Neap:

Trial Examination 2013

VCE Mathematical Methods (CAS) Units 3 & 4

Written Examination 2

Suggested Solutions

SECTION 1

1	A	B	**C**	D	E
2	A	B	C	D	**E**
3	A	B	C	D	**E**
4	**A**	B	C	D	E
5	A	B	**C**	D	E
6	A	**B**	C	D	E
7	A	B	C	**D**	E
8	**A**	B	C	D	E
9	A	**B**	C	D	E
10	A	B	**C**	D	E
11	**A**	B	C	D	E

12	**A**	B	C	D	E
13	A	B	**C**	D	E
14	A	B	C	**D**	E
15	A	B	**C**	D	E
16	**A**	B	C	D	E
17	A	B	C	**D**	E
18	A	B	**C**	D	E
19	A	B	C	**D**	E
20	**A**	B	C	D	E
21	**A**	B	C	D	E
22	A	B	**C**	D	E

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SECTION 1

Question 1 C

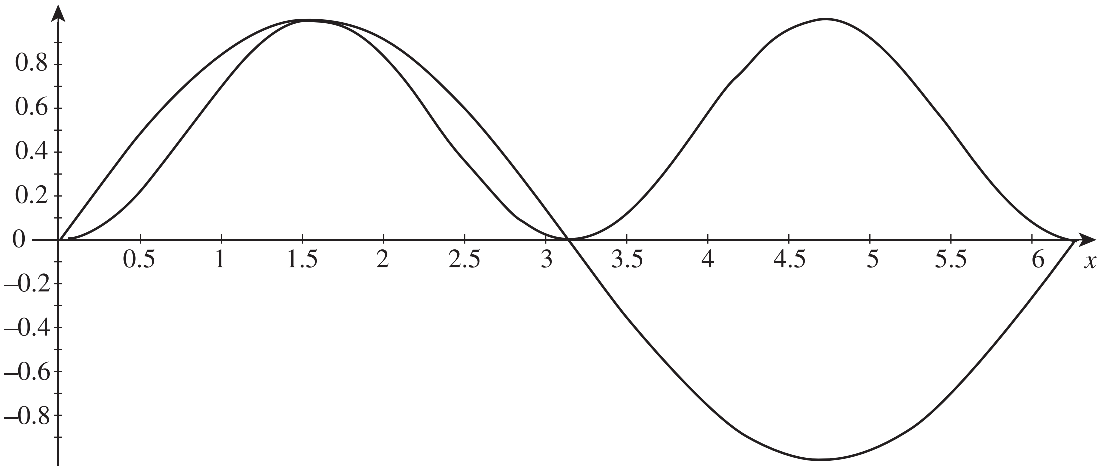
$$\sin(x) = \sin^2(x)$$

$$\sin(x)(1 - \sin(x)) = 0$$

$$\sin(x) = 0 \text{ or } 1$$

$x = 0, \pi, 2\pi$ or $\frac{\pi}{2}$, i.e. a total of 4 solutions.

Alternatively, a graph shows 4 intersections over the domain.



Question 2 E

We have $\int_{-2}^{4} f(x)dx = \int_{-2}^{3} f(x)dx + \int_{3}^{4} f(x)dx$

$$\text{So } a = \int_{-2}^{3} f(x)dx + b$$

$$a = -\int_{3}^{-2} f(x)dx + b$$

$$\int_{3}^{-2} f(x)dx = b - a$$

Question 3 E

$$f\left(-\frac{1}{3}\right) = \frac{1}{-\frac{1}{3}} = -3$$

$$f(-3) = \frac{1}{-3} + \frac{1}{2} = \frac{1}{6}$$

Question 4 A

If h is differentiable at $x = 1$, then

$h(1) = 1 + A + B = -1 + 8 + 4$

$A + B = 10$

also $h'(x) = \begin{cases} 2x + A & x < 1 \\ -2x + 8 & x \geq 1 \end{cases}$

$h'(1) = 2 + A = -2 + 8$

$A = 4$

Thus $B = 6$.

Question 5 C

Checking each function:

Inverse of $f(x) = x$ is clearly $f^{-1}(x) = x$.

$g(x) = \dfrac{4}{x}$ so inverse is given by $x = \dfrac{4}{y}$, i.e. $y = \dfrac{4}{x}$ so $g^{-1}(x) = \dfrac{4}{x}$.

$h(x) = \dfrac{x}{x-1}$ so inverse is given by $x = \dfrac{y}{y-1}$. CAS solve gives $y = \dfrac{x}{x-1}$ so $h^{-1}(x) = \dfrac{x}{x-1}$.

$j(x) = \dfrac{x-2}{x}$ so inverse is given by $x = \dfrac{y-2}{x}$. CAS solve gives $y = \dfrac{-2}{x-1} \neq j(x)$.

Question 6 B

The wheel has a diameter of 18 cm so $h_{max} = 18$ and $h_{min} = 0$.

The period of the function is 12 seconds so, for a sine or cosine function, $\dfrac{2\pi}{n} = 12 \Rightarrow n = \dfrac{\pi}{6}$

Now $t = 0$ corresponds to $h_{max} = 18$, which suggests a cosine function with amplitude 9 and vertical translation 9.

Thus $h(t) = 9 + 9\cos\left(\dfrac{\pi t}{6}\right)$.

As this is not an alternative given, use $\cos(x) = \sin\left(\dfrac{\pi}{2} - x\right)$.

Thus $h(t) = 9 + 9\sin\left(\dfrac{\pi}{2} - \dfrac{\pi t}{6}\right)$.

$h(t) = 9 + 9\sin\left(\dfrac{\pi}{6}(3-t)\right) = 9\left(1 + \sin\left(\dfrac{\pi}{6}(3-t)\right)\right)$.

Question 7 D

$$\frac{f}{g}(x) = \frac{\sqrt{x+9}}{x-6}$$

For f

$$x+9 \geq 0$$
$$x \geq -9$$

We must exclude $x = 6$ because we cannot divide by zero.

Thus $[-9, 6) \cup (6, \infty)$.

Question 8 A

Given $h(x) = g(f(x))$,

$$h'(x) = g'(f(x))f'(x)$$
$$h'(2) = g'(f(2))f'(2)$$
$$h'(2) = g'(1)f'(2) = (-3)(6) = -18$$

Question 9 B

$$\sqrt{x} + \sqrt{y} = 5$$

Solving for y by CAS gives $y = 25 - 10\sqrt{x} + x$.

Differentiating, $\frac{dy}{dx} = -\frac{5}{\sqrt{x}} + 1$.

At $x = 16$, gradient of tangent is $-\frac{5}{4} + 1 = -\frac{1}{4}$.

Equation of tangent:

$y - 1 = -\frac{1}{4}(x - 16)$ which has a y intercept of 5. Therefore $k = 5$.

Equation of normal:

$y - 1 = 4(x - 16)$ which has a y intercept of –63. Therefore $h = -63$.

$k - h = 68$

Question 10 C

The average rate of change of $f(x) = 3x^2 + 2x + k$ over the interval [0, 2] is given by

$$\frac{f(2) - f(0)}{2} = \frac{(12 + 4 + k) - k}{2} = 8$$

Thus $\frac{1}{2}\int_0^2 (3x^2 + 2x + k)\, dx = 8$

$$\left[x^3 + x^2 + kx\right]_0^2 = 16$$
$$8 + 4 + 2k = 16$$
$$k = 2$$

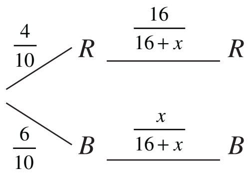
Question 11 A

Define the events R_i and B_i, such that R_i represents a red ball drawn from urn i and B_i represents a blue ball drawn from urn i, $i = 1, 2$

Let x be the number of blue balls in urn 2.

$$\frac{11}{25} = \Pr(R_1 \cap R_2) + \Pr(B_1 \cap B_2)$$
$$\frac{11}{25} = \Pr(R_1)\Pr(R_2) + \Pr(B_1)\Pr(B_2)$$

OR



$$\frac{11}{25} = \frac{4}{10}\left(\frac{16}{x+16}\right) + \frac{6}{10}\left(\frac{x}{x+16}\right)$$

Solving on CAS gives $x = 4$.

Question 12 A

Let the random variable X represent the number of successful first serves.

$X \sim Bi(n = 180,\ p = 0.65)$

$\mu = 180 \times 0.65 = 117$

$\sigma = \sqrt{180 \times 0.65 \times 0.35} = \dfrac{3\sqrt{455}}{10}$

Question 13 C

The initial state matrix is $S_O = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$

The win-lose probabilities can be tabulated:

$$\begin{array}{cc|cc} & & \multicolumn{2}{c}{\textit{Tomorrow}} \\ & & \textit{Win} & \textit{Lose} \\ \hline \textit{Today} & \textit{Win} & 0.80 & 0.25 \\ & \textit{Lose} & 0.20 & 0.75 \end{array}$$

Thus the transition matrix is $T = \begin{bmatrix} 0.8 & 0.25 \\ 0.2 & 0.75 \end{bmatrix}$

The probability that the team will win its fourth match equals $T^3 S_O = \begin{bmatrix} 0.8 & 0.25 \\ 0.2 & 0.75 \end{bmatrix}^3 \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$

Question 14 D

Let the increase in unit price per hat be $$x$.

Number of hats sold is $200-5x$.

Revenue from selling this number of hats is $(200-5x)(90+x)$.

Cost from manufacturer for this number of hats is $60(200-5x)$.

As profit = revenue – cost,

profit = $P(x)=(200-5x)(90+1x)-60(200-5x)$

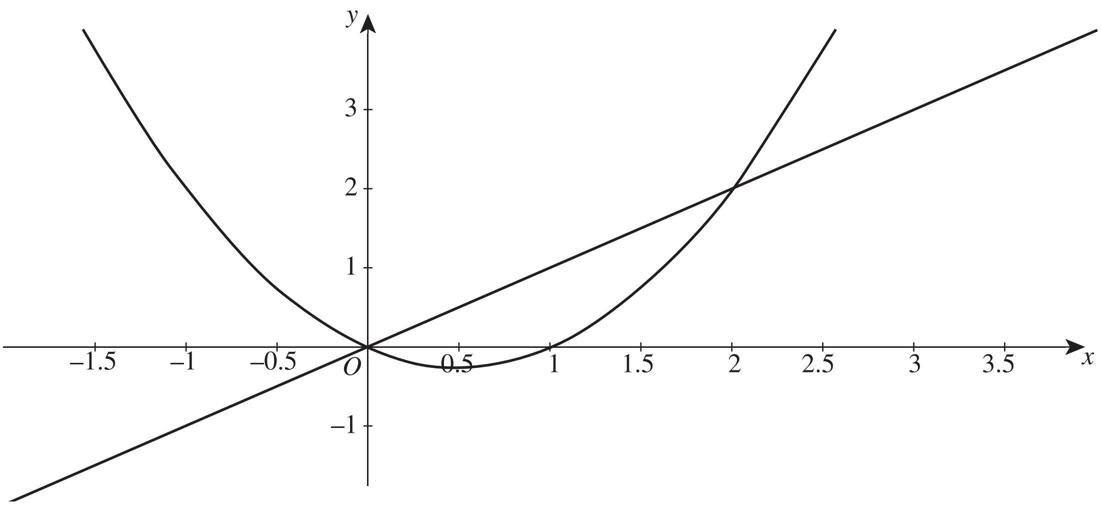
Simplifying on CAS gives $P(x)=-5x^2+50x+6000$.

Maximum of when $P(x)$ when $P'(x)=0$,

$$-10x+50=0$$
$$x=5$$

Number of hats sold is $200-5x=200-25=175$

Question 15 C



The graphs meet when $x=x^2-x$ $\quad x=0,\ 2$

Area bounded by the graphs equals $\int_0^2 x-(x^2-x)dx=\int_0^2(2x-x^2)dx=\frac{4}{3}$.

As $x = k$ divides the region in half, $\int_0^k(2x-x^2)dx=\frac{2}{3}$

$$\left[x^2-\frac{x^3}{3}\right]_0^k=\frac{2}{3}$$

$$k^2-\frac{k^3}{3}=\frac{2}{3}$$

Solving gives $k = 1$.

Question 16 A

The graph of the derivative needs to change from positive to negative within the domain.

This only occurs for the graph of *f*.

Question 17 D

Range of $f(x)-2$ will be $[-9,3]$.

So the range of $|f(x)-2|$ will be $[0,9]$ since the absolute value turns the negative results positive.

Finally, the range of $2|f(x)-2|+1$ equals $[1,19]$, by doubling the range and adding 1.

Question 18 C

$x^2+kx+k=0$

As $x=-\frac{1}{2}$ is a root, the equation can be written in factored form as $\left(x+\frac{1}{2}\right)(x+2k)=0$

Expanding gives $x^2+2kx+\frac{1}{2}x+k=0$

Equating coefficients of the *x* term

$2k+\frac{1}{2}=k \qquad k=-\frac{1}{2}$

Now the other root is $x=-2k=-2\times-\frac{1}{2}=1$

Alternatively, solving on CAS:

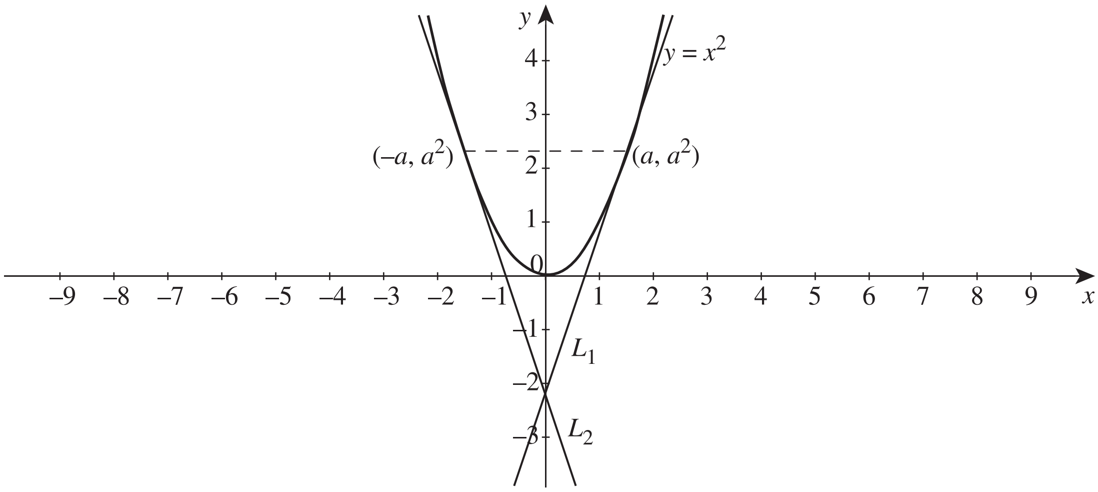
$\left(-\frac{1}{2}\right)^2+k\left(-\frac{1}{2}\right)+k=0$

gives $k=-\frac{1}{2}$

So $x^2-\frac{1}{2}x-\frac{1}{2}=0$.

Solving on CAS gives $x=-\frac{1}{2}$ or $x=1$

Question 19 D



Let the vertex of the triangle at the point of contact for L_1 have coordinates (a, a^2). In quadrant 2 the corresponding coordinates of the point of contact for L_2 will have coordinates $(-a, a^2)$.

Consider $L_1 : \dfrac{dy}{dx} = 2x = 2a$

But we know the triangle is equilateral so $m_1 = \tan(60^\circ) = \sqrt{3}$.

$$2a = \sqrt{3}$$

Thus $a = \dfrac{\sqrt{3}}{2}$

The length of each side of the triangle is $2a$.

Using the Sine rule for area formula (on formula sheet):

$$A = \frac{1}{2} \times (2a)(2a)\sin(60^\circ)$$

$$A = 2a^2 \frac{\sqrt{3}}{2} = \sqrt{3} \times \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3\sqrt{3}}{4}$$

Question 20 A

We require $\Pr(V < 2 \mid V \geq 1.5) = \dfrac{\Pr(1.5 \leq V \leq 2)}{\Pr(V \geq 1.5)}$.

Using CAS, compute $\dfrac{\int_{1.5}^{2} \frac{3}{v^4}\,dv}{1 - \int_{1}^{1.5} \frac{3}{v^4}\,dv}$ which gives $= 0.5781$.

Question 21 A

The gradient function $y = f'(x)$ has 4 x intercepts symmetrically placed either side of the y-axis. The function $f(x)$ has stationary points at those locations.

Both **A** and **C** satisfy this condition completely.

Also notice $f'(0)$ is undefined corresponding to the cusp on each of the graphs in **A** and **C**.

Notice that $f'(x) < 0$ for positive x values up to approximately 0.7. The gradient of a tangent to graph **A** is negative for these x values, but graph **C** has a positive gradient for these x values

Question 22 C

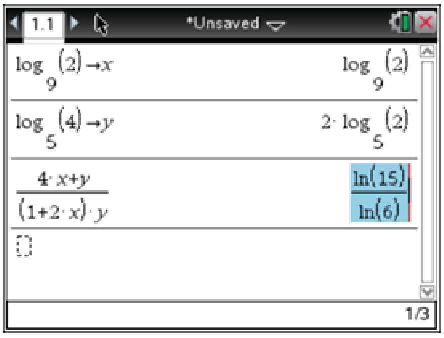
On CAS, define $x = \log_9(2)$ and $y = \log_5(4)$

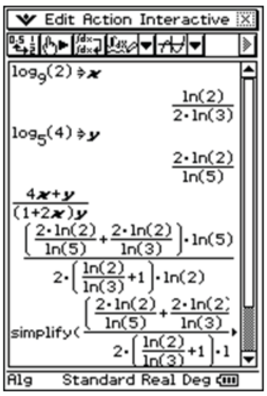
Check each alternative systematically.

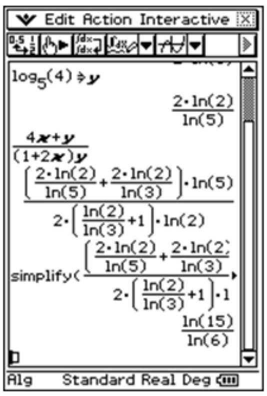
$$\frac{4x + y}{(1+2x)y} = \frac{\log_e(15)}{\log_e(6)}$$

By the change of base rule,

$$\frac{\log_e(15)}{\log_e(6)} = \log_6(15)$$







SECTION 2

Question 1 (15 marks)

a. $x = -\frac{3}{4}$ and $y = \frac{1}{2}$ represent the vertical and horizontal asymptotes respectively.

Thus $dom(f) = R \mid \left\{-\frac{3}{4}\right\}$. A1

The graph touches the x axis and is otherwise above it. We do not exclude $y = \frac{1}{2}$.

Thus $ran(f) = [0, \infty)$. A1

b. Given $\frac{1}{2}\left|1 - \frac{5}{4x+3}\right| = \left|\frac{ax+b}{cx+d}\right|$,

LHS $= \frac{1}{2}\left|\frac{4x+3-5}{4x+3}\right| = \frac{1}{2}\left|\frac{4x-2}{4x+3}\right| = \left|\frac{2x-1}{4x+3}\right|$. M1

This gives $a = 2$, $b = -1$, $c = 4$ and $d = 3$. A1

c. **i.** g must be a one-to-one function with range $[0, \infty)$.

$m = -\frac{3}{4}$, $n = \frac{1}{2}$ A1 A1

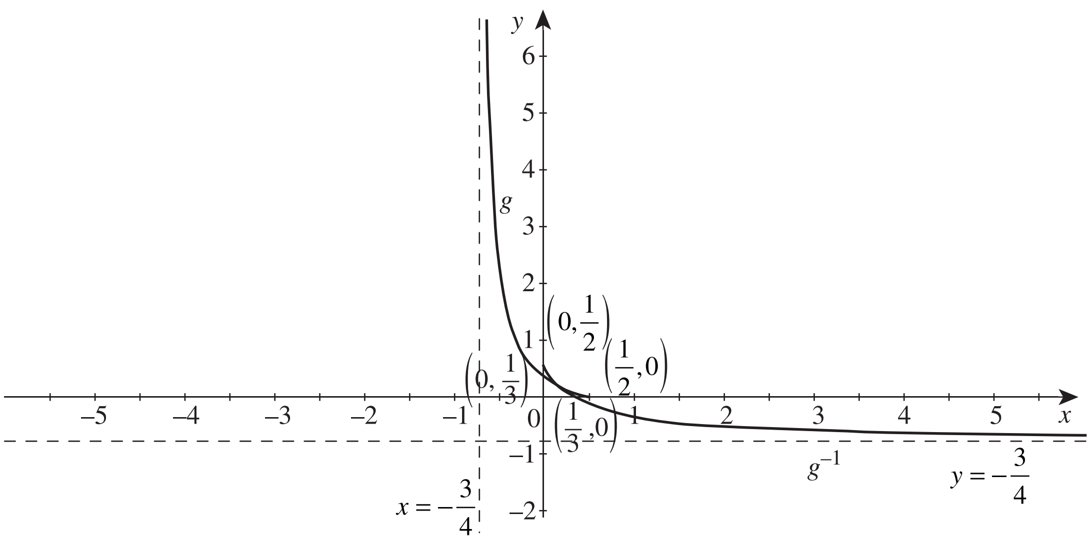
ii. For $x \in \left(-\frac{3}{4}, \frac{1}{2}\right]$, $g(x) = -\frac{2x-1}{4x+3} = \frac{1-2x}{4x+3}$.

The inverse is given by solving $x = \frac{1-2y}{4y+3}$. M1

Use CAS : $y = \frac{1-3x}{4x+2}$

Thus $g^{-1} : [0, \infty) \to R$, $g^{-1}(x) = \frac{1-3x}{4x+2}$. A1

iii.



Graph of g and g^{-1} A1
Correct intercepts A1
Correct *asymptotes* A1

iv. $g^{-1}(x)-g(x)=0$

Solve on CAS the equation $\dfrac{1-3x}{4x+2}=\dfrac{1-2x}{4x+3}$, giving $x=\dfrac{\sqrt{41}-5}{8}$. A1

d. For $x \le 0.5$, $f(x)=\dfrac{1-2x}{4x+3}$.

Using CAS, $f'(x)=\dfrac{-10}{(4x+3)^2}$. M1

Thus $f'(0)=-\dfrac{10}{9}$ and the equation of the tangent here is $y=-\dfrac{10}{9}x+\dfrac{1}{3}$. A1

Solving simultaneously on CAS:

$y=-\dfrac{10}{9}x+\dfrac{1}{3}$ and $y=\left|\dfrac{2x-1}{4x+3}\right|$ and gives intersection at (–1.25777, 1.73086)

This gives $p=-1.258,\ q=1.731$. A1

Question 2 (15 marks)

a. Using CAS, $y=\dfrac{x^3}{e^x}$ gives $\dfrac{dy}{dx}=\dfrac{(3x^2-x^3)}{e^x}$. A1

b. Stationary points occur when $\dfrac{dy}{dx}=\dfrac{(3x^2-x^3)}{e^x}=0$

$x^2(3-x)=0 \Rightarrow x=0,\ 3$ M1

Thus a maximum at $\left(3, \dfrac{27}{e^3}\right)$ and a stationary point of inflection at (0, 0). A1 A1

c. $\dfrac{x^3}{e^x} \le \dfrac{27}{e^3}$

As $e^3 > 0$, we rewrite the in-equation:

$x^3 e^{3-x} \le 27$ M1

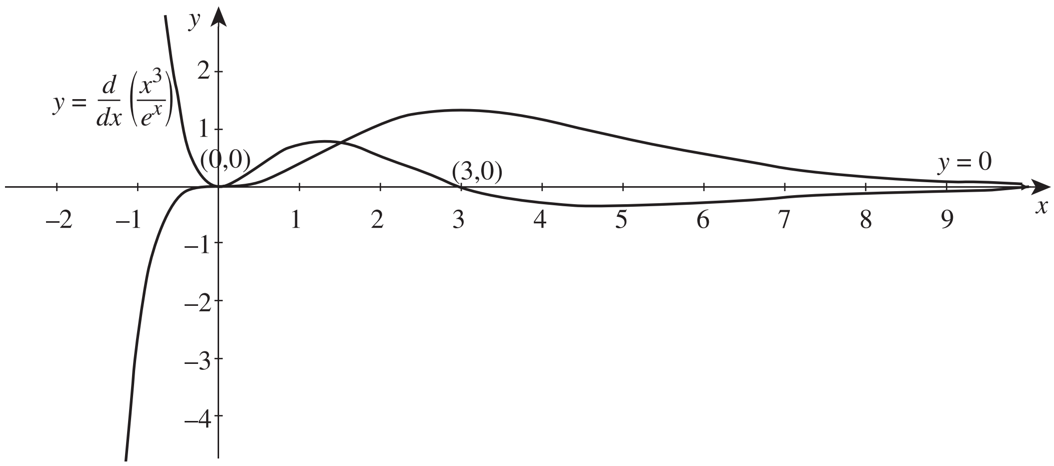
Taking logs of both sides:

$\log_e(x^3)+\log_e(e^{3-x}) \le \log_e(27)$ M1

$3\log_e(x)+3-x \le 3\log_e(3)$

Thus $3\log_e(x) \le x+3\log_e(3)-3$. A1

d. **i.**



Intercept coordinates and asymptote A1
Shape A1

ii. $\frac{dy}{dx} = \frac{(3x^2 - x^3)}{e^x}$

The maximum and minimum will occur when $\frac{d}{dx}\left(\frac{3x^2 - x^3}{e^x}\right) = 0$.

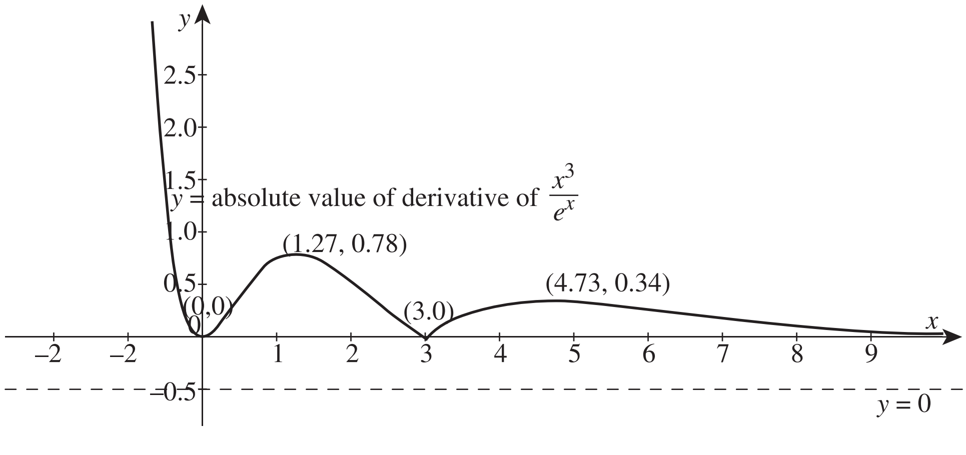
Thus $\frac{x(x^2 - 6x + 6)}{e^x} = 0$

$x = 3 \pm \sqrt{3}$ ($x = 0$ is also a solution.) M1

From graph, maximum occurs at $x = 3 - \sqrt{3}$, giving $\frac{dy}{dx} = 6(2\sqrt{3} - 3)e^{\sqrt{3}-3}$.

From graph, minimum occurs at $x = 3 + \sqrt{3}$, giving $\frac{dy}{dx} = -6(2\sqrt{3} + 3)e^{-\sqrt{3}-3}$. A1

iii. A function is strictly decreasing if for all, $a < b$, $f(a) > f(b)$.



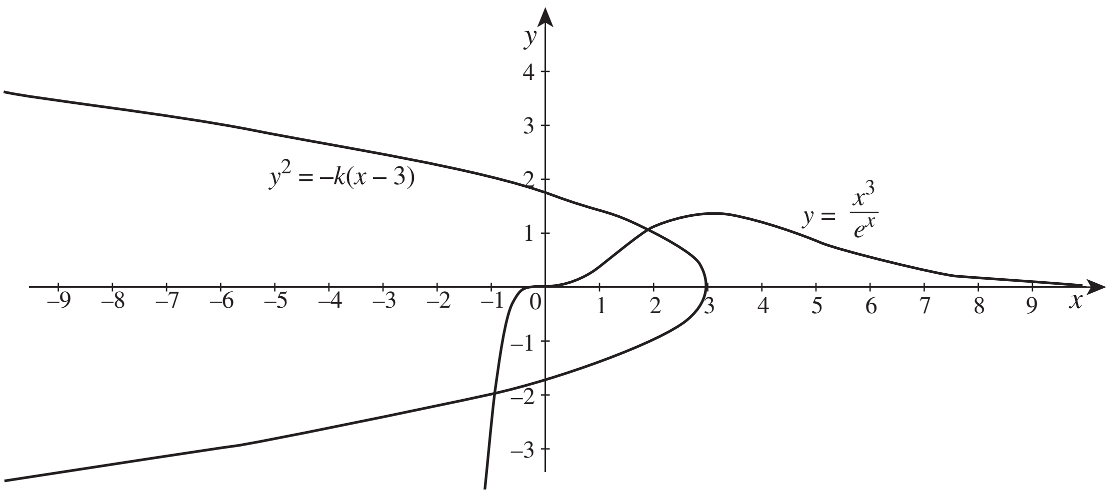
From graph, $[1.27, 3] \cup [4.73, \infty)$. A1 M1

e. Solving the equation $\frac{x^6}{e^{2x}} = k(x-3)$ is equivalent to solving $\left(\frac{x^3}{e^x}\right)^2 = k(x-3)$

i.e. $\left(\frac{x^3}{e^x}\right) = \pm\sqrt{k(x-3)}$. M1

The graph below illustrates, that for a negative k, 2 solutions are obtained.

Thus $k < 0$. A1



Question 3 (18 marks)

a. i. Let X represent the number of these enquiries which came through the phone.

$X \sim Bi(n = 100,\ p = 0.4)$

$E(X) = np = 100 \times 0.4 = 40$ A1

$\sigma_X = \sqrt{np(1-p)} = \sqrt{100 \times 0.4 \times 0.6} = 4.90$ A1

ii. Require $\Pr(X \geq 30) = 0.9852$. Using CAS the answer is directly obtained from binomialcdf:

binomCdf(100, 0.4, 30, 100) A1

b. i. For the eighth phone call to result in the first booking from phone enquiries on that day we need to have no bookings from the first seven phone calls, then a booking on the eighth call.

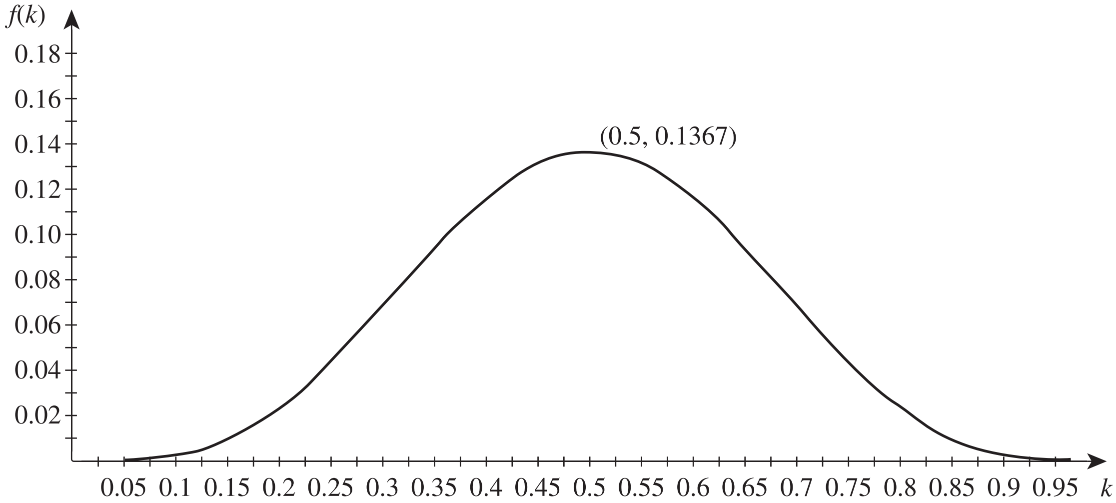
Thus required probability $= (1-k)^7 \times k$ or $k(1-k)^7$. A1

ii. Pr(eighth phone call results in fourth booking) = $k \times$ Pr(three bookings from seven phone calls) M1

$k \times {}^7C_3 k^3 (1-k)^4 = 35k^4(1-k)^4$ A1

iii. We need to locate the maximum of the function $f(k) = 35k^4(1-k)^4$

A graph sketch from a CAS shows the maximum turning point at (0.5, 0.1367) M1



The maximum probability occurs when $k = 0.5$ and equals 0.1367. A1

c. Pr (no booking) = 0.42

Thus $0.4(1-k) + 0.5(1-k^2) + 0.1(1-k^3) = 0.42$. M1

Solving on CAS and noting $0 < k < 1$, $k = 0.72$ A1

d. Pr(email via internet booking agency | a booking is made) = $\dfrac{\Pr(\text{internet} \cap \text{booking})}{\Pr(\text{booking})}$

Thus $0.25 = \dfrac{0.5k^2}{0.1k^3 + 0.4k + 0.5k^2}$. M1

Solving on CAS and noting $0 < k < 1$, $k = 0.27$. A1

e. **i.** We require 3 transitions to go from Sunday to Wednesday, i.e. $T^3 \times \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Thus $\begin{bmatrix} \frac{3}{5} & \frac{1}{3} \\ \frac{2}{5} & \frac{2}{3} \end{bmatrix}^3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{523}{1125} \\ \frac{602}{1125} \end{bmatrix}$. M1

So the required probability of dining in the restaurant on Wednesday night equals $\dfrac{523}{1125}$. A1

ii. Consider T^n for large n. For example $\begin{bmatrix} \frac{3}{5} & \frac{1}{3} \\ \frac{2}{5} & \frac{2}{3} \end{bmatrix}^{50} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.4545 \\ 0.5454 \end{bmatrix}$.

The percentage of nights the hotel can assume guests will dine in the hotel restaurant is 45%. A1

f. Define Y as the random variable "weight of lobster".

Then $Y \sim N(\mu, \sigma^2)$.

We are given that $\Pr(|Y-\mu| \le m) = 0.25$

$\Pr(-m \le Y-\mu \le m) = 0.25$ A1

Applying the Z transformation, $Z = \dfrac{Y-\mu}{\sigma}$ gives $\Pr(-\dfrac{m}{\sigma} \le Z \le \dfrac{m}{\sigma}) = 0.25$

Thus $\Pr(Z \le -\dfrac{m}{\sigma}) = 0.375$.

$-\dfrac{m}{\sigma} = invnorm(0.375) = -0.3186$ Thus $\dfrac{m}{\sigma} = 0.3186$ M1

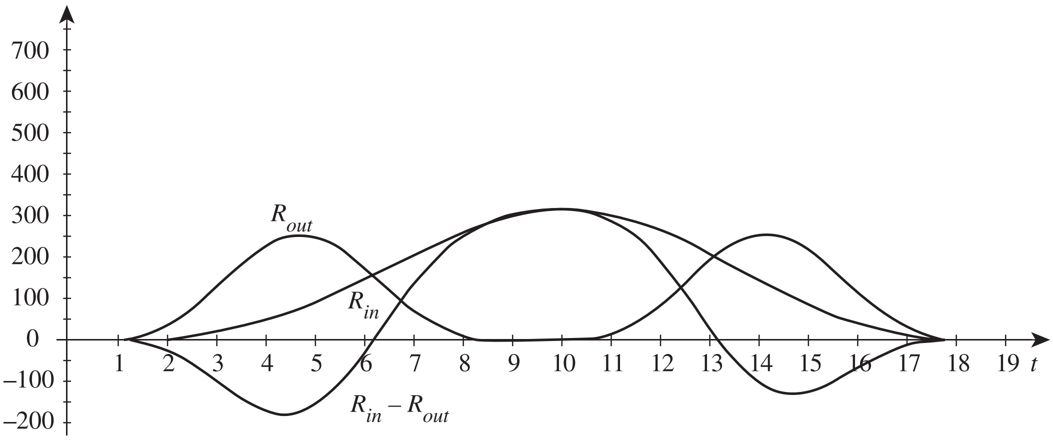
We require $\Pr(Y-\mu) \le 3m$.

This is equivalent to finding $\Pr(Z) \le \dfrac{3m}{\sigma}$, i.e. $\Pr(Z) \le 3 \times 0.3186$.

$= \Pr(Z \le 0.9559) = 0.8304$ A1

Question 4 (10 marks)

a.



Shape of graph A1
Critical points correctly located A1

b. Using the graph and CAS, we require

$y = R_{in} - R_{out}$ to be above axis. Intersection points occur at $t = 6.15095, 13.1152$, so between $t = 6.151$ and 13.115 A1

c. The tank contains the initial quantity plus an increase or decrease according to

$1200 + \int_0^{15} (R_{in} - R_{out})dt = 1200 + 804.71 = 2004.71$. M1

So 2005 litres. A1

d. The inflow rate and outflow rate are equal at $t = 6.15095,\ 13.1152$

At $t = 6.15095$,

volume = $12000+\int_0^{6.15095}(R_{in}-R_{out})\,dt = 1200-498.97 = 701.024$. A1

At $t = 13.1152$,

volume = $1200+\int_0^{13.1152}(R_{in}-R_{out})\,dt = 1200+984.516 = 2184.516\cdot$

At $t = 18$

volume = $1200+\int_0^{18}(R_{in}-R_{out})\,dt = 1200+655.2641 = 1855.2639$

Hence the absolute minimum quantity of liquid occurs at $t = 6.15$. A1

Using appropriate integrals to compute volume M1

e. At $t = 18$, there is a volume of 1855.26 litres remaining in the tank.

Thus $\int_{18}^{T} 250\sin^4\left(\frac{t}{6}\right)dt = 1855.26$. A1

Solving on CAS gives $T = 42.72$.

So tank is empty after 42 hours 43 minutes. A1