Year 2013

VCE Mathematical Methods CAS

Trial Examination 2

Suggested Solutions

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SECTION 1

ANSWERS

1	A	**B**	C	D	E
2	A	**B**	C	D	E
3	A	**B**	C	D	E
4	**A**	B	C	D	E
5	**A**	B	C	D	E
6	A	B	**C**	D	E
7	**A**	B	C	D	E
8	A	B	C	D	**E**
9	A	B	**C**	D	E
10	A	B	C	D	**E**
11	A	B	**C**	D	E
12	A	B	**C**	D	E
13	A	B	C	**D**	E
14	A	B	C	D	**E**
15	A	B	C	**D**	E
16	A	B	C	D	**E**
17	A	B	C	**D**	E
18	A	**B**	C	D	E
19	**A**	B	C	D	E
20	A	B	C	**D**	E
21	A	B	C	D	**E**
22	A	**B**	C	D	E

SECTION 1

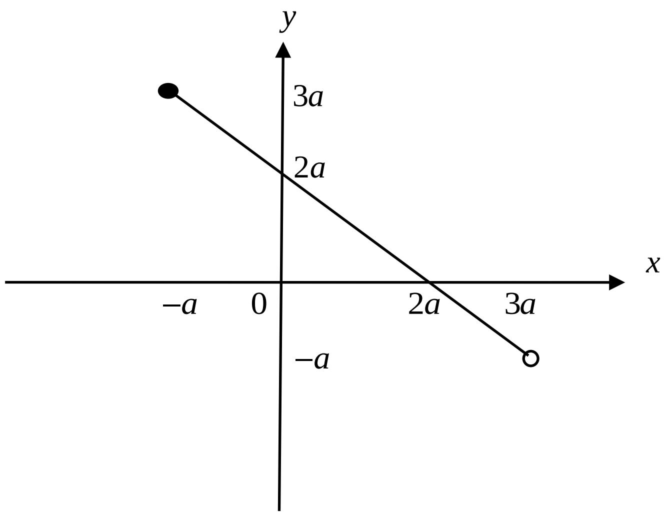
Question 1 **Answer B**

$f(x)=\sqrt{b^2-x^2}$

$f(0)=b \qquad f\left(\frac{b}{2}\right)=\sqrt{b^2-\frac{b^2}{4}}=\frac{\sqrt{3}\,b}{2}$

average rate of change $\quad \bar{f}=\dfrac{f\left(\frac{b}{2}\right)-f(0)}{\frac{b}{2}-0}=\dfrac{\frac{\sqrt{3}\,b}{2}-b}{\frac{b}{2}}=\dfrac{b\left(\frac{\sqrt{3}-2}{2}\right)}{\frac{b}{2}}=\sqrt{3}-2$

Question 2 **Answer B**



$f:[-a,3a)\to R\ ,\ f(x)=2a-x$

$f(3a)=-a$

$f(-a)=3a$

The range is $(-a,3a]$

Question 3 **Answer B**

The period $T=\dfrac{\pi}{n}=\dfrac{\pi}{\frac{b\pi}{3}}=\dfrac{3}{b}$

Question 4 **Answer A**

$f(x)=g(x)\log_e(h(x))$ using the product rule

$f'(x)=g'(x)\log_e(h(x))+\dfrac{g(x)h'(x)}{h(x)}$

$f'(2)=g'(2)\log_e(h(2))+\dfrac{g(2)h'(2)}{h(2)}$

Now $g(2)=3$, $g'(2)=4$, $h(2)=e^2$ and $h'(2)=2$

$f'(2)=4\times\log_e\left(e^2\right)+\dfrac{3\times 2}{e^2}=8+\dfrac{6}{e^2}$

Question 5 **Answer A**

$$\Delta = \begin{vmatrix} 1 & -k & 1 \\ 2 & -1 & k \\ 1 & 1 & 1 \end{vmatrix} = -k^2 + k + 2 = -(k-2)(k+1)$$

When $\Delta = 0$ or $k = 2$, $k = -1$ there is no unique solution.
When $k = -1$ there is no solution.
When $k = 2$ there is infinitely many solutions.

$$\det\left(\begin{bmatrix} 1 & -k & 1 \\ 2 & -1 & k \\ 1 & 1 & 1 \end{bmatrix}\right) \qquad -(k-2)\cdot(k+1)$$

$$\text{solve}\left(\det\left(\begin{bmatrix} 1 & -k & 1 \\ 2 & -1 & k \\ 1 & 1 & 1 \end{bmatrix}\right)=0,k\right) \qquad k=-1 \text{ or } k=2$$

$$eq1 := x-k\cdot y+z=14 \qquad x-k\cdot y+z=14$$

$$eq2 := 2\cdot x-y+k\cdot z=10 \qquad 2\cdot x-y+k\cdot z=10$$

$$eq3 := x+y+z=-2\cdot k \qquad x+y+z=-2\cdot k$$

$$\text{linSolve}\left(\begin{cases} eq1 \\ eq2 \\ eq3 \end{cases},\{x,y,z\}\right) \qquad \left\{-2\cdot(k+1), \frac{-2\cdot(k+7)}{k+1}, \frac{4\cdot(k+4)}{k+1}\right\}$$

$$\text{linSolve}\left(\begin{cases} eq1 \\ eq2 \\ eq3 \end{cases},\{x,y,z\}\right)|k=-1 \qquad \text{"No solution found"}$$

$$\text{linSolve}\left(\begin{cases} eq1 \\ eq2 \\ eq3 \end{cases},\{x,y,z\}\right)|k=2 \qquad \{-(\mathbf{c1}-2),-6,\mathbf{c1}\}$$

Question 6 **Answer C**

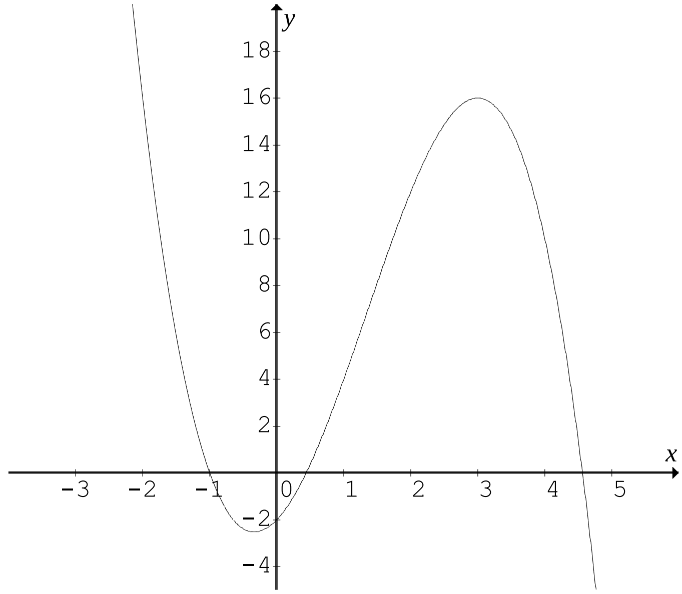
$$f(x) = \tan(x) \qquad f'(x) = \frac{1}{\cos^2(x)}$$

$$x = 60^0 = \frac{\pi}{3} \qquad h = 1^0 = \frac{\pi}{180}$$

$$f\left(\frac{\pi}{3}\right) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3} \qquad f'\left(\frac{\pi}{3}\right) = \frac{1}{\cos^2\left(\frac{\pi}{3}\right)} = \frac{1}{\left(\frac{1}{2}\right)^2} = 4$$

$$f(x+h) \approx f(x) + hf'(x)$$

$$\tan(61^0) = \sqrt{3} + \frac{\pi}{180} \times 4 = 1.8019$$



Question 7 **Answer A**

$f(x) = -x^3 + 4x^2 + 3x - 2$

$f'(x) = -3x^2 + 8x + 3$

$= -(3x^2 - 8x - 3)$

$= -(3x+1)(x-3)$

Turning points at $x = -\frac{1}{3}$ and $x = 3$

To restrict the domain to make f a one-one function we require $a \le -\frac{1}{3}$

Question 8 **Answer E**

$y' = 2 - 2\log_e(2x' + 2)$

$\frac{y'-2}{-2} = \log_e(2x'+2)$ $\quad y = \log_e(x)$

$\Rightarrow y = \frac{y'-2}{-2}$ and $x = 2x' + 2$ $\quad\Rightarrow\quad y' = 2 - 2y$ and $x' = \frac{x}{2} - 1$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Question 9 **Answer C**

$y = \cos(\pi b x)$ $\quad \frac{dy}{dx} = -\pi b \sin(\pi b x)$ when $x = \frac{1}{6b}$ $\quad m_T = -\pi b \sin\left(\frac{\pi}{6}\right) = -\frac{\pi b}{2}$

normal $m_N = \frac{2}{\pi b} = 4 \quad\Rightarrow\quad b = \frac{1}{2\pi}$

Question 10 **Answer E**

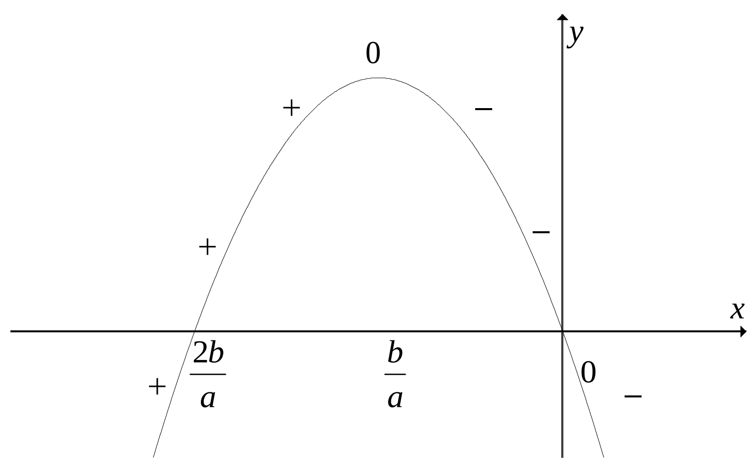
$f(x) = ax^2 - 2bx = x(ax - 2b)$, the graph crosses the x-axis at $x = 0$ and $x = \frac{2b}{a}$.

$f'(x) = 2ax - 2b$.

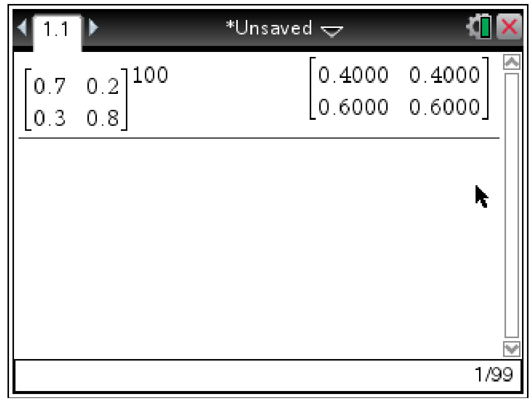
A decreasing function has a negative gradient, so that $f'(x) < 0 \quad\Rightarrow\quad 2ax - 2b < 0$.

$\Rightarrow \quad ax < b$ if $a < 0$ and $b > 0$

then $x > \frac{b}{a}$.



Question 11 **Answer C**



T catches the train and D drives to work

$T \to T = 0.7 \quad \Rightarrow \; T \to D = 0.3$

$D \to T = 0.2 \quad \Rightarrow \; D \to D = 0.8$

$$\begin{array}{cc} & \begin{array}{cc} T & D \end{array} \\ \begin{array}{c} T \\ D \end{array} & \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}^{100} = \begin{bmatrix} 0.4 & 0.4 \\ 0.6 & 0.6 \end{bmatrix} \end{array}$$

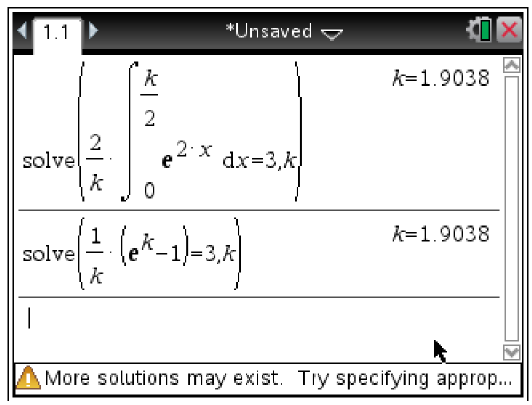
long-term drives is $0.6 = \dfrac{3}{5}$

Question 12 **Answer C**

$A = \dfrac{3\sqrt{3}}{2}x^2 \quad \Rightarrow \quad \dfrac{dA}{dx} = 3\sqrt{3}\,x \quad$ given $\dfrac{dA}{dt} = 36$

$\dfrac{dx}{dt} = \dfrac{dx}{dA}\dfrac{dA}{dt} = \dfrac{36}{3\sqrt{3}\,x} \qquad \left.\dfrac{dx}{dt}\right|_{x=2} = \dfrac{36}{3\sqrt{3}\times 2} = \dfrac{6}{\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}} = 2\sqrt{3}$ cm/min

Question 13 **Answer D**



$$\frac{1}{\frac{k}{2} - 0}\int_0^{\frac{k}{2}} e^{2x}\,dx = 3$$

$$\frac{2}{k}\left[\frac{1}{2}e^{2x}\right]_0^{\frac{k}{2}} = \frac{1}{k}\left(e^k - 1\right) = 3$$

solving for k gives, $k = 1.904$

Question 14 **Answer E**

$f(x) = |x|$ and $g(x) = x^2 \quad \Rightarrow \; g^{-1}(x) = \sqrt{x}$

$g^{-1}(x)\,g(x) = \sqrt{x} \times x^2 = x^{\frac{5}{2}} \neq f(x)$ Peter is incorrect

$g^{-1}(g(x)) = g^{-1}(x^2) = \sqrt{x^2} = |x| = f(x)$ Quentin is correct

$g(g^{-1}(x)) = g(\sqrt{x}) = (\sqrt{x})^2 = x \neq f(x)$ Sara is incorrect

$f(g(x)) = f(x^2) = |x^2| = x^2 = g(x)$ Tanya is correct

Question 15 **Answer D**

Total area under the curve is one $k\int_0^{\frac{\pi}{3}} \sin(3x)\,dx = 1$

$$k\left[-\frac{1}{3}\cos(3x)\right]_0^{\frac{\pi}{3}} = k\left[\left(-\frac{1}{3}\cos(\pi)+\frac{1}{3}\cos(0)\right)\right] = \frac{2k}{3} = 1 \;\Rightarrow\; k = \frac{3}{2}$$

The mean and median are both symmetrical at $x = \frac{\pi}{6}$

D. Is false $E\left(X^2\right) = \frac{\pi^2 - 4}{18}$

$$\text{solve}\left(k\cdot\int_0^{\frac{\pi}{3}} \sin(3\cdot x)\,dx = 1, k\right) \qquad k = \frac{3}{2}$$

$$\frac{3}{2}\cdot\int_0^{\frac{\pi}{3}} \left(x\cdot\sin(3\cdot x)\right)dx \qquad \frac{\pi}{6}$$

$$\frac{3}{2}\cdot\int_0^{\frac{\pi}{3}} \left(x^2\cdot\sin(3\cdot x)\right)dx \qquad \frac{\pi^2-4}{18}$$

$$\frac{3}{2}\cdot\int_0^{\frac{\pi}{4}} \sin(3\cdot x)\,dx \qquad \frac{\sqrt{2}+2}{4}$$

Question 16 **Answer E**

$$\Pr(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

$$\Pr(X = 0) = e^{-\lambda}\;,\;\Pr(X = 1) = \lambda e^{-\lambda}\;,\;\Pr(X = 2) = \frac{\lambda^2 e^{-\lambda}}{2!}$$

$$\Pr(X > 2) = \Pr(X \geq 3)$$

$$\Pr(X \geq 3) = 1 - \left[\Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2)\right]$$

$$\Pr(X \geq 3) = 1 - \left[e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2 e^{-\lambda}}{2!}\right]$$

$$\Pr(X \geq 3) = 1 - \frac{e^{-\lambda}}{2}\left(\lambda^2 + 2\lambda + 2\right)$$

Question 17 **Answer D**

$v(t)=\dfrac{27}{(3t+4)^2}$ m/s initial velocity $v(0)=\dfrac{27}{16}$ m/s **A.** is true.

$\displaystyle\int_0^2 \frac{27}{(3t+4)^2}\,dt=\frac{27}{20}$ distance travelled in the first two seconds, **B.** is true.

$a=\dfrac{dv}{dt}=\dfrac{d}{dt}\left[\dfrac{27}{(3t+4)^2}\right]=\dfrac{-162}{(3t+4)^3}$ m/s^2 acceleration **C.** is true.

$$x(t)=\int \frac{27}{(3t+4)^2}\,dt=\frac{-27}{3(3t+4)}+c=\frac{-9}{3t+4}+c$$

$$x(0)=0 \Rightarrow 0=-\frac{9}{4}+c \quad c=\frac{9}{4}$$

$x(t)=\dfrac{9}{4}-\dfrac{9}{3t+4}=\dfrac{9(3t+4)-9\times 4}{4(3t+4)}=\dfrac{27t}{4(3t+4)}$ position **E.** is true , **D**. **is false**

Define $v(t)=\dfrac{27}{(3\cdot t+4)^2}$	*Done*
$v(0)$	$\dfrac{27}{16}$
$\displaystyle\int_0^2 v(t)\,dt$	$\dfrac{27}{20}$
$\dfrac{d}{dt}(v(t))$	$\dfrac{-162}{(3\cdot t+4)^3}$
$\displaystyle\int_0^t v(t)\,dt$	$\dfrac{9}{4}-\dfrac{9}{3\cdot t+4}$
comDenom$\left(\dfrac{9}{4}-\dfrac{9}{3\cdot t+4}\right)$	$\dfrac{27\cdot t}{12\cdot t+16}$

Question 18 **Answer B**

A cubic with no turning points can be expressed in the form $f(x)=a(x+h)^3+k$

$$f(x)=a(x+h)^3+k=a\left(x^3+3x^2h+3xh^2+h^3\right)+k=ax^3+3ahx^2+3ah^2x+ah^3+k$$

$$f(x)=ax^3+bx^2+cx+d \quad \Rightarrow b=3ah \ , \ c=3ah^2 \ \text{and} \ d=ah^3+k$$

The point of inflexion is at $(-h,k)$ and $f'(x)=3ax^2+2bx+c=3a(x+h)^2$

$f'(x)=0$ when $x=-h$ and this equation has only one root, or for no turning points, no real factors, when $\Delta\le 0$, thus

$$\Delta=(2b)^2-4\times 3a\times c=4b^2-12ac=4\left(b^2-3ac\right)\le 0 \ \Rightarrow b^2\le 3ac$$

Question 19 **Answer A**

The area bounded by the graph of $y = f(x)$ between the *x*-values of $x = a$ and $x = b$, is the same if the area is translated *b* units to the right, that is the area bounded by the graph of $y = f(x-b)$ between the *x*-values of $x = a+b$ and $x = 2b$. This is also the same area if the area is translated *b* units to the left, that is the area bounded by the graph of $y = f(x+b)$ between the *x*-values of $x = a-b$ and $x = 0$, since $f(x) > 0$ for $x \in [a-b, 2b]$ where $a, b \in R$ and $b > a > 0$.

Question 20 **Answer D**

$X \stackrel{d}{=} N(\mu = 15, \sigma^2 = 16)$ $\sigma = 4$ $Z \stackrel{d}{=} N(\mu = 0, \sigma^2 = 1)$ $Z = \dfrac{x-\mu}{\sigma}$

X	7	11	15	19	23
Z	–2	–1	0	1	2

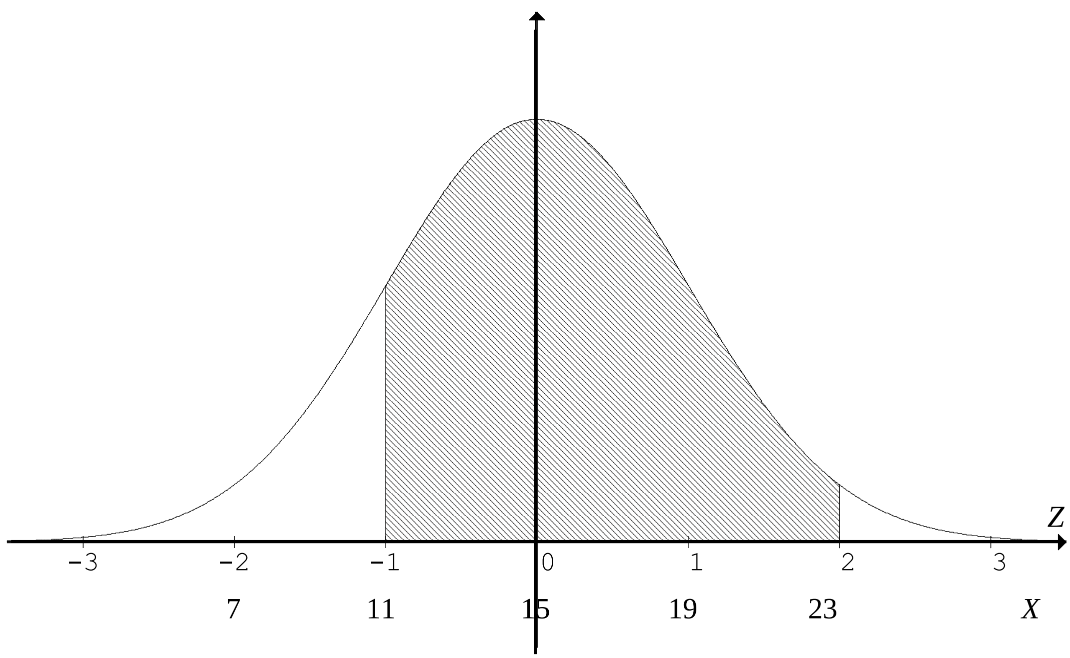
$\Pr(-1 < Z < 2) = 1 - [\Pr(Z > 2) + \Pr(Z < -1)]$ **A.** is true

$= \Pr(-1 < Z < 0) + \Pr(0 < Z < 2) = \Pr(0 < Z < 1) + \Pr(0 < Z < 2)$ **B.** is true

$= \Pr(11 < X < 23) = 1 - [\Pr(X > 23) + \Pr(X < 11)]$ **C.** is true

$= 1 - [\Pr(X < 7) + \Pr(X > 19)$ by symmetry, **E.** is true

D. is false, all of **A. B. C. E.** are true



Question 21 **Answer E**

Since it is a discrete random variable, the probabilities add to one, so that

$$a+\frac{a}{2}+\frac{a}{3}=\frac{11a}{6}=1 \Rightarrow a=\frac{6}{11}$$

$$E(X)=\sum x\Pr(X=x)=1\times a+2\times\frac{a}{2}+3\times\frac{a}{3}=3a=\frac{18}{11}$$

$$E(X^2)=\sum x^2\Pr(X=x)=1\times a+4\times\frac{a}{2}+9\times\frac{a}{3}=6a=\frac{36}{11}$$

$$\text{var}(X)=E(X^2)-(E(X))^2=\frac{36}{11}-\left(\frac{18}{11}\right)^2=\frac{72}{121}$$

All of **A. B. C.** and **D.** are true, **E.** is false

$$E\left(\frac{1}{X}\right)=\sum\frac{1}{x}\Pr(X=x)=1\times a+\frac{1}{2}\times\frac{a}{2}+\frac{1}{3}\times\frac{a}{3}=\frac{49}{36}a=\frac{49}{36}\times\frac{6}{11}=\frac{49}{66}$$

Question 22 **Answer B**

$$f(x)=\log_e(3x+4)$$

x	0	1	2	3	4
$f(x)$	$\log_e(4)$	$\log_e(7)$	$\log_e(10)$	$\log_e(13)$	$\log_e(16)$

consider four right rectangles, each of width one unit.

$$A_L=1\times\log_e(7)+1\times\log_e(10)+1\times\log_e(13)+1\times\log_e(16)$$

$$A_L=\log_e(7\times10\times13\times16)=\log_e(14560)$$

END OF SECTION 1 SUGGESTED ANSWERS

SECTION 2

Question 1

a.i. Let R pays by credit card, and C pays using cash

$$C \to C = 0.35 \quad \Rightarrow C \to R = 0.65 \text{ and } R \to R = 0.45 \quad \Rightarrow R \to C = 0.55$$

$$\Pr(\text{uses credit card three times and the first is credit card})$$

$$= \Pr(RRRC) + \Pr(RRCR) + \Pr(RCRR) \qquad \text{M1}$$

$$= 0.45^2 \times 0.55 + 0.45 \times 0.55 \times 0.65 + 0.55 \times 0.65 \times 0.45$$

$$= 0.433 \qquad \text{A1}$$

ii.

$$\Pr(\text{first and fourth are credit cards})$$

$$= \Pr(RCCR) + \Pr(RCRR) + \Pr(RRCR) + \Pr(RRRR)$$

$$= 0.55 \times 0.35 \times 0.65 + 0.55 \times 0.65 \times 0.45 + 0.45 \times 0.55 \times 0.65 + 0.45^3$$

$$= 0.538 \qquad \text{A1}$$

alternatively

$$\begin{array}{c} \\ C \\ R \end{array} \begin{array}{c} \begin{array}{cc} C & R \end{array} \\ \begin{bmatrix} 0.35 & 0.55 \\ 0.65 & 0.45 \end{bmatrix}^3 \end{array} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.462 \\ 0.538 \end{bmatrix}$$

fourth is credit card 0.538

b.i.

$$X \overset{d}{=} Bi(n = 16, p = ?)$$

$$\Pr(X = 8) + \Pr(X = 9) = 0.25$$

$$\binom{16}{8} p^8 (1-p)^8 + \binom{16}{9} p^9 (1-p)^7 = 0.25 \quad \text{by CAS} \qquad \text{M1}$$

$$1430 p^8 (p-9)(p-1)^7 = 0.25$$

solving with $0 < p < 1$ gives $p = 0.4132$ or 0.6473

but since $E(X) > 8 \ \Rightarrow \ p > 0.5$ so $p = 0.6473$ A1

ii.

$$X \overset{d}{=} Bi(n = 16, p = 0.65)$$

$$\Pr(X > 8) = \Pr(X \geq 9) = 0.8406 \qquad \text{A1}$$

nCr(16,8)· p^8· $(1-p)^8$+nCr(16,9)· p^9· $(1-p)^7$	1430· p^8· $(p-9)$· $(p-1)^7$
solve(1430· p^8· $(p-9)$· $(p-1)^7$=0.25,p)|0<p<▸	◂=0.4132 or p=0.6473
binomCdf(16,0.65,9,16)	0.8406

c. X is the time in hours spent shopping, $X \overset{d}{=} N\left(\mu = ?, \sigma^2 = ?\right)$

$(1) \quad \Pr(X > 3700) = 0.14$

$(2) \quad \Pr(X < 2990) = 0.26$

$(1) \;\Rightarrow\; \dfrac{3700-\mu}{\sigma} = 1.0803$ M1

$(2) \;\Rightarrow\; \dfrac{2990-\mu}{\sigma} = -0.6433$ A1

$(1) \quad 3700 - \mu = 1.0803\sigma$

$(2) \quad 2990 - \mu = -0.6433\sigma$

now subtract equations $(1)-(2)$ solving gives

$\sigma = \$412$ substituting gives $\mu = \$3,255$ A1

$eq1 := \dfrac{3700-m}{s} = \text{invNorm}(0.86,0,1)$	$\dfrac{3700-m}{s} = 1.0803$
$eq2 := \dfrac{2990-m}{s} = \text{invNorm}(0.26,0,1)$	$\dfrac{2990-m}{s} = -0.6433$
$\text{solve}\left(\begin{cases} eq1 \\ eq2 \end{cases}, \{m,s\}\right)$	$s=411.9130$ and $m=3255.002$

d.i. Since the function is continuous at $t=3$ $\; a e^{1} = b\sin\left(\dfrac{3\pi}{6}\right) = b \quad \Rightarrow b = ae$ A1

$$\int_0^3 a e^{\frac{t}{3}}\,dt + \int_3^6 b\sin\left(\frac{\pi t}{6}\right)dt = 1$$

Since the total area under the curve is equal to one, substituting $b = ae$

$$a\left[\int_0^3 e^{\frac{t}{3}}\,dt + \int_3^6 e\sin\left(\frac{\pi t}{6}\right)dt\right] = 1$$

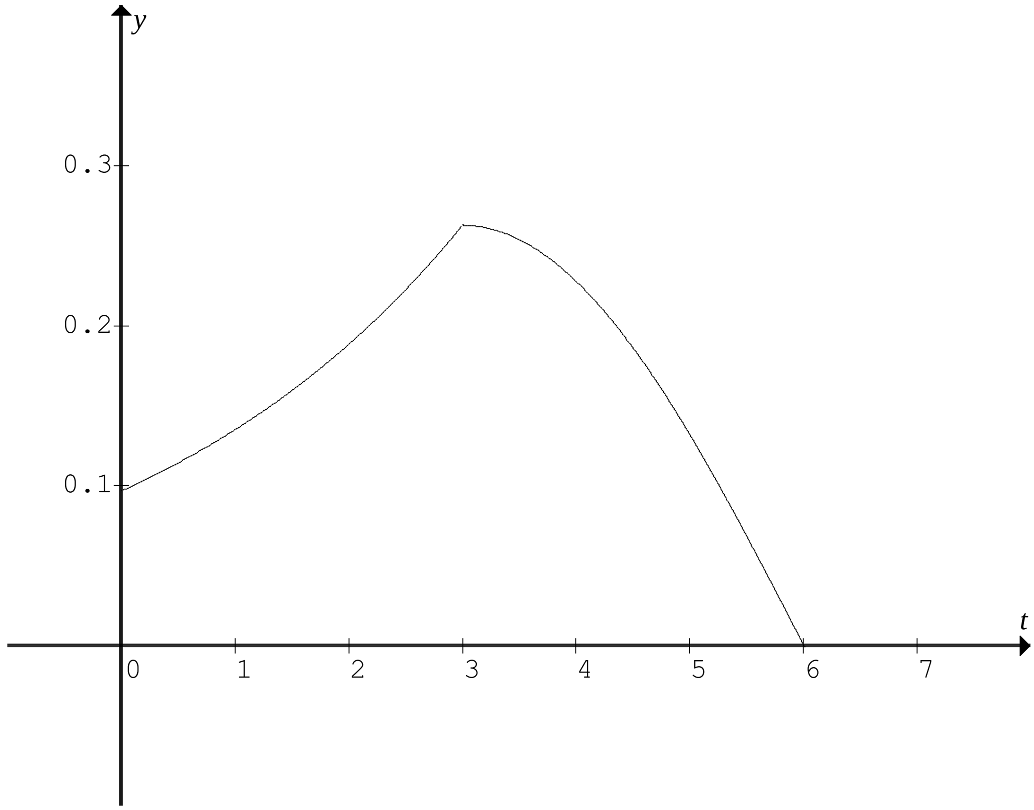
$$a\left(\left[3e^{\frac{t}{3}}\right]_0^3 - \left[\frac{6e}{\pi}\cos\left(\frac{\pi t}{6}\right)\right]_3^6\right) = 1$$ A1

$$a\left(3(e-1) - \frac{6e}{\pi}\left(\cos(\pi) - \cos\left(\frac{\pi}{2}\right)\right)\right) = a\left(3(e-1) + \frac{6e}{\pi}\right) = 1$$

$$a\left(\frac{3\pi(e-1)+6e}{\pi}\right) = 1 \quad \Rightarrow \quad a = \frac{\pi}{3\pi e + 6e - 3\pi}$$

$$a = \frac{\pi}{3\left(e(\pi+2)-\pi\right)}$$

ii. Graph passes through $(0,0.097)$ $(3,0.26)$ $(6,0)$ and is continuous and zero elsewhere G2



iii. $\Pr(X>4)=b\int_4^6 \sin\left(\frac{\pi t}{6}\right)dt$

$=\left[-\frac{6b}{\pi}\cos\left(\frac{\pi t}{6}\right)\right]_4^6=\left(-\frac{6b}{\pi}\left(\cos(\pi)-\cos\left(\frac{2\pi}{3}\right)\right)\right)$ with $b=\frac{\pi e}{3(e(\pi+2)-\pi)}$

$=\frac{e}{e(\pi+2)-\pi}$ A1

iv. $\Pr(T<2|T<3)=\frac{\Pr(T<2)}{\Pr(T<3)}$ M1

$=\frac{\int_0^2 a e^{\frac{t}{3}}\,dt}{\int_0^3 a e^{\frac{t}{3}}\,dt}=\frac{\left[3e^{\frac{t}{3}}\right]_0^2}{\left[3e^{\frac{t}{3}}\right]_0^3}$

$=\frac{e^{\frac{2}{3}}-1}{e-1}$ A1

v. $E(T) = a\int_0^3 t\,e^{\frac{t}{3}}\,dt + \int_3^6 et\sin\left(\frac{\pi t}{6}\right)dt$ with $a = \frac{\pi}{3\left(e(\pi+2)-\pi\right)}$ M1

mean time $E(T) = 2.92$ hours A1

Define $f(t)=\begin{cases} a\cdot e^{\frac{t}{3}}, & 0\le t\le 3 \\ b\cdot \sin\left(\frac{\pi\cdot t}{6}\right), & 3\le t\le 6\end{cases}$	*Done*
$f(3)$	$a\cdot e$
$\text{solve}\left(\int_0^6 f(t)\,dt=1,a\right)\Big\vert b=a\cdot e$	$a=\frac{\pi}{3\cdot(e\cdot(\pi+2)-\pi)}$
$a:=\frac{\pi}{3\cdot(e\cdot(\pi+2)-\pi)}$	$\frac{\pi}{3\cdot(e\cdot(\pi+2)-\pi)}$
$b:=a\cdot e$	$\frac{e\cdot\pi}{3\cdot(e\cdot(\pi+2)-\pi)}$
Define $f(t)=\begin{cases} a\cdot e^{\frac{t}{3}}, & 0\le t\le 3 \\ b\cdot \sin\left(\frac{\pi\cdot t}{6}\right), & 3\le t\le 6\end{cases}$	*Done*

$\int_4^6 f(t)\,dt$	$\frac{e}{e\cdot(\pi+2)-\pi}$
$\frac{\int_0^2 f(t)\,dt}{\int_0^3 f(t)\,dt}$	$\frac{e^{\frac{2}{3}}-1}{e-1}$
$\int_0^6 (t\cdot f(t))\,dt$	2.9222

Question 2

i. $f(x)=x^3+bx^2+cx+6$

$f'(x)=3x^2+2bx+c$ A1

ii. $f(2)=(2)^3+b(2)^2+2c+6=8+4b+2c+6=4b+2c+14$

$f'(2)=3(2)^2+4b+c=4b+c+12$ M1

Equation of the tangent at P

$y-(4b+2c+14)=(4b+c+12)(x-2)$

$tp(x)=y=(4b+c+12)x-2(4b+c+12)+(4b+2c+14)$

$tp(x)=y=(4b+c+12)x-4b-10$ A1

iii. Solving $tp(x)=f(x)$ when $x=-1$ $tp(-1)=6=f(-1)$

$tp(-1)=-(4b+c+12)-4b-10=-8b-c-22=6$

$\Rightarrow (1)\quad -8b-c=28$ M1

$f(-1)=6 \Rightarrow -1+b-c+6=6$

$\Rightarrow (2)\quad b-c=1$ A1

solving these simultaneous equations,

$(2)-(1) \Rightarrow 9b=-27 \quad \Rightarrow b=-3$ and $c=-4$ $P(2,-6)$

iv. so substitute $b=-3$ and $c=-4$

$f(x)=x^3-3x^2-4x+6$ and $tp(x)=-4x+2$ A1

$$A_1=\int_{-1}^{2}\left(f(x)-tp(x)\right)dx$$

$$A_1=\int_{-1}^{2}\left(x^3-3x^2+4\right)dx$$ A1

v. Equation of the tangent at Q

$f(-1)=6$ and $f'(x)=3x^2-6x-4 \Rightarrow f'(-1)=3+6-4=5$ M1

$y-6=5(x+1)$

$tq(x)=y=5x+11$ A1

vi. $tq(x)=f(x)$

$5x+11=x^3-3x^2-4x+6$ solving gives $\Rightarrow x=5$ or $x=-1$

$f(5)=(5)^3-3(5)^2-20+6=36$ or $tq(5)=25+11=36$

$R(5,36)$ A1

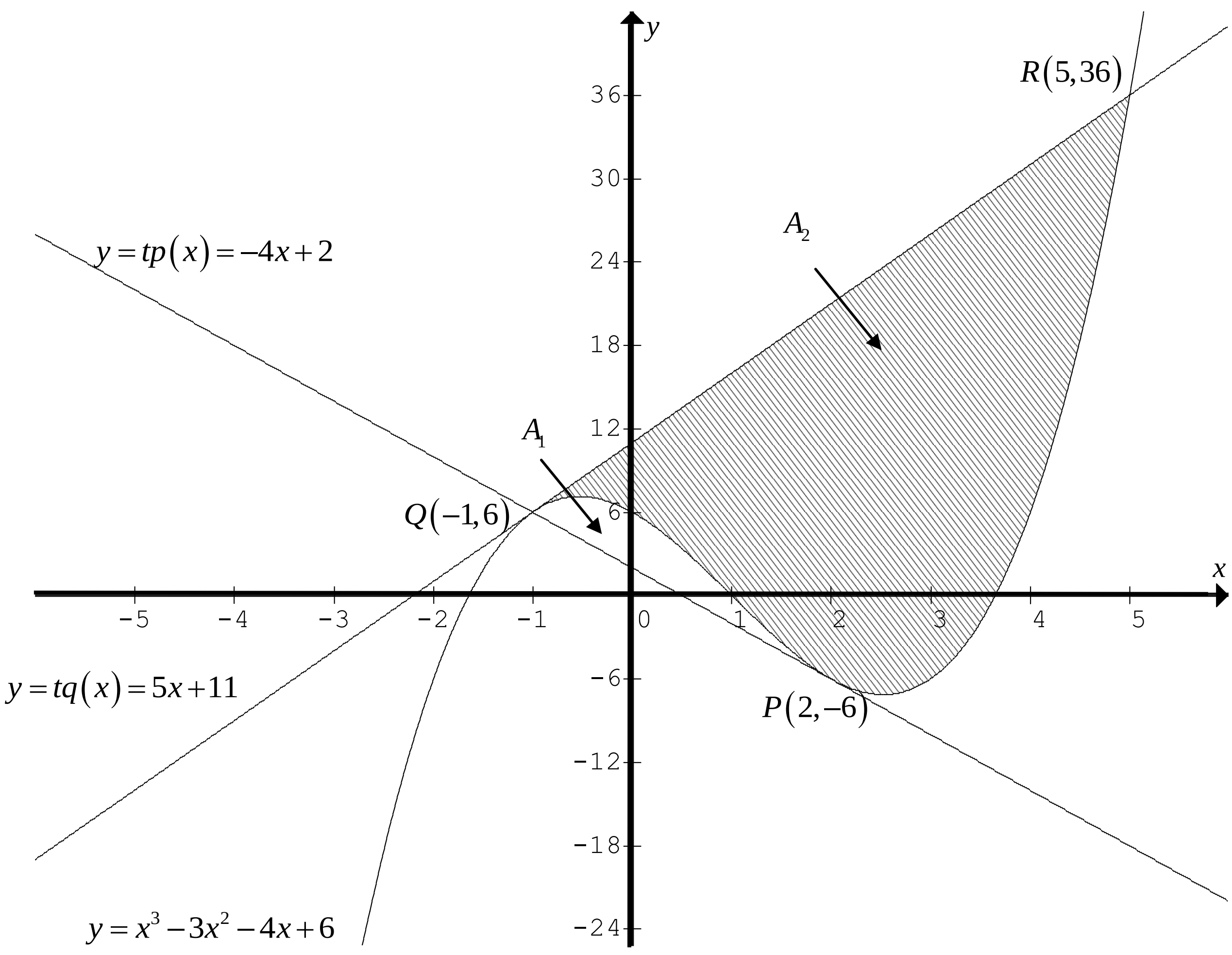
vii. $A_2 = \int_{-1}^{5} \left(tq(x) - f(x)\right) dx$

$A_2 = \int_{-1}^{5} \left(-x^3 + 3x^2 + 9x + 5\right) dx$ A1

viii. $A_1 = \frac{27}{4} \quad A_2 = 108$

$\frac{A_2}{A_1} = 16$ A1

ix. correct graphs, tangents at points P, and Q G2



Define $f(x)=x^3+b\cdot x^2+c\cdot x+6$	*Done*
Define $df(x)=\frac{d}{dx}(f(x))$	*Done*
$f(2)$	$4\cdot b+2\cdot c+14$
$df(2)$	$4\cdot b+c+12$
solve$(y-f(2)=df(2)\cdot(x-2),y)$	$y=(4\cdot b+c+12)\cdot x-4\cdot b-10$
Define $tp(x)=(4\cdot b+c+12)\cdot x-4\cdot b-10$	*Done*
solve$(tp(x)=f(x),x)$	$x=-(b+4)$ or $x=2$
solve$(-(b+4)=-1$ and $f(-1)=6,\{b,c\})$	$b=-3$ and $c=-4$
Define $f(x)=x^3+b\cdot x^2+c\cdot x+6 \mid b=-3$ and $c=-4$	*Done*
Define $df(x)=\frac{d}{dx}(f(x))$	*Done*
Define $tp(x)=(4\cdot b+c+12)\cdot x-4\cdot b-10 \mid b=-3$ and $c=-4$	*Done*
$f(x)-tp(x)$	$x^3-3\cdot x^2+4$
$\int_{-1}^{2}(f(x)-tp(x))dx$	$\frac{27}{4}$

solve$(y-f(-1)=df(-1)\cdot(x+1),y)$	$y=5\cdot x+11$
Define $tq(x)=5\cdot x+11$	*Done*
solve$(tq(x)=f(x),x)$	$x=-1$ or $x=5$
$tq(x)-f(x)$	$-x^3+3\cdot x^2+9\cdot x+5$
$tq(5)$	36
$f(5)$	36
$tq(x)-f(x)$	$-x^3+3\cdot x^2+9\cdot x+5$
$\int_{-1}^{5}(tq(x)-f(x))dx$	108
$\frac{108}{\frac{27}{4}}$	16

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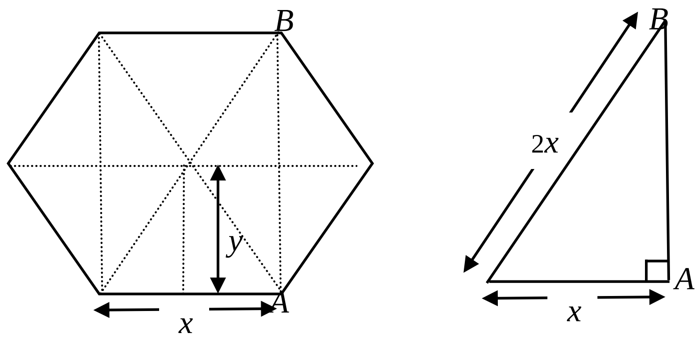
Question 3

a. Consider the hexagonal base of the vase



M1

$d(AB)=\sqrt{3}\,x \quad y=\frac{1}{2}d(AB)=\frac{\sqrt{3}\,x}{2}$

One triangle has area $\frac{1}{2}x\,y=\frac{1}{2}x\frac{\sqrt{3}\,x}{2}=\frac{\sqrt{3}}{4}x^2$

or area of one equilateral triangle $\frac{1}{2}x^2\sin\left(60^0\right)=\frac{1}{2}x^2\times\frac{\sqrt{3}}{2}=\frac{\sqrt{3}}{4}x^2$

Area of the base is six triangles $6\times\frac{\sqrt{3}}{4}x^2=\frac{3\sqrt{3}}{2}x^2$ A1

Total surface area is the area of the base and six rectangles of length x and height h.

$S=\frac{3\sqrt{3}}{2}x^2+6xh$

b. Volume is area of the base multiplied by the height

$V=\frac{3\sqrt{3}}{2}x^2h$ A1

c.i. $S=\frac{3\sqrt{3}}{2}x^2+6xh \Rightarrow 6xh=S-\frac{3\sqrt{3}}{2}x^2=\frac{2S-3\sqrt{3}\,x^2}{2}$

$h=\frac{2S-3\sqrt{3}x^2}{12x}$ M1

$V(x)=\frac{3\sqrt{3}}{2}x^2h=\frac{3\sqrt{3}}{2}x^2\left(\frac{2S-3\sqrt{3}x^2}{12x}\right)$

$V(x)=\frac{\sqrt{3}}{8}\left(2S\,x-3\sqrt{3}\,x^3\right)$

M1

$V(x)=\frac{\sqrt{3}\,x}{8}\left(2S-3\sqrt{3}\,x^2\right)>0 \Rightarrow 2S-3\sqrt{3}\,x^2>0 \quad 2S>3\sqrt{3}\,x^2$ and $x>0$

so that $0<x<\sqrt{\frac{2S}{3\sqrt{3}}}$ or $0<x<\frac{\sqrt{2\sqrt{3}\,S}}{3}$ A1

ii. $\dfrac{dV}{dx}=\dfrac{\sqrt{3}}{8}\left(2S-9\sqrt{3}x^2\right)$ A1

for a maximum volume $\dfrac{dV}{dx}=0 \;\Rightarrow\; 2S=9\sqrt{3}x^2$

$x=\sqrt{\dfrac{2S}{9\sqrt{3}}}=\dfrac{1}{3}\sqrt{\dfrac{2S}{\sqrt{3}}}$ A1

d.i. $V=\dfrac{3\sqrt{3}}{2}x^2h \quad\Rightarrow\quad h=\dfrac{2V}{3\sqrt{3}\,x^2}$ A1

$S=\dfrac{3\sqrt{3}}{2}x^2+6xh$

$S=\dfrac{3\sqrt{3}}{2}x^2+6x\left(\dfrac{2V}{3\sqrt{3}\,x^2}\right)$

$S(x)=\dfrac{3\sqrt{3}}{2}x^2+\dfrac{4V}{\sqrt{3}\,x}=\dfrac{3\sqrt{3}}{2}x^2+\dfrac{4V}{\sqrt{3}}x^{-1}=\sqrt{3}\left(\dfrac{3}{2}x^2+\dfrac{4V}{3}x^{-1}\right)$ M1

ii. $\dfrac{dS}{dx}=\sqrt{3}\left(3x-\dfrac{4V}{3}x^{-2}\right)=\sqrt{3}\left(3x-\dfrac{4V}{3x^2}\right)$ A1

for a minimum surface area

$\dfrac{dS}{dx}=0 \;\Rightarrow\; 3x-\dfrac{4V}{3x^2}=0 \;\Rightarrow\; x^3=\dfrac{4V}{9}$

$x=\sqrt[3]{\dfrac{4V}{9}}$ A1

Note that from CAS, there are many equivalent forms, for all these answers and equations.

e. solving the five non-linear equations using CAS, for four unknowns, S, V, h , x

$(1)\;\; S=\dfrac{3\sqrt{3}}{2}x^2+6xh \qquad (2)\;\; V=\dfrac{3\sqrt{3}}{2}x^2h$ M1

$(3)\;\; x=\sqrt{\dfrac{2S}{9\sqrt{3}}} \qquad (4)\;\; x=\sqrt[3]{\dfrac{4V}{9}}$ and $(5)\;\; V=9S$

gives $x=18\sqrt{3}$ and $h=27$ cm A1

$s=\frac{3\cdot\sqrt{3}}{2}\cdot x^2+6\cdot x\cdot h$	$s=\frac{3\cdot\sqrt{3}\cdot x^2}{2}+6\cdot h\cdot x$
$v=\frac{3\cdot\sqrt{3}}{2}\cdot x^2\cdot h$	$v=\frac{3\cdot h\cdot\sqrt{3}\cdot x^2}{2}$
$\text{solve}\left(s=\frac{3\cdot\sqrt{3}}{2}\cdot x^2+6\cdot x\cdot h,h\right)$	$h=\frac{2\cdot s-3\cdot\sqrt{3}\cdot x^2}{12\cdot x}$
$v=\frac{3\cdot\sqrt{3}}{2}\cdot x^2\cdot h\mid h=\frac{2\cdot s-3\cdot\sqrt{3}\cdot x^2}{12\cdot x}$	$v=\frac{\sqrt{3}\cdot s\cdot x}{4}-\frac{9\cdot x^3}{8}$
$\frac{d}{dx}\left(\frac{\sqrt{3}\cdot s\cdot x}{4}-\frac{9\cdot x^3}{8}\right)$	$\frac{\sqrt{3}\cdot s}{4}-\frac{27\cdot x^2}{8}$
$\text{solve}\left(\frac{\sqrt{3}\cdot s}{4}-\frac{27\cdot x^2}{8}=0,x\right)$	$x=\frac{3^{\frac{3}{4}}\cdot\sqrt{2\cdot s}}{9}$ and $s\geq 0$ or $x=\frac{-3^{\frac{3}{4}}\cdot\sqrt{2\cdot s}}{9}$ and $s\geq($ ▸
$\text{solve}\left(v=\frac{3\cdot\sqrt{3}}{2}\cdot x^2\cdot h,h\right)$	$h=\frac{2\cdot\sqrt{3}\cdot v}{9\cdot x^2}$
$s=\frac{3\cdot\sqrt{3}}{2}\cdot x^2+6\cdot x\cdot h\mid h=\frac{2\cdot\sqrt{3}\cdot v}{9\cdot x^2}$	$s=\frac{4\cdot\sqrt{3}\cdot v}{3\cdot x}+\frac{3\cdot\sqrt{3}\cdot x^2}{2}$
$\frac{d}{dx}\left(\frac{4\cdot\sqrt{3}\cdot v}{3\cdot x}+\frac{3\cdot\sqrt{3}\cdot x^2}{2}\right)$	$3\cdot\sqrt{3}\cdot x-\frac{4\cdot\sqrt{3}\cdot v}{3\cdot x^2}$

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$\frac{d}{dx}\left(\frac{4\cdot\sqrt{3}\cdot v}{3\cdot x}+\frac{3\cdot\sqrt{3}\cdot x^2}{2}\right)$	$3\cdot\sqrt{3}\cdot x-\frac{4\cdot\sqrt{3}\cdot v}{3\cdot x^2}$
$\text{solve}\left(3\cdot\sqrt{3}\cdot x-\frac{4\cdot\sqrt{3}\cdot v}{3\cdot x^2}=0,x\right)$	$x=\frac{2^{\frac{2}{3}}\cdot(3\cdot v)^{\frac{1}{3}}}{3}$
$eq1:=s=\frac{3\cdot\sqrt{3}}{2}\cdot x^2+6\cdot x\cdot h$	$s=\frac{3\cdot\sqrt{3}\cdot x^2}{2}+6\cdot h\cdot x$
$eq2:=v=\frac{3\cdot\sqrt{3}}{2}\cdot x^2\cdot h$	$v=\frac{3\cdot h\cdot\sqrt{3}\cdot x^2}{2}$
$eq3:=x=\frac{1}{3}\cdot\sqrt{\frac{2\cdot s}{\sqrt{3}}}$	$x=\frac{3^{\frac{3}{4}}\cdot\sqrt{2\cdot s}}{9}$
$eq4:=x=\sqrt[3]{\frac{4\cdot v}{9}}$	$x=\frac{2^{\frac{2}{3}}\cdot(3\cdot v)^{\frac{1}{3}}}{3}$
$eq5:=v=9\cdot s$	$v=9\cdot s$
$\text{solve}\left(\begin{cases}eq1\\eq2\\eq3\\eq4\\eq5\end{cases},\{x,h,s,v\}\right)\mid x>0$ and $h>0$ and $s>0$ and $v>$ ▸	◂$=4374\cdot\sqrt{3}$ and $v=39366\cdot\sqrt{3}$ and $x=18\cdot\sqrt{3}$ and $h=27$

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Question 4

a.i. $g(x)=\int_0^x f(t)\,dt-2x$

$g(-2)=\int_0^{-2} f(t)\,dt+4$

$=4-\int_{-2}^{0} f(t)\,dt$

Now $\int_{-2}^{0} f(t)\,dt$ is the area of a triangle of side lengths 2 and 4

$=4-\frac{1}{2}\times 2\times 4$ A1

$=0$

ii. $g(4)=\int_0^4 f(t)\,dt-8$

Now $\int_0^4 f(t)\,dt$ represents one-quarter of the area of a circle of radius 4

$=\frac{1}{4}\pi\times 4^2-8$

$=4\pi-8$ A1

iii. $g(-3)=\int_0^{-3} f(t)\,dt+6$

$=6-\int_{-3}^{0} f(t)\,dt=6-\int_{-3}^{0}(2t+4)\,dt$

$=6-\left[t^2+4t\right]_{-3}^{0}=6-\left[0-(9-12)\right]$

$=3$ A1

b. $g'(x)=f(x)-2$

where $f(x)=\begin{cases}2x+4 & \text{for} \quad -3\le x\le 0\\ \sqrt{16-x^2} & \text{for} \quad 0\le x\le 4\end{cases}$ M1

$g'(x)=\begin{cases}2x+2 & \text{for} \quad -3\le x\le 0\\ \sqrt{16-x^2}-2 & \text{for} \quad 0\le x\le 4\end{cases}$ A1

c. The function $g(x)$ has a maximum or minimum turning point when

$g'(x)=0$ or $f(x)=2$

(1) for $-3\le x\le 0 \Rightarrow 2x+2=0 \Rightarrow x=-1$ M1

$$g(-1)=\int_0^{-1} f(t)\,dt+2=2-\int_{-1}^{0}(2t+4)\,dx=2-\left[t^2+4t\right]_{-1}^{0}$$

$$=2-\left[(0)-(1-4)\right]$$

$$=-1$$

(2) for $0\le x\le 4 \Rightarrow \sqrt{16-x^2}-2=0$

$16-x^2=4 \Rightarrow x^2=12 \Rightarrow x=2\sqrt{3}$ since $0\le x\le 4$ M1

$$g\left(2\sqrt{3}\right)=\int_0^{2\sqrt{3}} f(t)\,dt-4\sqrt{3}=\int_0^{2\sqrt{3}}\sqrt{16-t^2}\,dt-4\sqrt{3}$$

$$=\frac{8\pi}{3}+2\sqrt{3}-4\sqrt{3}=\frac{8\pi}{3}-2\sqrt{3}\approx 4.9135$$

$\left(2\sqrt{3},\frac{8\pi}{3}-2\sqrt{3}\right)$ is the maximum turning point A1

$(-1,-1)$ is the minimum turning point A1

d. The point $(0,0)$ is the inflexion point A1

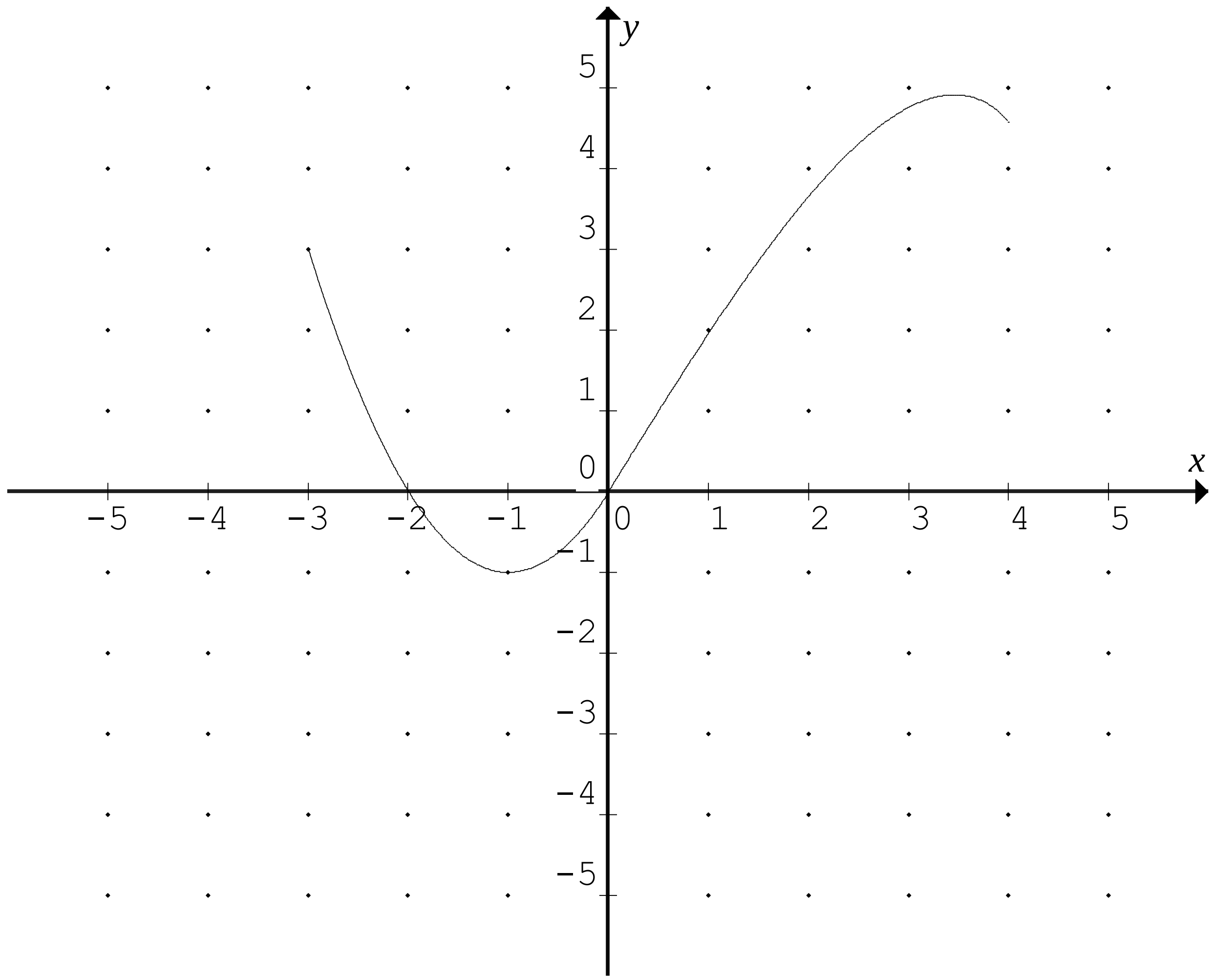
e. Now g has a domain $[-3,4]$

$g(4)=4\pi-8\approx 4.5664$ and $g(-3)=3$ are endpoints.

The range is $\left[-1,\frac{8\pi}{3}-2\sqrt{3}\right]$, crosses x-axis at $x=-2$ and $x=0$, $(-2,0)$ $(0,0)$

the point $(0,0)$ is the inflexion point

correct graph over correct domain and range, G1
all stationary points and intercepts G1



Define $f(x)=\begin{cases} 2\cdot x+4, & -3\le x\le 0 \\ \sqrt{16-x^2}, & 0\le x\le 4 \end{cases}$	*Done*
Define $g(x)=\int_0^x f(t)\,dt-2\cdot x$	*Done*
$g(-2)$	0
$g(4)$	$4\cdot\pi-8$
$g(-3)$	3
$g(-1)$	-1
$g(2\cdot\sqrt{3})$	$\frac{8\cdot\pi}{3}-2\cdot\sqrt{3}$
$g(0)$	0

END OF SECTION 2 SUGGESTED ANSWERS