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MATHS METHODS (CAS) 3 & 4 TRIAL EXAMINATION 1 SOLUTIONS 2014

Question 1 (4 marks)

a.

$$y = 3x^2 e^{4x}$$

$$\frac{dy}{dx} = 6xe^{4x} + 3x^2 \times 4e^{4x} \text{ (product rule)}$$

$$= 6xe^{4x} + 12x^2 e^{4x}$$

(1 mark) – correct first term
(1 mark) – correct second term

b.

$$g(x) = \frac{\sin(x)}{2x}$$

$$g'(x) = \frac{2x \times \cos(x) - 2 \times \sin(x)}{(2x)^2} \quad \text{(quotient rule)}$$ **(1 mark)**

$$g'\left(\frac{\pi}{2}\right) = \frac{2 \times \frac{\pi}{2} \times 0 - 2 \times 1}{\left(2 \times \frac{\pi}{2}\right)^2}$$

since $\cos\left(\frac{\pi}{2}\right) = 0$ and $\sin\left(\frac{\pi}{2}\right) = 1$

$$= \frac{0-2}{\pi^2}$$

$$= \frac{-2}{\pi^2}$$

(1 mark)

Question 2 (2 marks)

$$\int \frac{1}{2x-3}\, dx = \frac{1}{2}\log_e|2x-3|$$

Note, because we were asked for 'an' antiderivative *c* is not required (because *c* could, in this case, equal zero)

(1 mark) – recognition that a logarithm was required
(1 mark) – correct answer

Question 3 (2 marks)

$f : R \setminus \{0\} \to R, f(x) = \dfrac{5}{x}$

To show, $f(f(x)) = f(f^{-1}(x))$

$$\begin{aligned} LS &= f(f(x)) \\ &= f\left(\frac{5}{x}\right) \\ &= \frac{5}{\frac{5}{x}} \\ &= 5 \div \frac{5}{x} \\ &= 5 \times \frac{x}{5} \\ &= x \end{aligned}$$

(1 mark)

Now find $f^{-1}(x)$.

$$f(x) = \frac{5}{x}$$

Let $y = \dfrac{5}{x}$

Swap x and y for inverse.

$$x = \frac{5}{y}$$

$$y = \frac{5}{x}$$

$$f^{-1}(x) = \frac{5}{x}$$

$$\begin{aligned} RS &= f(f^{-1}(x)) \\ &= f\left(\frac{5}{x}\right) \\ &= x \\ &= LS \end{aligned}$$

Have shown.

(1 mark)

Question 4 (5 marks)

a.

$$\log_6(3) - 2\log_6(x) + \log_6(2) = 1$$
$$\log_6(3) - \log_6(x^2) + \log_6(2) = 1$$
$$\log_6\left(\frac{3 \times 2}{x^2}\right) = 1$$

(1 mark)

$$6^1 = \frac{6}{x^2}$$
$$6x^2 = 6$$
$$x^2 = 1$$
$$x = \pm 1$$

(1 mark)

but $x > 0$ (for the term $-2\log_6(x)$ to be defined)

So $x = 1$.

(1 mark)

b.

$$8^{1-2x} = 2^{4+x}$$
$$(2^3)^{1-2x} = 2^{4+x}$$
$$2^{3-6x} = 2^{4+x}$$

(1 mark)

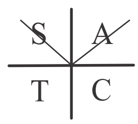
So $3 - 6x = 4 + x$

$$-1 = 7x$$
$$x = -\frac{1}{7}$$

(1 mark)

Question 5 (2 marks)

$$\sin\left(\frac{x}{3}\right) = \frac{1}{\sqrt{2}} \qquad x \in [0, 6\pi]$$
$$\frac{x}{3} = \frac{\pi}{4}, \frac{3\pi}{4} \qquad \frac{x}{3} \in [0, 2\pi]$$
$$x = \frac{3\pi}{4}, \frac{9\pi}{4}$$



sin is positive in the 1st and 2nd quadrants

(1 mark) – one correct answer
(1 mark) – second correct answer

Question 6 (5 marks)

a. $\Pr(X=0)+\Pr(X=1)+\Pr(X=2)=0.5$
and, $\Pr(X=3)+\Pr(X=4)=0.5$.
The median of X is therefore halfway between 2 and 3 so the median of X is 2.5

(1 mark)

b. Method 1 – using intuition and the table.

$$\Pr(X \le 2 \mid X > 0) = \frac{0.3}{0.8}$$ **(1 mark)**

$$= \frac{3}{8}$$

(1 mark)

Since we're told $X > 0$, then we eliminate $\Pr(X=0)$ which gives us a denominator of 0.8.
We want $\Pr(X \le 2)$, so we want $\Pr(X=1)$ and $\Pr(X=2)$. Remember that $\Pr(X=0)$ is eliminated.
So the numerator is $0.2+0.1=0.3$.

Method 2 – using the conditional probability formula

$$\Pr(X \le 2 \mid X > 0) = \frac{\Pr(X \le 2 \cap X > 0)}{\Pr(X > 0)}$$ (from the formula sheet) **(1 mark)**

$$= \frac{\Pr(X=1)+\Pr(X=2)}{\Pr(X=1)+\Pr(X=2)+\Pr(X=3)+\Pr(X=4)}$$

$$= \frac{0.2+0.1}{0.2+0.1+0.4+0.1}$$

$$= \frac{0.3}{0.8}$$

$$= \frac{3}{8}$$ **(1 mark)**

c. $$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= 0^2 \times 0.2 + 1^2 \times 0.2 + 2^2 \times 0.1 + 3^2 \times 0.4 + 4^2 \times 0.1 - 2^2$$ **(1 mark)**

$$= 0.2 + 0.4 + 3.6 + 1.6 - 4$$

$$= 1.8$$

(1 mark)

This formula is not given on the formula sheet. It is easier to use than the one on the formula sheet so therefore worth remembering.

Question 7 (4 marks)

a. Since f is a probability density function,

$$\int_0^1 a\cos\left(\frac{\pi x}{2}\right)dx = 1$$ **(1 mark)**

$$a\left[\frac{1}{\pi/2}\sin\left(\frac{\pi x}{2}\right)\right]_0^1 = 1 \qquad (a \text{ is a constant})$$ **(1 mark)**

$$a\left\{\frac{2}{\pi}\sin\left(\frac{\pi}{2}\right) - \frac{2}{\pi}\sin(0)\right\} = 1$$

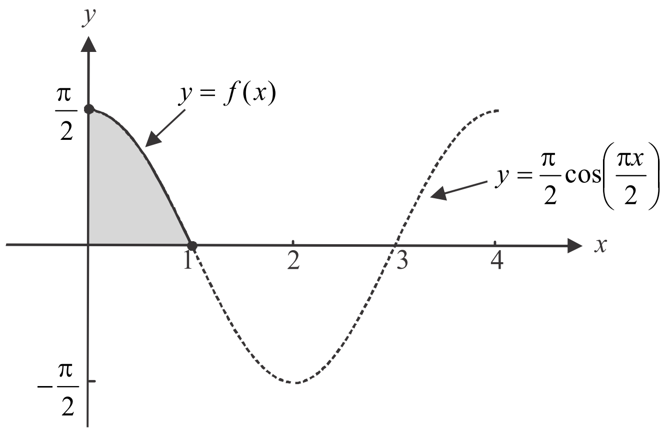
$$a\left(\frac{2}{\pi}\times 1 - 0\right) = 1$$

$$\frac{2a}{\pi} = 1$$

$$a = \frac{\pi}{2}$$

(1 mark)

b. The mode of X is the value of x for which $f(x)$ is a maximum.
Do a quick sketch.



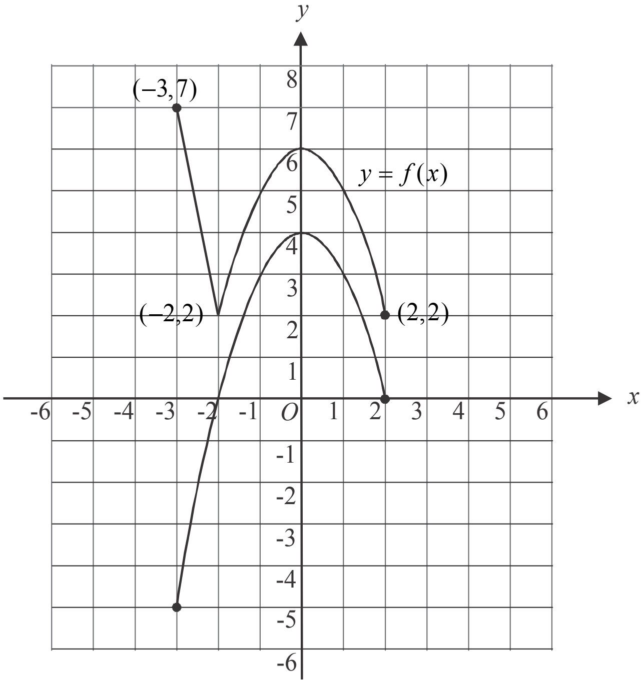
The maximum value of $f(x)$ is $\frac{\pi}{2}$ and it occurs when $x = 0$.

The mode is zero.

(1 mark)

Question 8 (6 marks)

a.



(1 mark) – correct shape
(1 mark) – correct position including endpoints

b. **i.** $f(x) = \left|4 - x^2\right| + 2$

When the graph of *f* is translated 2 units in the negative direction of the *y*-axis, its rule becomes

$y = \left|4 - x^2\right| + 2 - 2$, that is, $y = \left|4 - x^2\right|$

When this graph is then translated 2 units in the positive direction of the *x*-axis, its rule becomes $y = \left|4 - (x-2)^2\right|$

When this graph is then dilated from the *y*-axis by a factor of $\frac{1}{2}$, its rule

becomes $y = \left|4 - \left(\frac{x}{1/2} - 2\right)^2\right|$, that is, $h(x) = \left|4 - (2x-2)^2\right|$.

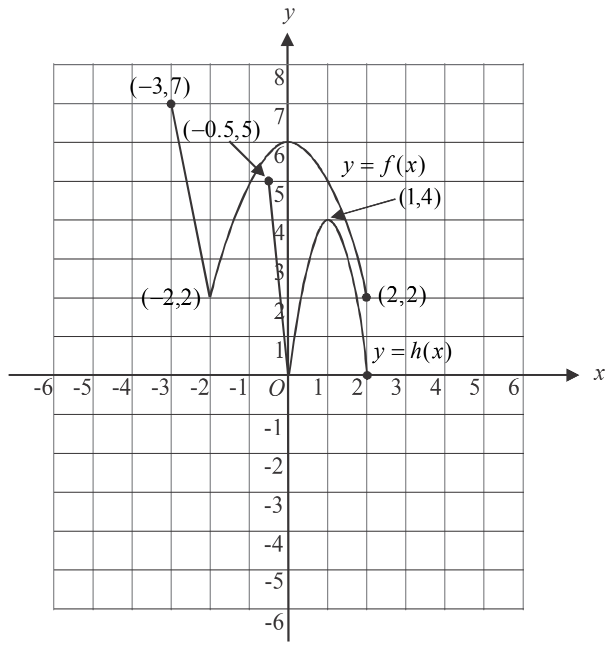
(1 mark) – for $y = \left|4 - (x-2)^2\right|$

(1 mark) – for $h(x) = \left|4 - (2x-2)^2\right|$

ii. Method 1 – drawing a graph

To find the domain, do a quick sketch of $y = f(x)$ after it has undergone the two translations and the dilation to become $y = h(x)$.

$d_h = [-0.5, 2]$ **(1 mark)**



Method 2 – using the rule

$f(x)=\left|4-x^2\right|+2 \qquad d_f=[-3,2] \qquad r_f=[2,7]$ from part **a.**

After a translation 2 units in the negative direction of the y-axis, we have:

$y=\left|4-x^2\right| \qquad d=[-3,2] \qquad r=[2-2,7-2]=[0,5]$

After a translation 2 units in the positive direction of the x-axis, we have:

$y=\left|4-(x-2)^2\right| \qquad d=[-3+2,2+2]=[-1,4] \qquad r=[0,5]$

After a dilation from the y-axis by a factor of $\frac{1}{2}$, we have:

$h(x)=\left|4-(2x-2)^2\right| \qquad d_h=[-1\times\frac{1}{2},4\times\frac{1}{2}]=[-\frac{1}{2},2] \qquad r_h=[0,5]$

So $d_h=\left[-\frac{1}{2},2\right]$ **(1 mark)**

iii. Using either method in part **ii.**, $r_h=[0,5]$ **(1 mark)**

Question 9 (4 marks)

a.
$$\frac{d}{dx}(1-x)\log_e(x)=-1\times\log_e(x)+(1-x)\times\frac{1}{x} \quad \text{(product rule)}$$
$$=-\log_e(x)+\frac{1}{x}-\frac{x}{x}$$
$$=-\log_e(x)+x^{-1}-1$$

(1 mark)

b. The area of the shaded region is given by the integral $\int_{0.5}^{1}-\log_e(x)dx$.

From part **a.**, we have
$$\frac{d}{dx}(1-x)\log_e(x)=-\log_e(x)+x^{-1}-1$$

Rearrange this.
$$-\log_e(x)=\frac{d}{dx}(1-x)\log_e(x)-x^{-1}+1$$

Finding the antiderivative of each and every term on both sides of the equation gives:
$$\int_{0.5}^{1}-\log_e(x)dx=\int_{0.5}^{1}\frac{d}{dx}(1-x)\log_e(x)dx-\int_{0.5}^{1}x^{-1}dx+\int_{0.5}^{1}1dx \quad \textbf{(1 mark)}$$
$$=\left[(1-x)\log_e(x)\right]_{0.5}^{1}-\left[\log_e|x|\right]_{0.5}^{1}+\left[x\right]_{0.5}^{1} \quad \textbf{(1 mark)}$$
$$=\{0-(1-0.5)\log_e(0.5)\}-\{\log_e(1)-\log_e(0.5)\}+\{1-0.5\}$$
$$=-0.5\log_e(0.5)+\log_e(0.5)+0.5$$
$$=0.5\log_e(0.5)+0.5$$

So area $=0.5(\log_e(0.5)+1)$ square units as required.

(1 mark)

Note that $\log_e(1)=0$ and that $\int_{0.5}^{1}\frac{d}{dx}(1-x)\log_e(x)dx$
$$=\left[(1-x)\log_e(x)\right]_{0.5}^{1}$$
because the antiderivative ‘undoes’ the derivative.

Question 10 (6 marks)

a. **i.** In ΔCDE, $\sin(\theta) = \dfrac{DE}{CD}$

$$= \frac{DE}{p} \text{ since } AB = DC = p \text{ cm}$$

So $DE = p\sin(\theta)$ **(1 mark)**

ii. In ΔCDE, $\cos(\theta) = \dfrac{CE}{CD}$

$$= \frac{CE}{p}$$

$CE = p\cos(\theta)$ **(1 mark)**

b. $A =$ area of rectangle $ABCD -$ area of ΔCDE

$$= 4 \times p - \frac{1}{2} \times DE \times CE$$

$$= 4p - \frac{1}{2} \times p\sin(\theta) \times p\cos(\theta)$$

$$A = 4p - \frac{p^2}{2}\sin(\theta)\cos(\theta)$$ **(1 mark)**

c. $\dfrac{dA}{d\theta} = -\dfrac{p^2}{2}(\cos(\theta) \times \cos(\theta) + \sin(\theta) \times {}^{-}\sin(\theta))$ (product rule)

$$= \frac{-p^2}{2}(\cos^2(\theta) - \sin^2(\theta))$$ **(1 mark)**

Min occurs when $\dfrac{dA}{d\theta} = 0$ so

$$\frac{-p^2}{2}(\cos^2(\theta) - \sin^2(\theta)) = 0$$

$$\cos^2(\theta) - \sin^2(\theta) = 0 \quad \text{since} \quad p > 0$$

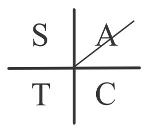
$$\cos^2(\theta) = \sin^2(\theta)$$

$$1 = \frac{\sin^2(\theta)}{\cos^2(\theta)}$$

$$1 = \tan^2(\theta)$$

$\tan(\theta) = 1$ or $\tan(\theta) = -1$

$\theta = \dfrac{\pi}{4}$ not possible since $0 < \theta < \dfrac{\pi}{2}$



(1 mark)

d. From part **c.**, the minimum area occurs when $\theta = \dfrac{\pi}{4}$. From part **b.**,

$$A = 4p - \frac{p^2}{2}\sin(\theta)\cos(\theta)$$

$$16 = 4p - \frac{p^2}{2}\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right)$$

$$16 = 4p - \frac{p^2}{2} \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}$$

$$16 = 4p - \frac{p^2}{4}$$

$$64 = 16p - p^2$$

$$p^2 - 16p + 64 = 0$$

$$(p-8)^2 = 0$$

$$p = 8$$

(1 mark)