Year 2013

VCE Mathematical Methods CAS

Trial Examination 1

Suggested Solutions

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Question 1

a. If $f(x)=\log_e(\cos(3x))$ using the chain rule

$y=\log_e(u)$ where $u=\cos(3x)$

$\frac{dy}{du}=\frac{1}{u}$ $\quad\quad \frac{du}{dx}=-3\sin(3x)$

$f'(x)=\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=\frac{-3\sin(3x)}{\cos(3x)}=-3\tan(3x)$ M1

$f'\left(\frac{\pi}{18}\right)=-3\tan\left(\frac{\pi}{6}\right)=-3\times\frac{\sqrt{3}}{3}$

$f'\left(\frac{\pi}{18}\right)=-\sqrt{3}$ A1

b. If $y=\frac{\sin(4x)}{2x^2}$ using the quotient rule

$u=\sin(4x)$ $\quad\quad v=2x^2$

$\frac{du}{dx}=4\cos(4x)$ $\quad \frac{dv}{dx}=4x$ M1

$\frac{dy}{dx}=\frac{8x^2\cos(4x)-4x\sin(4x)}{\left(2x^2\right)^2}$

$\frac{dy}{dx}=\frac{4x\left(2x\cos(4x)-\sin(4x)\right)}{4x^4}=\frac{2x\cos(4x)-\sin(4x)}{x^3}=\frac{g(x)}{x^3}$

$g(x)=2x\cos(4x)-\sin(4x)$ A1

Question 2

$kx\ -\ 4y=2$

$3x-(k+4)y=k+1$

$\Delta=\begin{vmatrix}k & -4\\ 3 & -(k+4)\end{vmatrix}=-k(k+4)+12=-k^2-4k+12$ M1

$\Delta=-\left(k^2+4k-12\right)=-(k+6)(k-2)$

i. There is a unique solution when $k\in R\setminus\{2,-6\}$ A1

When $k=2$ the equations become $\begin{matrix}2x-4y=2\\3x-6y=3\end{matrix}$ these lines are both the same line as $x-2y=1$, therefore we have an infinite number of solutions when $k=2$

ii. When $k=-6$ the equations become $\begin{matrix}-6x-4y=2\\3x+2y=-5\end{matrix}$ these lines are parallel with different *y*-intercepts, therefore there is no solution when $k=-6$ A1

Question 3

$$f'(x)=\frac{dy}{dx}=\frac{2}{\sqrt{4x+9}}$$

$$y=f(x)=\int\frac{2}{\sqrt{4x+9}}dx=\int 2\times(4x+9)^{-\frac{1}{2}}dx$$ A1

$$y=f(x)=\frac{2}{4}\times 2\times(4x+9)^{\frac{1}{2}}+c=\sqrt{4x+9}+c$$ M1

$$f(0)=0 \quad \Rightarrow \quad 0=\sqrt{9}+c \quad \Rightarrow c=-3$$

$$y=f(x)=\sqrt{4x+9}-3$$ A1

Question 4

a. $f:R\to R\,,\ f(x)=1-2\cos\left(\frac{\pi x}{6}\right)$

crosses the *x*-axis, when $y=0$, finding the general solution

$$1-2\cos\left(\frac{\pi x}{6}\right)=0$$

$$2\cos\left(\frac{\pi x}{6}\right)=1$$ M1

$$\cos\left(\frac{\pi x}{6}\right)=\frac{1}{2}$$

$$\frac{\pi x}{6}=2n\pi\pm\cos^{-1}\left(\frac{1}{2}\right)=2n\pi\pm\frac{\pi}{3}$$

$x=12n\pm 2$, where $n\in Z$ A1

b. $g:[0,12]\to R\,,\ g(x)=1-2\cos\left(\frac{\pi x}{6}\right)$

amplitude is 2, period $T=\frac{2\pi}{\frac{\pi}{6}}=12$ and the range is $[-1,3]$ A1

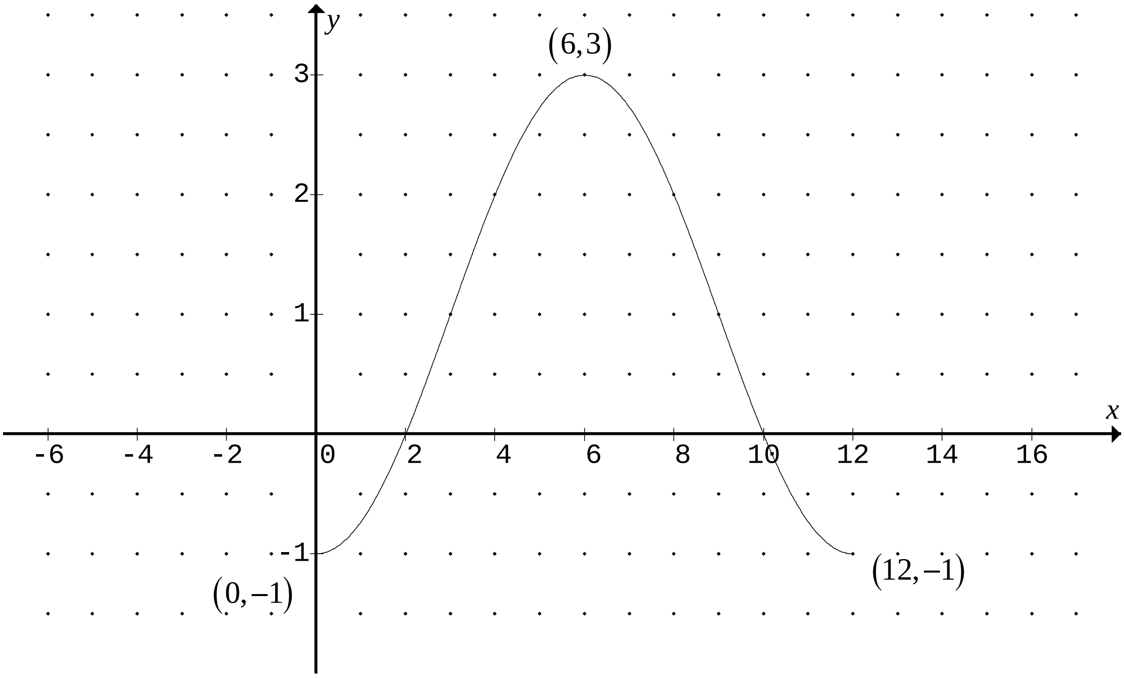
crosses the *x*-axis at $x=2$ and $x=10$ $\quad n=0$ and $n=1$ from **a.**

end-points $f(0)=-1 \quad f(12)=-1 \quad (0,-1) \quad (12,-1)$

maximum, when $y=3$ when $\cos\left(\frac{\pi x}{6}\right)=-1$

$$\frac{\pi x}{6}=\pi \quad \text{so} \quad x=6 \quad (6,3)$$ A1

correct graph, on restricted domain, end-points, shape. G1



Question 5

$\log_x 5 + \log_5 x = \frac{5}{2}$ to solve let $u = \log_x 5$ then $\log_5 x = \frac{1}{\log_x 5} = \frac{1}{u}$

$u + \frac{1}{u} = \frac{5}{2}$ multiply both sides by $2u$

$2u^2 + 2 = 5u$ M1

$2u^2 - 5u + 2 = 0$

$(2u-1)(u-2) = 0$

$u = \log_x 5 = \frac{1}{2}, 2$ M1

$\log_x 5 = \frac{1}{2}$ $\log_x 5 = 2$

in index form $x^{\frac{1}{2}} = \sqrt{x} = 5$ $x^2 = 5$ since $x > 0$

$x = 25, \sqrt{5}$ A1

Question 6

$f\ :\quad y = 3e^{-2x} - 4$ interchange x and y

$f^{-1}:\quad x = 3e^{-2y} - 4$ re-arrange to make y the subject

$3e^{-2y} = x + 4$ M1

$e^{-2y} = \dfrac{x+4}{3} \quad \Rightarrow \quad -2y = \log_e\left(\dfrac{x+4}{3}\right)$

$y = -\dfrac{1}{2}\log_e\left(\dfrac{x+4}{3}\right)$ or $y = \dfrac{1}{2}\log_e\left(\dfrac{3}{x+4}\right)$ A1

but domain f = range $f^{-1} = R$ and range f = domain $f^{-1} = (-4, \infty)$

we must state the maximal domain of the inverse function

$f^{-1}: (-4, \infty) \to R\ ,\ f^{-1}(x) = -\dfrac{1}{2}\log_e\left(\dfrac{x+4}{3}\right)$ A1

Question 7

a. $y = 3 - \dfrac{12}{(x+2)^2}$ when $x = 0$ $y = 3 - \dfrac{12}{2^2} = 0$

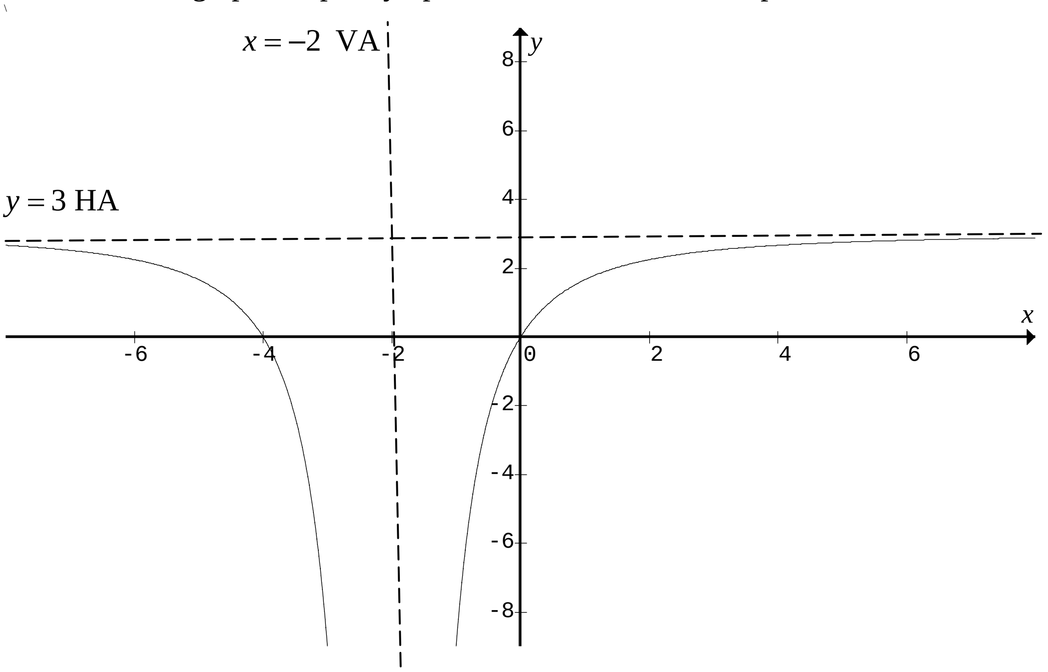
crosses the x-axis when $y = 0 \Rightarrow 3 - \dfrac{12}{(x+2)^2} = 0 \quad \Rightarrow (x+2)^2 = 4$

$x + 2 = \pm 2 \quad \Rightarrow x = 0$ and $x = -4$ $(0,0)$ $(-4,0)$ A1

$x = -2$ is a vertical asymptote and $y = 3$ is a horizontal asymptote

domain $R\backslash\{-2\}$ range $(-\infty, 3)$ A1

correct graph, shape asymptotes, correct axial intercepts G1



b. $y = 3 - \dfrac{12}{(x+2)^2}$ from the graph of $y = \dfrac{1}{x^2}$

$\dfrac{1}{2}$ mark for each correct transformation, the translations must come last.

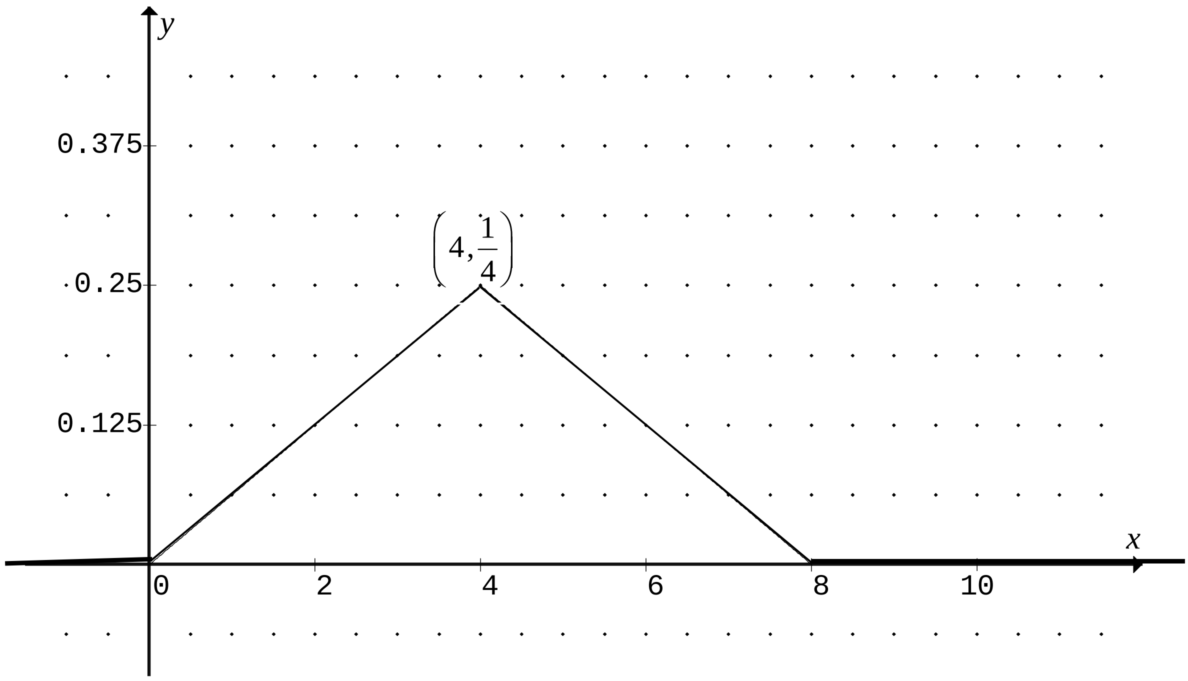
- reflect in the x-axis $y = -\dfrac{1}{x^2}$
- dilate by a factor of 12 parallel to the y-axis (or away from the x-axis) $y = -\dfrac{12}{x^2}$
- translate 2 units to the left parallel to the x-axis (or away from the y-axis) $y = -\dfrac{12}{(x+2)^2}$
- translate 3 units up parallel to the y-axis (or away from the x-axis) $y = 3 - \dfrac{12}{(x+2)^2}$

Question 8

a. $f(x) = \begin{cases} \dfrac{1}{16}\left(4 - |x-4|\right) & \text{for } 0 \le x \le 8 \\ 0 & \text{elsewhere} \end{cases}$

$f(0) = f(8) = 0$ endpoints $(0,0)$ $(8,0)$

correct graph, shape, point at $\left(4, \dfrac{1}{4}\right)$, zero elsewhere G2



b. $\Pr(X<2\,|\,X<6)=\dfrac{\Pr(X<2)}{\Pr(X<6)}$

$\Pr(X<2\,|\,X<6)=\dfrac{\frac{1}{2}\times 2\times\frac{1}{8}}{1-\frac{1}{2}\times 2\times\frac{1}{8}}=\dfrac{\frac{1}{8}}{\frac{7}{8}}$ by area of triangles

$\Pr(X<2\,|\,X<6)=\dfrac{1}{7}$ A1

Question 9

$\Pr(A)=\dfrac{1}{3}$ and $\Pr(B)=\dfrac{2}{5}$

Since *A* and *B* are independent, then A' and B' are also independent

$\Pr(A'\cap B')=\Pr(A')\Pr(B')=\dfrac{2}{3}\times\dfrac{3}{5}=\dfrac{2}{5}$ M1

$\Pr(A'\cup B')=\Pr(A')+\Pr(B')-\Pr(A'\cap B')$

$=\dfrac{2}{3}+\dfrac{3}{5}-\dfrac{2}{5}=\dfrac{10+9-6}{15}$

$=\dfrac{13}{15}$ A1

Question 10

a. $f(x)=xe^{-2x}$ product rule

$u=x \qquad v=e^{-2x}$

$\dfrac{du}{dx}=1 \qquad \dfrac{dv}{dx}=-2e^{-2x}$

$f'(x)=e^{-2x}-2xe^{-2x}$ does not need to be simplified A1

b. for a stationary point $f'(x)=0$

$f'(x)=e^{-2x}(1-2x)=0$ A1

since $e^{-2x}\neq 0 \ \Rightarrow 1-2x=0 \ \Rightarrow \ x=\dfrac{1}{2}$ and $f\left(\dfrac{1}{2}\right)=\dfrac{1}{2}e^{-1}$

stationary point is a maximum at $\left(\dfrac{1}{2},\dfrac{1}{2e}\right)$ A1

c. from **a.** $\frac{d}{dx}\left(xe^{-2x}\right)=e^{-2x}-2xe^{-2x}$

$\int\left(e^{-2x}-2xe^{-2x}\right)dx=xe^{-2x}$

$\int e^{-2x}dx-\int 2xe^{-2x}dx=xe^{-2x}$

$-2\int xe^{-2x}dx=xe^{-2x}-\int e^{-2x}dx=xe^{-2x}+\frac{1}{2}e^{-2x}$ M1

$\int xe^{-2x}\,dx=-\frac{1}{2}xe^{-2x}-\frac{1}{4}e^{-2x}=-\frac{1}{4}e^{-2x}(2x+1)$

The required area is $A=\int_0^2 xe^{-2x}\,dx$ A1

$A=-\left[\frac{1}{4}(2x+1)e^{-2x}\right]_0^2$

$A=\left(-\frac{5}{4}e^{-4}+\frac{1}{4}\right)$

$A=\frac{1}{4}\left(1-5e^{-4}\right)$ A1

Question 11

Since the mode is 5, $\Pr(X=5)>\Pr(X=4)\ \Rightarrow\ \frac{\Pr(X=5)}{\Pr(X=4)}>1$

substitute into $\frac{\Pr(X=k+1)}{\Pr(X=k)}=\frac{(n-k)p}{(k+1)(1-p)}$ with $k=4$ and $n=8$

$\frac{4p}{5(1-p)}>1\quad\Rightarrow\quad 4p>5-5p\quad\Rightarrow\quad 9p>5\quad\Rightarrow\quad p>\frac{5}{9}$ M1

Also since the mode is 5, $\Pr(X=6)<\Pr(X=5)\ \Rightarrow\ \frac{\Pr(X=6)}{\Pr(X=5)}<1$

substitute into $\frac{\Pr(X=k+1)}{\Pr(X=k)}=\frac{(n-k)p}{(k+1)(1-p)}$ with $k=5$ and $n=8$

$\frac{3p}{6(1-p)}<1\quad\Rightarrow\quad p<2(1-p)\quad\Rightarrow\quad p<2-2p$ M1

$3p<2\quad\Rightarrow\quad p<\frac{2}{3}$

so $\frac{5}{9}<p<\frac{2}{3}$ $\quad p_1=\frac{5}{9}$ and $p_2=\frac{2}{3}$ A1

END OF SUGGESTED SOLUTIONS