e 2} \frac{dy}{dx} dx = \int_a^{\log_e 2} e^{-x}\, dx - \int_a^{\log_e 2} x e^{-x}\, dx$ where $0 < a < \log_e 2$

$\therefore \int_a^{\log_e 2} x e^{-x}\, dx = \int_a^{\log_e 2} e^{-x}\, dx - \int_a^{\log_e 2} \frac{dy}{dx} dx = \left[-e^{-x}\right]_a^{\log_e 2} - \left[x e^{-x}\right]_a^{\log_e 2}$

$= -e^{-\log_e 2} + e^{-a} - (\log_e 2)e^{-\log_e 2} + a e^{-a}$

$= -\frac{1}{2} + e^{-a} - \frac{1}{2}\log_e 2 + a e^{-a}$

$\int_{-\log_e 2}^{\log_e 2} |x| e^{-|x|}\, dx = 2 \times \lim_{a \to 0^+} \int_a^{\log_e 2} x e^{-x}\, dx$

$\therefore \int_{-\log_e 2}^{\log_e 2} |x| e^{-|x|}\, dx = 2 \times \lim_{a \to 0^+}\left(-\frac{1}{2} + e^{-a} - \frac{1}{2}\log_e 2 + a e^{-a}\right)$

$= 1 - \log_e 2$

Q9 $\frac{2^{2x} + 2^{-2x} - 2}{2^x - 2^{-x}} = 2$ where $2^x - 2^{-x} \ne 0$

$\frac{\left(2^x - 2^{-x}\right)\left(2^x - 2^{-x}\right)}{2^x - 2^{-x}} = 2$, $\therefore\ 2^x - 2^{-x} = 2$

$\left(2^x - 2^{-x}\right) \times 2^x = 2 \times 2^x$, $\therefore\ \left(2^x\right)^2 - 2 \times 2^x - 1 = 0$

$\therefore\ 2^x = 1 + \sqrt{2}$ by the quadratic formula

$\therefore\ x = \log_2\left(1 + \sqrt{2}\right)$

Q10a $\int_0^2 \left(0.75 - k|x-1|\right) dx = 1$, $\int_0^2 0.75\, dx - \int_0^2 k|x-1|\, dx = 1$

$\int_0^2 0.75\, dx - 2 \times \int_1^2 k(x-1)\, dx = 1$

$\left[0.75x\right]_0^2 - 2k\left[\frac{(x-1)^2}{2}\right]_1^2 = 1$, $1.5 - k = 1$, $k = 0.5$

Q10b $f(x) = \begin{cases} 0.75 - 0.5(-x+1), & 0 \le x \le 1 \\ 0.75 - 0.5(x-1), & 1 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$

$= \begin{cases} 0.5x + 0.25, & 0 \le x \le 1 \\ -0.5x + 1.25, & 1 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$

$E(X) = \int_0^1 x(0.5x + 0.25)\, dx + \int_1^2 x(-0.5x + 1.25)\, dx$

$= \int_0^1 \left(0.5x^2 + 0.25x\right) dx + \int_1^2 \left(-0.5x^2 + 1.25x\right) dx$

$= \left[\frac{x^3}{6} + \frac{x^2}{8}\right]_0^1 + \left[-\frac{x^3}{6} + \frac{5x^2}{8}\right]_1^2 = 1$

Q10c $\Pr(X < b) = 1 - \Pr(X > b)$ where $0 < b < 2$

$\therefore\ \Pr(X > b) = 1 - \Pr(X < b) = \frac{3}{16}$

$\int_b^2 (-0.5x + 1.25)\, dx = \frac{3}{16}$, $\therefore \left[-\frac{x^2}{4} + \frac{5x}{4}\right]_b^2 = \frac{3}{16}$

$(-1+5) - \left(-\frac{b^2}{4} + \frac{5b}{4}\right) = \frac{3}{16}$, $4b^2 - 20b + 21 = 0$, $\therefore\ b = \frac{3}{2}$

Please inform mathline@itute.com re conceptual and/or mathematical errors