MATHEMATICAL MATHODS (CAS)

Unit 4
Targeted Evaluation Task for School-assessed Coursework 4

TSSM
Creating VCE Success

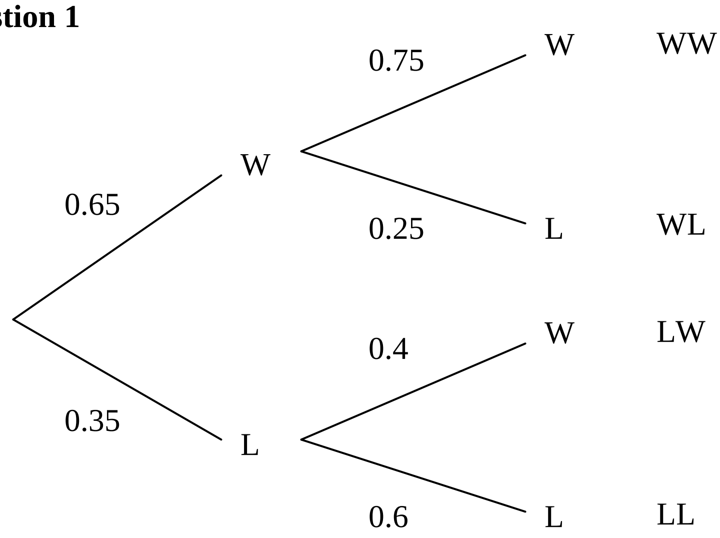
2012 Item Analysis Task on Probability for Outcomes 1, 2 & 3

SOLUTIONS & RESPONSE GUIDE

The marks given are allocated to the 3 outcomes according to the following:
A – Outcome 1, B – Outcome 2, C – Outcome 3

Question 1

a.



A2

b. Pr(Wins one set only) = Pr(WL $\cup$ LW)

$= 0.65 \times 0.25 + 0.35 \times 0.4$

$= 0.1625 + 0.14$

$= 0.3025$

A1

c. The probability of only one way of winning one set is found rather than both ways.

B1

Question 2

a. This is a conditional probability

$$\Pr(\text{Wins first set}|\text{Wins one set only}) = \frac{\Pr(\text{Wins first set} \cap \text{Wins one set only})}{\Pr(\text{Wins one set only})}$$

$= \dfrac{0.1625}{0.3025}$ (Using results from **Question 1**)

$= 0.5372$ and alternative **D is correct.**

A1, B1

b. Interpreting the question as Pr(Wins one set only|Wins first set)

This leads to $\dfrac{\Pr(\text{Wins first set} \cap \text{Wins one set only})}{\Pr(\text{Wins first set})} = \dfrac{0.1625}{0.65} = 0.25$

B1

c. Adding the probabilities of winning the first set and winning one set only.

B1

Question 3

a. $\mu = \sum x \Pr(X = x)$

$\mu = 2\times0.1 + 3\times0.3 + 4\times0.4 + 5\times0.2 = 3.7$

A1

b. Finding the mean of the values of *x*.

A1

c. Increasing each *x*-value by 2.

B1

d. Let $x = \Pr(X = 2)$ and $y = \Pr(X = 4)$

$\sum \Pr(X = x) = 1$

$\therefore x + 0.2 + y + 0.4 = 1$

$x + y = 0.4 \qquad (1)$

$\sum x \Pr(X = x) = 3.7$

$\therefore 2x + 0.6 + 4y + 2.0 = 3.7$

$2x + 4y = 1.1 \qquad (2)$

B2

$(2) - 2\times(1) \Rightarrow 2y = 0.3$

$\therefore y = 0.15$ and $x = 0.25$

So $\Pr(X = 2) = 0.25$ and $\Pr(X = 4) = 0.15$

A1

Question 4

a. $E(X^2) = \sum x^2 \Pr(X = x)$

$E(X^2) = 4\times0.1 + 9\times0.3 + 16\times0.4 + 25\times0.2 = 14.5$

A1

$Var(X) = E(X^2) - \left[E(X)\right]^2$

$Var(X) = 14.5 - 3.7^2 = 0.81$

$\sigma = \sqrt{Var(X)} = \sqrt{0.81} = 0.90$

Therefore **D** is the correct answer.

A1

b.

i. Finding the variance but then failing to take the square root to find the standard deviation.

ii. Finding the variance using $Var(X) = E(X^2) - \left[E(X)\right]$ rather than

$Var(X) = E(X^2) - \left[E(X)\right]^2$

A2

c. One or both of $\Pr(X = 2)$ and $\Pr(X = 5)$ would need to be made larger and one or both of $\Pr(X = 3)$ and $\Pr(X = 4)$ would need to be made smaller.

B2

Question 5

a. For a binomial distribution $\Pr(X = x) = {}^nC_x p^x (1-p)^{n-x}$

For this distribution $n = 8$

$\therefore \Pr(X = 3) = {}^8C_3 p^3 (1-p)^5 = 0.2668$

$56p^3(1-p)^5 = 0.2668$ or $p^3(1-p)^5 = 0.004764$

B1

b. Using the Numeric Solver application gives $p = 0.32$

C2

c. $\mu = np = 8 \times 0.32 = 2.56$

$\sigma = \sqrt{np(1-p)} = \sqrt{8 \times 0.32 \times 0.68} = 1.319$

$\therefore$ Alternative **B** is the correct answer.

A1

d.

i. Finding the variance instead of the standard deviation

ii. Using $\mu = nq = n(1-p)$ to find the mean. That is, using the probability of failure rather than the probability of success for p.

A2

e.

i. $\sqrt{np(1-p)} = 1.2$

$8p(1-p) = 1.44$

$p^2 - p + 0.18 = 0$

$$p = \frac{1 \pm \sqrt{1 - 4 \times 1 \times 0.18}}{2 \times 1} = 0.2354 \text{ or } 0.7646$$

But $p = 0.2354$ gives $\mu = 1.88$ which is less than the value found in part **a.**

$\therefore p = 0.7646$

B2

ii. Bi(8, 0.7646, 3) = 0.0181

C1

Question 6

a. Let X represent the number of people who like the product. X is a binomial distribution with $n = 20$ and $p = 0.6$.

$\Pr(11 \le X \le 15) = \Pr(X \le 15) - \Pr(X \le 10)$

A1

$\text{binomcdf}(20, 0.6, 15) - \text{binomcdf}(20, 0.6, 10)$

$= 0.94905 - 0.24466$

$= 0.7044$

Therefore alternative **C** is correct

C1

b. The probabilities are added. $\Pr(X \le 15) + \Pr(X \le 10) = 0.94905 + 0.24466 = 1.1937$
However this value is greater than 1 and as all probabilities must lie between 0 and 1 it can be rejected as an impossible alternative.

B1, C1

Question 7

a. $S_0 = \begin{bmatrix} 0 \\ 500 \end{bmatrix}$ and $T = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}$ or $S_0 = \begin{bmatrix} 500 \\ 0 \end{bmatrix}$ and $T = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}$

A1

b. $S_2 = T^2 S_0$

$S_2 = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}^2 \begin{bmatrix} 0 \\ 500 \end{bmatrix} = \begin{bmatrix} 260 \\ 240 \end{bmatrix}$ or $S_2 = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix}^2 \begin{bmatrix} 500 \\ 0 \end{bmatrix} = \begin{bmatrix} 240 \\ 260 \end{bmatrix}$

Therefore 260 people go to Ausfilm cinema in the second week and alternative **E** is correct.

C2

c.

i. Misreading the answer matrix and taking the Boyts value as the Ausfilm value.

B1

ii. Using an incorrect T matrix due to entering the probabilities in the incorrect positions in the matrix. If $S_0 = \begin{bmatrix} 0 \\ 500 \end{bmatrix}$ then using $T = \begin{bmatrix} 0.3 & 0.6 \\ 0.7 & 0.4 \end{bmatrix}$ gives $S_2 = \begin{bmatrix} 210 \\ 290 \end{bmatrix}$ and hence 210 people going to Ausfilm cinema in the second week

C2

Question 8

a. When the numbers attending each cinema have reached their long term values $S_{n-1} = S_n$
If x is the number of people attending the Ausfilm cinema in the long term then

$$\begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix} \begin{bmatrix} x \\ 500 - x \end{bmatrix} = \begin{bmatrix} x \\ 500 - x \end{bmatrix}$$

B1

$$\therefore 0.7x + 0.4(500 - x) = x$$
$$0.3x + 200 = x$$
$$x = \frac{200}{0.7} = 285.7$$

And rounding up gives 286 which is alternative **E**.

A2

b. The answer was incorrectly rounded down to 285

A1

c. Insufficient number of iterations. For example $T^5 S_0$ gives 285.

C1

Question 9

a. If the median value is *a* then:

$$\int_1^a \left(\frac{x}{2} - \frac{1}{2} \right) dx = \frac{1}{2}$$
$$\left[\frac{x^2}{4} - \frac{x}{2} \right]_1^a = \frac{1}{2}$$
$$\frac{a^2}{4} - \frac{a}{2} - \frac{1}{4} = 0$$
$$a^2 - 2a - 1 = 0$$

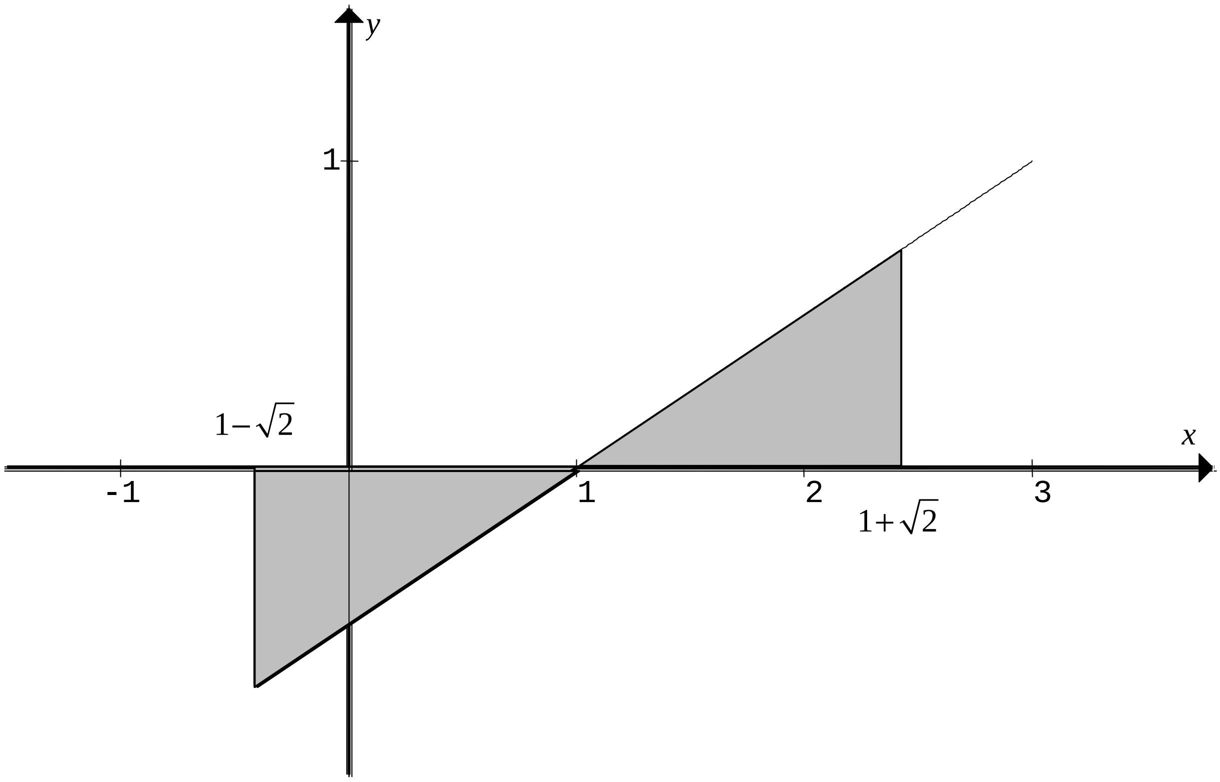
Solving this quadratic equation gives $a = 1 \pm \sqrt{2}$. But $1 - \sqrt{2} < 1$ so the only solution for the given function is $1 + \sqrt{2}$ which is alternative **E**.

A1, B1

b. Assuming that the median was halfway between the lower and upper limits of the domain of the probability density function. This may be true for a symmetrical function but the given function is not symmetrical.

B1

c.



If the graph of the function is extended back to the left it can be seen that the area between the graph and the *x*-axis from $x=1-\sqrt{2}$ to $x=1$ is also 0.5. Therefore $x=1-\sqrt{2}$ can also be a solution of the equation used in part **a.**

A1, B1

Question 10

a. The mean of a continuous random variable is given by $\int_{-\infty}^{\infty} xf(x)\,dx$

For the given function $\mu = E(X) = \int_0^{\pi} \frac{4x^2\sin^2 x}{\pi^2}\,dx$. Graphing this and using the numeric integration function on a graphics calculator gives a value of 1.7761which is alternative **C**.

A1, C1

b. $E(X^2) = \int_0^{\pi} \frac{4x^3\sin^2 x}{\pi^2}\,dx = 3.4348$ using a graphics calculator

$\sigma = \sqrt{E(X^2)-[E(X)]^2} = \sqrt{3.4348-1.7761^2} = 0.5294$

A1, C1

c. Using the maximum function of the graphics calculator the maximum can be shown to be at $x=1.8366$. The mean can sometimes be confused with the mode, which is the most frequently occurring value. In a continuous distribution this would be equivalent to the *x*-value of the maximum of the probability density function

B1, C1

Question 11

a. Let the random variable X denote the life of a battery. X is normally distributed with $\mu = 30$ and $\Pr(X \geq 36) = 0.05$

If Z is the standard normal distribution and $\Pr(Z \geq z) = 0.05$ then $\Pr(Z < z) = 0.95$

A1

$z = \text{InvNorm}(0.95) = 1.64485$

$$z = \frac{x - \mu}{\sigma}$$

$$\sigma = \frac{x - \mu}{z} = \frac{36 - 30}{1.64485} = 3.65 \text{ months}$$

Therefore alternative **C** is the correct answer.

B1, C1

b. Because $\Pr(Z < z) = 0.95$ this may be confused with the general rule that $\Pr(\mu - 2\sigma \leq Z \leq \mu + 2\sigma) = 0.95$ leading to the assumption that 36 is 2 standard deviations from the mean and hence that the standard deviation of is $\frac{36 - 30}{2} = 3$ months.

B1

c. If $\mu = 30$ and $\sigma = 6.32$

$\Pr(X \geq 36) = \textit{normalcdf}\,(36, \infty.30.6.32) = 0.1712$ using the graphics calculator.

C1

Summary of mark allocation per Outcome

Question	Part	Outcome 1	Outcome 2	Outcome 3
1	a	2		
	b	1		
	c		1	
2	a	1	1	
	b		1	
	c		1	
3	a	1		
	b	1		
	c		1	
	d	1	2	
4	a	2		
	b	2		
	c		2	
5	a		1	
	b			2
	c	1		
	d	2		
	e		2	1
6	a	1		1
	b		1	1
7	a	1		
	b			2
	c		1	2
8	a	2	1	
	b	1		
	c			1
9	a	1	1	
	b		1	
	c	1	1	
10	a	1		1
	b	1		1
	c		1	1
11	a	1	1	1
	b		1	
	c			1
Raw Marks		24	21	15
Adjusted Marks		8	7	5