QATs
Quality Assessment Tasks

UNIT 4 ■ OUTCOMES 1, 2 & 3

VCE Mathematics Methods (CAS)

SCHOOL-ASSESSED COURSEWORK

Introduction

Outcome 1
Define and explain key concepts as specified in the content from the areas of study, and apply a range of related mathematical routines and procedures.

Outcome 2
Apply mathematical processes in non-routine contexts, and analyse and discuss these applications of mathematics.

Outcome 3
Select and appropriately use technology to develop mathematical ideas, produce results and carry out analysis in situations requiring problem-solving, modeling or investigative techniques or approaches.

Task

Assignment

The task has been designed to allow achievement up to and including the highest level in the Performance Descriptors. It covers a broad range of **key knowledge** and **key skills** over the three outcomes for Unit 4.

It will contribute 20 out of the total (40) marks allocated for SAC in Unit 4.

This task will be marked out of 60 and then will be converted to a proportion of the contribution of this task to SAC in this unit.

The marks for each part are indicated in brackets. Answer in space provided or as directed.

You have 120 minutes over no more than two days. Work in progress will be collected.

You can access your logbook and an approved CAS calculator.

T Indicates where use of the technology is specifically required in order to answer the question.

Your teacher will advise you of any variation to these conditions.

NAME:

Task

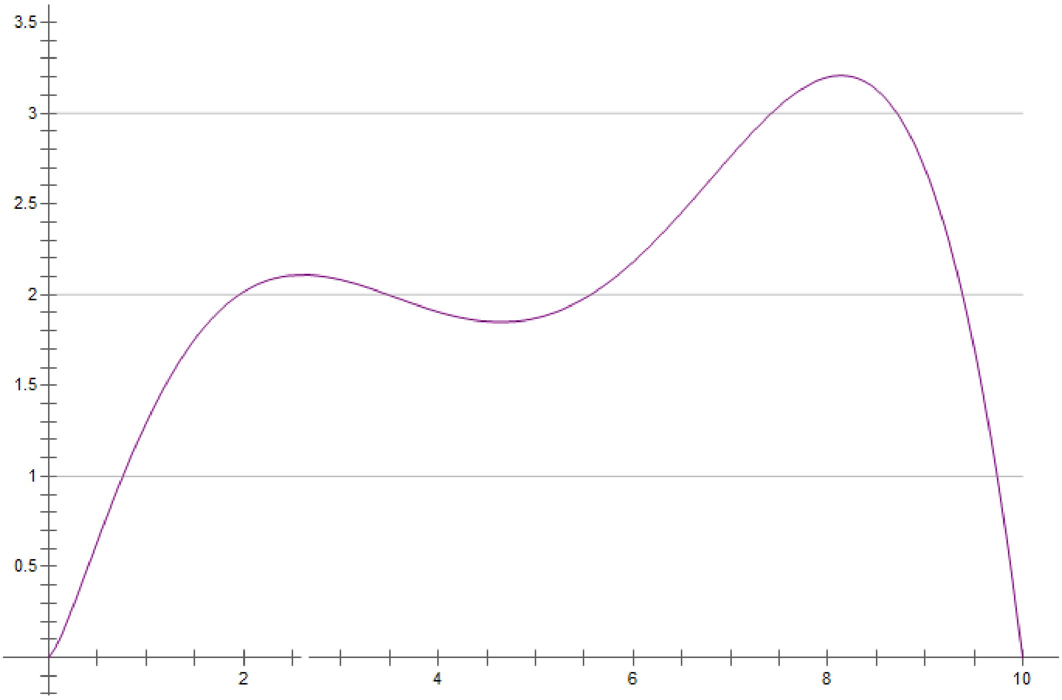
This assignment style task provides students with the opportunity to explore the concepts of Integral Calculus in greater detail.

SHORT ANSWER (17 marks)

Question 1

Approximation using rectangles

a) Find an approximation to the area under the curve shown using 5 rectangles. **(1 mark)**



b) Which of the two methods, left end point or right end point gives the larger estimate? Explain your reasoning to be awarded marks. **(2 marks)**

Task

Question 2

For the graph of a function with equation $f(x) = 10x - x^2$,

a) calculate the actual area bounded by the curve and values x = 0 and x = 10 **(2 marks)**

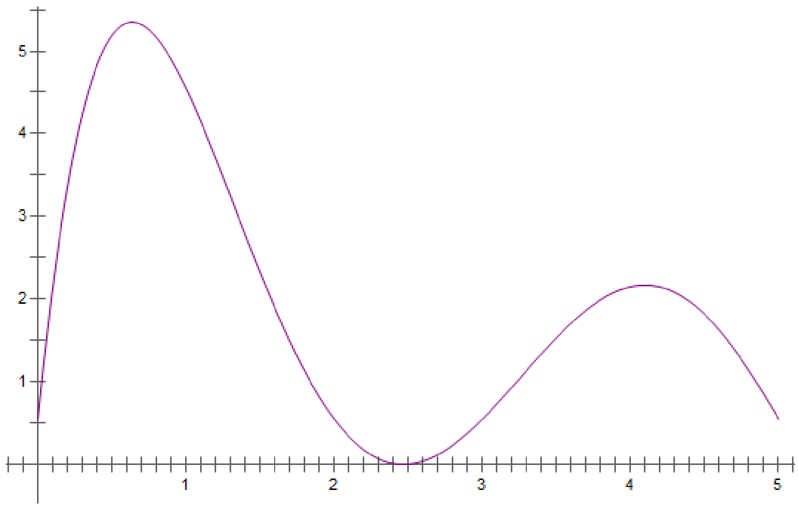
b) calculate the approximate area using 10 rectangles of width 1 unit. **(1 mark)**

c) the percentage error between the answers to a) and b) to the nearest percent. **(1 mark)**

Task

Question 3

When approximating the area under a curve, the more rectangles that are used the better the estimate becomes.



For the graph above give reasons why each of the following numbers of rectangles might lead to particularly poor estimates of area: **(3 marks)**

a) 1 rectangle

b) 2 rectangles

c) 5 rectangles

Task

Question 4

For the function $f(x) = x^3 - 6x^2 + 9x$

a) Find an expression for the definite integral between x = 0 and x = a **(2 marks)**

b) Show that the expression for the average value of this function is: **(2 marks)**

$$\frac{x^4 - 8x^3 + 18x^2}{4a}$$

c) Determine the value of the right terminal that gives an area under the curve equal to a rectangle with height 2 units. **(3 marks)**

Task

MULTIPLE CHOICE (8 x 1 = 8 marks)

Consider the function $f(x) = \sqrt{x^3 - 2x^2 + x}$. Questions 1 to 3 relate to *f(x)*.

1) This function is the same as:

A. $x^{3/2} - \sqrt{2}x + x^{1/2}$
B. $x^{3/2} - 2x + x^{1/2}$
C. $|\sqrt{x}(x-1)|$
D. $\sqrt{x}|x-1|$
E. $\sqrt{x}(x-1)$

☐

2) The function *f(x)* cannot be integrated by rule and so to find the area bounded by the curve it is necessary to use a numerical technique. The area bounded by the curve *y = f(x),* the *x* axis and *x = 0* and *x = 2* is closest to: **T**

A. 0.91046
B. 0.91045
C. 1.4142
D. 2
E. 1.7678

☐

3) The function *f(x)* has a local maximum between x = 0 and x = 1. The coordinates of this maximum are closest to: **T**

A. (0.3333, 0)
B. (0.3333, 0.3849)
C. (0.3, 0.3834)
D. (0.3849, 0.3333)
E. (0.3834, 0.3)

☐

4) If, $f'(x) = \frac{1}{2\sqrt{x+3}}$ and f(1) = 1 then the y-intercept of f(x) is closest to: **T**

A. (0, 0.28868)
B. (0, -1)
C. (0, -2)
D. (0, 0.73205)
E. (0, 0.25)

☐

Task

5) For the function $g(x) = (2x)^{1/3}$ the area bounded by the curve, the x-axis, x = 0 and x = 5 correct to four decimal places is: **T**

A. 8.0791 unit^2
B. 2.1544 unit^2
C. 0.1436 unit^2
D. 3.1420 unit^2
E. 2.1544 unit^2

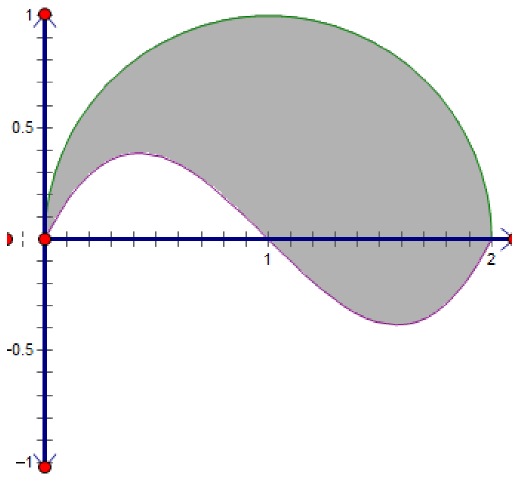
6) When evaluated, $\int_0^5 x^3 - 4x^2 + 2x + 1\, dx$, to two decimal places, is: **T**

A. 29.57
B. 23.66
C. 35.00
D. 19.59
E. 19.58

7) When evaluated, the integral $\int_{-1}^{5} |x^2 - 4x| dx$ is ***NOT*** the same as:

A. $\int_{-1}^{0}(x^2 - 4x)\, dx + \int_{0}^{4}(4x - x^2)\, dx + \int_{4}^{5}(x^2 - 4x)\, dx$
B. $\int_{-1}^{0}(x^2 - 4x)\, dx - \int_{0}^{4}(x^2 - 4x)\, dx + \int_{4}^{5}(x^2 - 4x)\, dx$
C. $\int_{-1}^{0}(x^2 - 4x)\, dx + \int_{0}^{4}(x^2 - 4x)\, dx + \int_{4}^{5}(x^2 - 4x)\, dx$
D. $\int_{-1}^{5}|(x-2)^2 - 4x|\, dx$
E. $\int_{-1}^{0}(x^2 - 4x)\, dx + \int_{4}^{0}(x^2 - 4x)\, dx + \int_{4}^{5}(x^2 - 4x)\, dx$

8) The area bounded by a semicircle of radius 1 unit and the curve $y = x^3 - 3x^2 + 2x$ is shown. The value of this area, to three decimal places, is: **T**



A. 1.321 units^2
B. 1.821 units^2
C. 1.071 units^2
D. 2.071 units^2
E. 1.571 units^2

Task

EXTENDED RESPONSE (35 marks)

Question 1

Two parabolas form an enclosed area. The equation of the first parabola is:

$$f(x) = x^2 - 12x + 27$$

The second parabola, *g(x)* is a transformation of *f(x)* described by the matrix equation:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 7 \end{bmatrix}$$

a) Explain the combination of transformations described by this equation.

(2 marks)

b) Show algebraically, that the equation of the second function is:

$$g(x) = -x^2 + 12x - 20$$

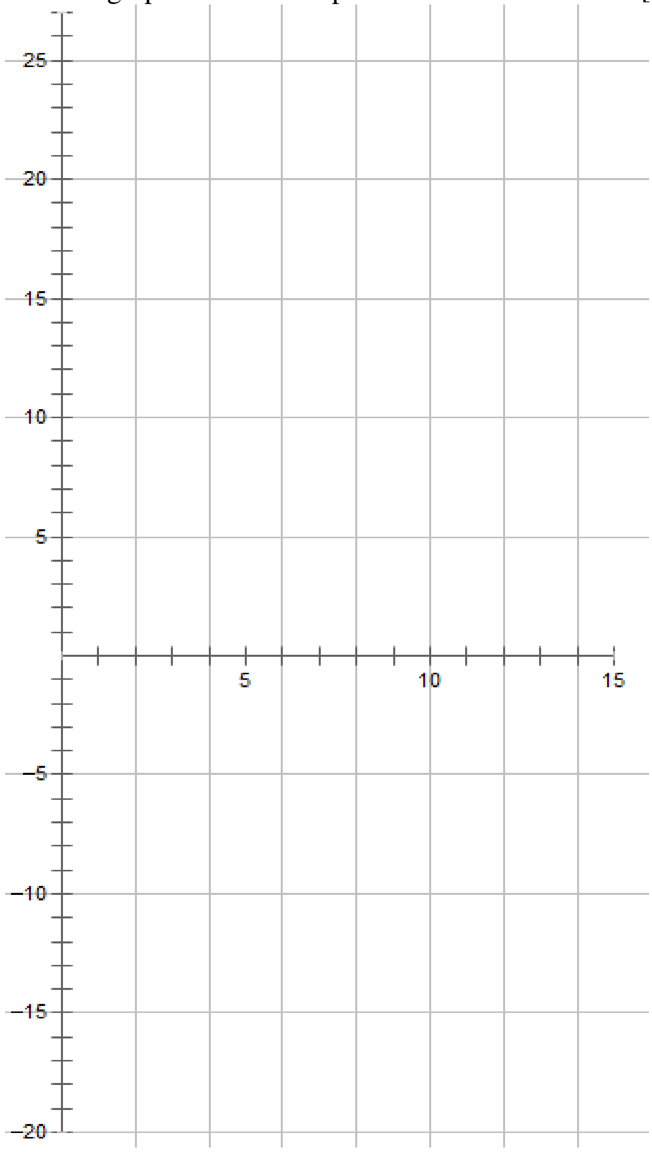
(3 marks)

c) Determine the coordinates of the points of intersection of *f(x)* and *g(x)* in exact form.

(3 marks)

Task

d) Sketch both graphs on the axes provided over the domain [0, 12].



(4 marks)

e) Set up an expression to determine the area between $f(x)$ and $g(x)$.

(3 marks)

Task

f) Determine the area enclosed between the two functions. State your answer to two decimal places. **T**

(2 marks)

Task

Question 2

The velocity of a particle can be modelled by the equation:

$$v(t) = 3t^2 - 19t + 6$$

a) At what time(s) is the particle stationary? Give your answer(s) in exact form. **(2 marks)**

b) Use calculus to determine an expression for acceleration, *a(t)*, of the particle at any given time *t*. **(1 mark)**

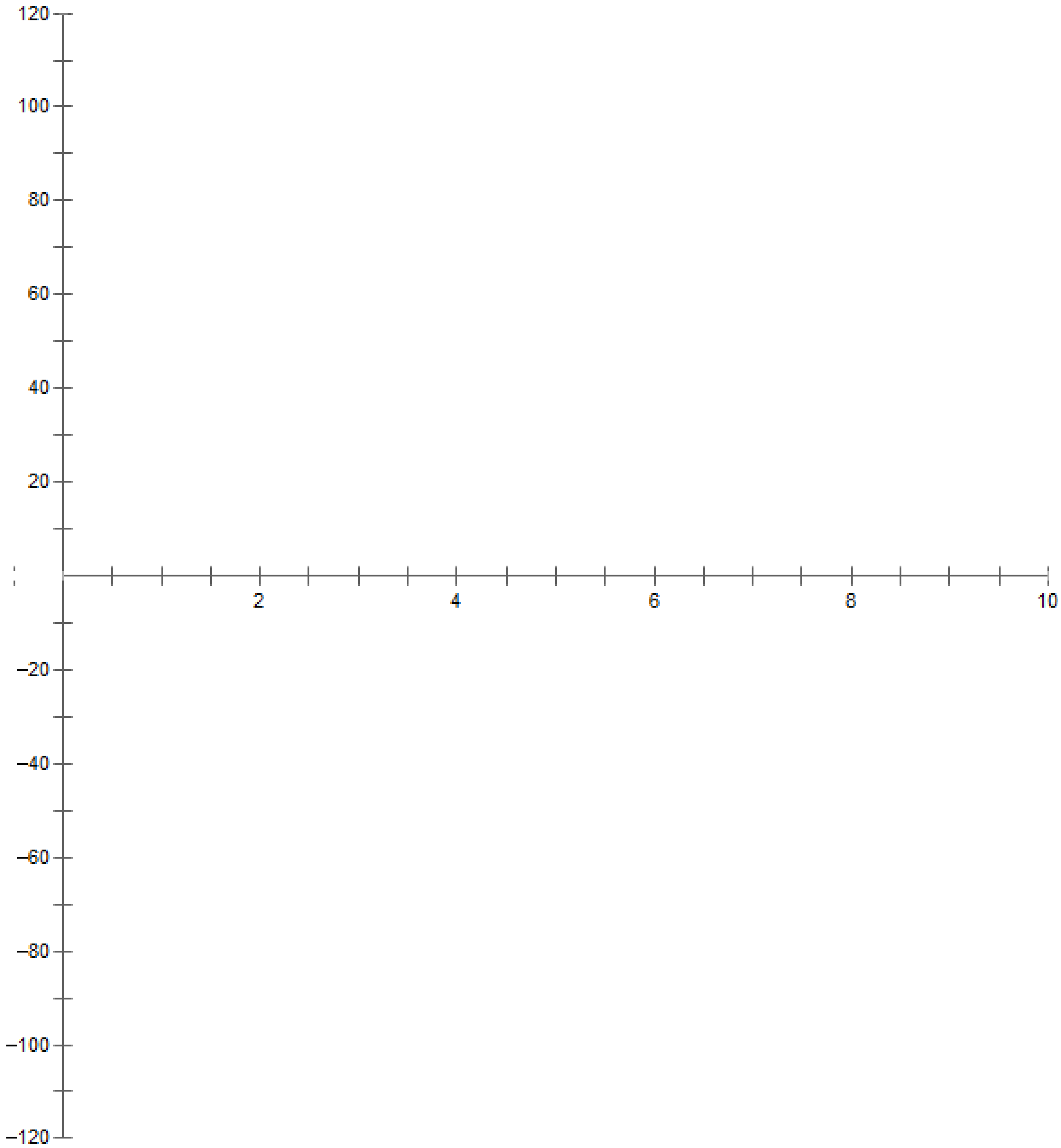
c) What is the particle's initial acceleration? **(1 mark)**

d) At what time is the acceleration of the particle zero and what is its velocity at this time? Give your answer for velocity to 2 decimal places. **T** **(2 marks)**

e) It is known that after 10 seconds the displacement of the particle is 100 metres. Use this information and calculus to determine an expression for the displacement of the particle, *x(t)*, at any time *t*. **(3 marks)**

Task

f) Sketch the graph of the displacement of the particle for the first 10 seconds of its motion. Include the coordinates of all intercepts, end points and stationary points. **(4 marks)**



g) What is the total distance travelled by the particle in the first 10 seconds? State your answer correct to the nearest centimetre. **T** **(2 marks)**

h) Describe the motion of the particle over the first 10 seconds. **(3 marks)**

Teacher Advice

Assessment planning

This is the Analysis Task and it is suggested to be undertaken in weeks 12 and 13 in the sample teaching sequence on page 193 of the VCAA Study Design. Two analysis tasks should be completed in Unit 4.

Key knowledge and key skills

This task covers a broad range of **key knowledge** and **key skills** across Outcomes 1, 2 and 3 as per the VCAA VCE Study Design 2006-14.

This task covers assessment in:

- Functions and Graphs; Linear, Quadratic, Cubic, Hyperbolas, Absolute Value, Square Root and combinations thereof
- Transformations using Matrices
- Algebra
- Differentiation and Integration.

This task contributes 20 of the 40 SAC marks in Unit 4.
The coursework scores for this task are:

Outcome 1	8 marks	40%
Outcome 2	7 marks	35%
Outcome 3	5 marks	25%
TOTAL	20 marks	

This weighting can be used in the conversion of a student mark out of 50.
For example, a score of 40 results in:

OUTCOME 1	OUTCOME 2	OUTCOME 3
40/60*20*0.4	40/60*20*0.35	40/60*20*0.25
= 5.333	= 4.666	= 3.333
= 5 (rounded down)	= 5 (rounded up)	= 3 (rounded down)

The above can be established in an Excel file.

Alternatively marks can be allocated according to the table on the next page.

Teacher Advice

Question	Outcome 1 (7)	Outcome 2 (8)	Outcome 3 (5)
Short Response			
1 a)	1		
b)		2	
2 a)	2		
b)	1		
c)	1		
3		3	
4 a)	2		
b)	2		
c)		3	
Multiple Choice			
1		1	
2			1
3			1
4			1
5			1
6			1
7		1	
8			1

Teacher Advice

Extended Response

1 a)		2	
b)	3		
c)	3		?
d)	4		
e)		3	
f)			2
2 a)	2		
b)		1	
c)		1	
d)			2
e)		3	
f)	3		1
g)		1	1
h)		3	

This QAT has been designed to meet the highest level in the performance descriptors provided by VCAA for each outcome in Unit 4 in the Assessment Handbook.

Solution Pathway

Short answer

Below are suggested responses. Teachers should consider the merits of alternative responses.

Question 1 (3 marks)

a) Approximate area = 2(2 + 1.9 + 2.2 + 3.2 + 0) using right end point

= 18. 60 u^2

(1 mark)

b) Neither. Because the terminals at x = 0 and x = 10 both have the same y-value the sum of the rectangles will be the same using both methods.

Using left endpoint gives 2(0 + 2 + 1.9 + 2.2 + 3.2) = 18.60 unit^2 as before. **(2 marks)**

Question 2

a) $\int_0^{10} 10x - x^2 dx = \left[5x^2 - \frac{x^3}{3}\right]_0^{10}$ **(1 mark)**

$= 500 - 333\frac{1}{3}$

$= 166\frac{2}{3}$ u^2 **(1 mark)**

b) 0+9+16+21+24+25+24+21+16+9 = 165 u^2 **(1 mark)**

c) $\frac{4/3}{165} \times 100 = 1\%$

Hence the approximation underestimates the area by 1%. **(1 mark)**

Question 3

For 1 Rectangle: use of either of the endpoints is necessary. For the graph shown, these are both significantly below the values of both of the local maximums and will greatly underestimate the area. **(1 mark)**

For 2 rectangles: Not only do the end points give significantly lower values but so does the central value where x = 2.5. **(1 mark)**

For 5 rectangles: 3 of the 5 rectangles will be significantly lower than the local maximums and hence the estimate will be a great deal lower than the actual area. **(1 mark)**

Solution Pathway

Question 4

a) $F(x) = \int_0^a x^3 - 6x^2 + 9x\,dx$

$$= \left[\frac{x^4}{4} - 2x^3 + \frac{9x^2}{2}\right]_0^a$$ **(1 mark)**

$$= \frac{a^4}{4} - 2a^3 + \frac{9a^2}{2}$$ **(1 mark)**

b) Average value $= \frac{1}{a-0}\left(\frac{x^4}{4} - 2x^3 + \frac{9x^2}{2}\right)$ **(1 mark)**

$$= \frac{x^4}{4a} - \frac{2x^3}{a} + \frac{9x^2}{2a}$$

$$= \frac{x^4}{4a} - \frac{8x^3}{4a} + \frac{18x^2}{4a}$$

$$= \frac{x^4 - 8x^3 + 18x^2}{4a}$$ (as required) **(1 mark)**

c) The average value of the curve is 2.

Therefore,

$$2 = \frac{a^4 - 8a^3 + 18a^2}{4a}$$ **(1 mark)**

$$2 = \frac{a^3 - 8a^2 + 18a}{4}$$

$$8 = a^3 - 8a^2 + 18a$$

$$0 = a^3 - 8a^2 + 18a - 8$$ **(1 mark)**

Using factor theorem and the factors of 8 (1, 2, 4, 8) finds

a = 4 units, **(1 mark)**

a = $2 - \sqrt{2}$ or $2 + \sqrt{2}$ are also valid answers to this part.

Solution Pathway

Multiple Choice

1) Answer is **D**.

Factorising gives $\sqrt{x\,(x-1)^2}$

Splitting the expression into parts gives $\sqrt{x}\,\sqrt{(x-1)^2}$

Now, remember the definition of absolute value is $|A| = \sqrt{A^2}$

So $\sqrt{x\,(x-1)^2} = \sqrt{x}\,|x-1|$

2) Answer is **A**.

Since the equation cannot be integrated in the normal fashion using the numeric integrator on the CAS is necessary, giving the result is 0.91046

$\int_0^2 \sqrt{x^3-2\cdot x^2+x}\, dx$	0.910457

3) Answer is **B**.

Differentiating gives $f'(x) = \frac{(3x^2-4x+1)}{2\sqrt{x^3-2x^2+x}}$

Let *f'(x)* = 0

Thus $0 = 3x^2 - 4x + 1$

Now solve for x using quadratic formula, factor theorem or a calculator to find x = 1/3

Substituting into *f(x)* gives y = 0.3849

$\text{fMax}\left(\sqrt{x^3-2\cdot x^2+x},x,0,1\right)$	$x=\frac{1}{3}$
$\sqrt{x^3-2\cdot x^2+x}\,\vert x=\frac{1}{3}$	$\frac{2\cdot\sqrt{3}}{9}$
$\sqrt{x^3-2\cdot x^2+x}\,\vert x=\frac{1}{3}$	0.3849

4) Answer is **D**.

Integrating gives $f(x) = (x+3)^{1/2} + C$

Substituting, x = 1 and *f(x)* = 1 gives C = 1

Now let x = 0 to find the y-intercept: (0, 0.73205)

5) Answer is **A**.

$\int_0^5 (2x)^{1/3}dx = \left[\frac{3(2x)^{4/3}}{8}\right]_0^5 = 8.0791 \text{ u}^2$

Solution Pathway

6)

Answer Is **E**.

$$\int_0^5 x^3 - 4x^2 + 2x + 1\,dx = \left[\frac{x^4}{4} - \frac{4x^3}{3} + x^2 + x\right]_0^5$$
$$= 19.58$$

7) Answer is **C**.

Evaluating an integral means that signed areas retain their sign. The section of f(x) between x = 0 and x = 4 is positive due to the modulus. In order to account for this A reverses terms, B changes the sign in front of the second integral, D retains the modulus and E reverses the positions of the terminals. Only C does not make any of the appropriate changes.

8) Answer is **E**.

Factorising the expression gives x(x – 1)(x – 2). This is symmetric over the interval x = 0 to x = 2. Thus the area is simply that of the semi-circle, ie 1.571 u^2

(8 x 1 = 8 marks)

Extended Response

Question 1

Two parabolas form an enclosed area. The equation of the first parabola is

$$f(x) = x^2 - 12x + 27$$

a) A reflection in the x-axis **(1 mark)**

followed by a translation of 7 units parallel to the y-axis in the positive direction. **(1 mark)**

b) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 7 \end{bmatrix}$ expands to give *x' = x* and *y' = -y + 7* **(1 mark)**

Rearranging gives *x = -x'* and *y = -y' + 7*

Substituting into f(x):

-y' + 7 = (-x') ^{2} *– 12x' + 27* **(1 mark)**

-y' = (x')$^{2-}$ *-12x' +20* **(1 mark)**

y = -x^2 *+12x – 20 (as required, removing ' symbols)*

Solution Pathway

c)

$f(x) = x^2 - 12x + 27$ and $g(x) = x^2 + 12x - 20$

Let *f(x) = g(x):* $x^2 - 12x + 27 = -x^2 + 12x - 20$

Rearranging gives: $2x^2 - 24x + 47 = 0$ **(1 mark)**

Now use the Quadratic Formula: a = 2, b = -24, c = 47

$$x = \frac{24 \pm \sqrt{24^2 - 4\times2\times47}}{2\times2}$$

$$x = \frac{24 \pm \sqrt{200}}{4} = 6 \pm \frac{5\sqrt{2}}{2}$$ **(1 mark)**

Now find the y values by substitution;

$f\left(6 + \frac{5\sqrt{2}}{2}\right) = 3.5$ and $f\left(6 - \frac{5\sqrt{2}}{2}\right) = 3.5$ **(1 mark)**

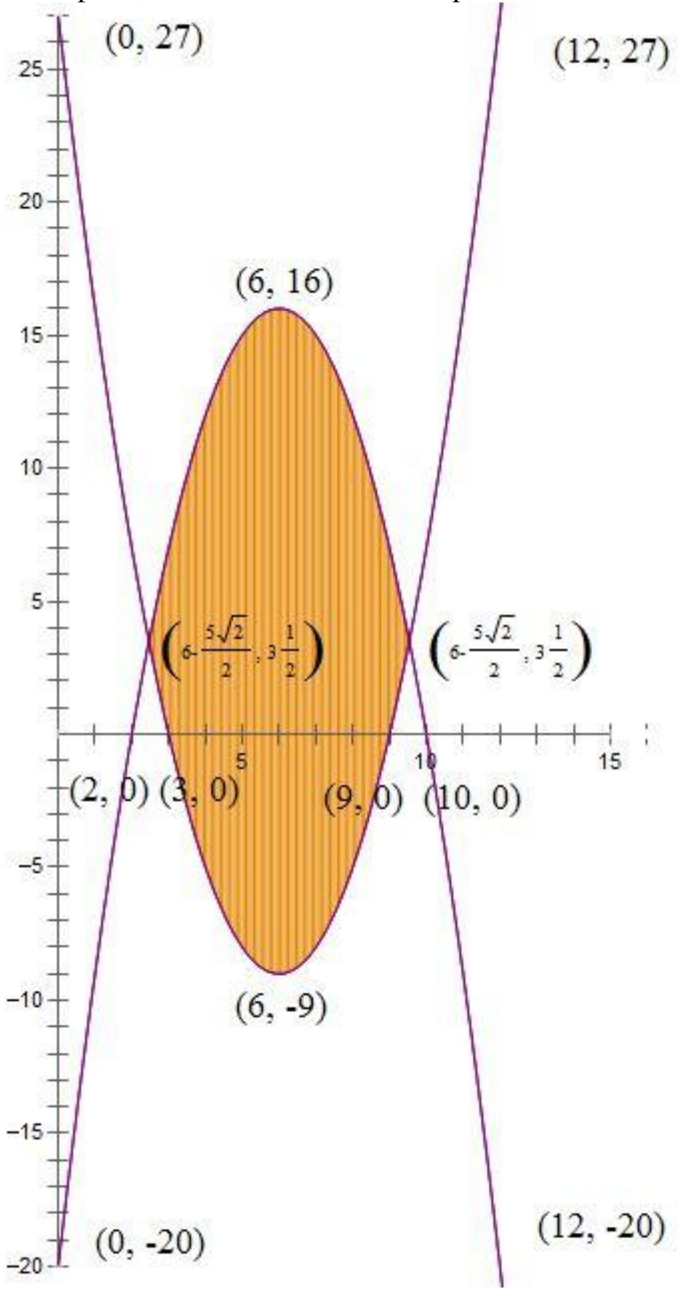
Alternatively students can use CAS to solve the function.

solve$(y=x^2-12\cdot x+27$ and $y=-x^2+12\cdot x-20, x, y)$

$x=2.46447$ and $y=3.5$ or $x=9.53553$ and $y=3.5$

Solution Pathway

d) Shapes and position – **1 mark**, Intersection points – **1 mark**, intercept points – **2 marks.**



e) Let $a = 6 - \frac{5\sqrt{2}}{2}$ and $b = 6 + \frac{5\sqrt{2}}{2}$ (this is not necessary for the solution but is included to make the values of the terminals more easily read here.)

$$\int_a^b g(x) - f(x)dx = \int_a^b -2x^2 + 24x - 47dx$$

Terminals correct – **1 mark**, Subtraction correct way around – **1 mark**, Final Expression correct – **1 mark**

Solution Pathway

f) $\int_a^b -2x^2 + 24x - 47dx = \left[\frac{-2x^3}{3} + 12x^2 - 47x\right]_a^b$ **(1 mark)**

$= 117.85 \text{ units}^2$ **(1 mark)**

Question 2

The velocity of a particle can be modelled by the equation:

$$v(t) = 3t^2 - 19t + 6$$

a) Let v(t) = 0 and factorise:

(3t – 1)(t – 6) **(1 mark)**

Hence when $t = \frac{1}{3}$ seconds, the particle is also stationary. **(1 mark)**

b) Acceleration is the derivative of velocity so:

$a(t) = \frac{d}{dt}(3t^2 - 19t + 6)$

a(t) = 6t - 19 **(1 mark)**

c) *When t = 0, a(0) = -19* ms^{-2} **(1 mark)**

d) Let a(t) = 0

0 = 6t – 19

19 = 6t

$t = \frac{19}{6}$ seconds **(1 mark)**

Hence, $v\left(\frac{19}{6}\right) = 3\left(\frac{19}{6}\right)^2 - 19\left(\frac{19}{6}\right) + 6$

$= -24.08 \text{ ms}^{-1}$ **(1 mark)**

Solution Pathway

e) Displacement is the integral of velocity so:

$$x(t) = \int 3t^2 - 19t + 6\, dt = t^3 - \frac{19t^2}{2} + 6t + C$$ **(1 mark)**

Now when *t = 10, x(10) = 100* so substitute to find C

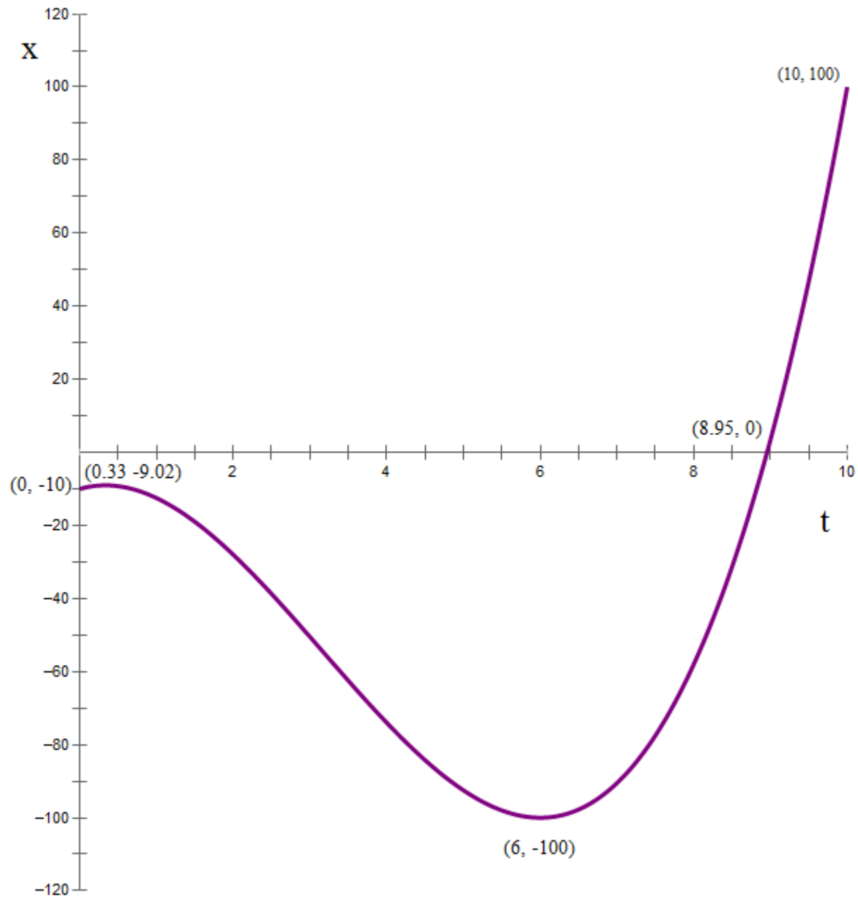
$$100 = (10)^3 - \frac{19(10)^2}{2} + 60 + C$$

$$\therefore\ C = 100 - (10)^3 + \frac{19(10)^2}{2} - 60$$

C = -10 **(1 mark)**

Hence $x(t) = t^3 - \frac{19t^2}{2} + 6t - 10$ **(1 mark)**

f) Shape – **1 mark**, End points – **1 mark**, Stationary points – **2 marks**



Solution Pathway

g) $0.98 + 90.98 + 200 = 291.96$ m

Correct values – **1 mark**, correct operations – **1 mark**

Note: distance is always a positive quantity, so to find distance travelled look at the positive difference between the end points and the stationary points.

h) The particle starts at a position 10 metres to the left of the origin and then moves toward the origin for the first 1/3 of a second. **(1 mark)**

For the next $5\frac{2}{3}$ seconds the particle travels away from the origin for a distance of 90.98 metres. **(1 mark)**

Over the last 4 seconds, the particle again travels in a forward direction through the origin and then past it until it reaches a point 100 metres to the right of the origin. This final leg is a distance of 200m. **(1 mark)**