MATHEMATICAL MATHODS (CAS)

Unit 4
Targeted Evaluation Task for School-assessed Coursework 2

2012 Modelling Analysis Task on Trigonometric Functions for

Outcomes 1, 2 & 3

<u>SOLUTIONS & RESPONSE GUIDE</u>

The marks given are allocated to the 3 outcomes according to the following:
A – Outcome 1, B – Outcome 2, C – Outcome 3

Question 1

a. The amplitude of the variation in depth is $\frac{9.5-0.5}{2}=4.5$ m so a = 4.5

The average depth of water is $\frac{9.5+0.5}{2}=5$ m so c = 5

Now the period of the trigonometric function $T=\frac{2\pi}{b}$

So $\frac{25}{2}=\frac{2\pi}{b}$

$b=\frac{4\pi}{25}$

A3

b. First maximum occurs when $\sin\left(\frac{4\pi t}{25}\right)$ first equals 1. That is, when $\frac{4\pi t}{25}=\frac{\pi}{2}$

$t=\frac{25}{8}=3.125$ hrs or 3 hrs 7.5 min. Therefore the first maximum depth will occur at 12.08 pm on Sept 2. (Accept 12.07 pm.)

A1

The first minimum occurs when $\sin\left(\frac{4\pi t}{25}\right)$ first equals -1. That is, when $\frac{4\pi t}{25}=\frac{3\pi}{2}$

$t=\frac{75}{8}=9.375$ hrs or 9 hrs 22.5 min. Therefore the first minimum depth will occur at 6.23 pm on Sept 2. (Accept 6.22 pm.).
This may also be calculated as being 6 hrs 15 min (that is, half a period) after the time of the first maximum.

A1

c. This time is 45 hrs and 15 min after 9.00 am on Sept 2 therefore t = 45.25

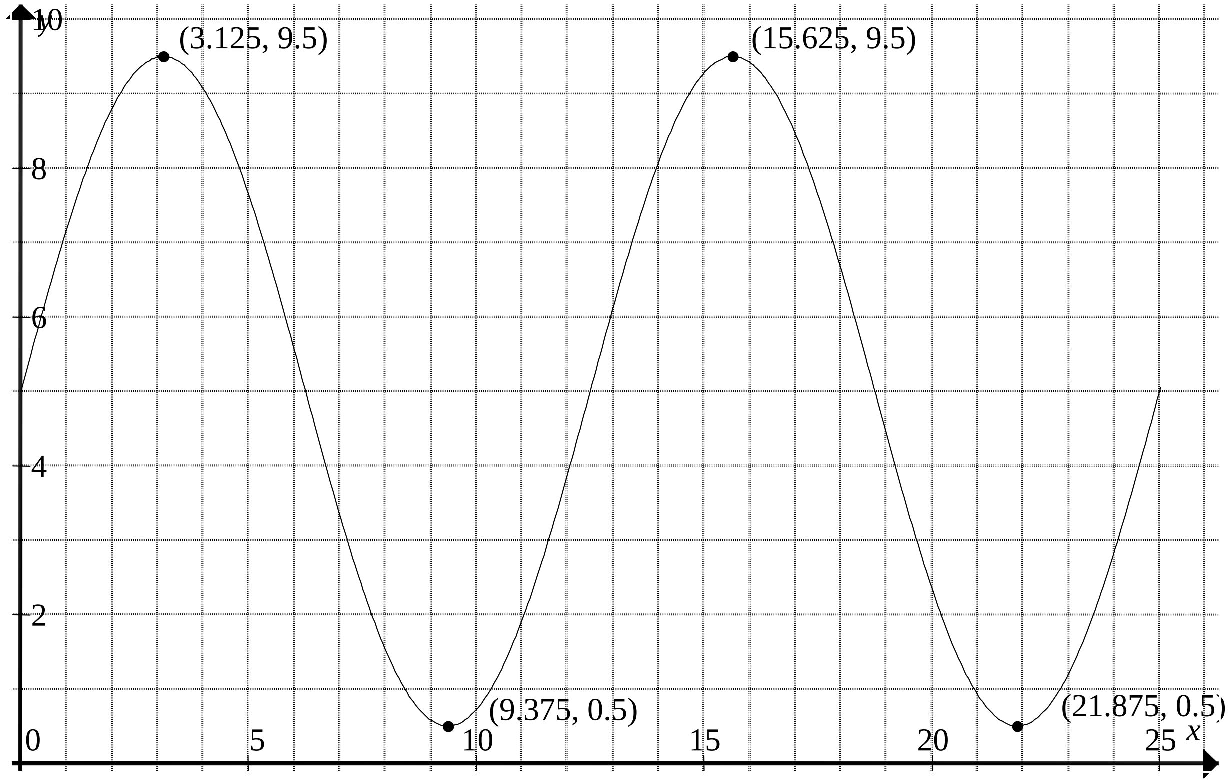
B1

$d_1(45.25)=4.5\sin\left(\frac{181\pi}{25}\right)+5=1.91954$

Depth is 192 cm at 6.15 am on Sept 4

A1

d.



Correct graph and scale A2
Turning points correctly labelled B2

Question 2

a. $4.5\sin\left(\frac{4\pi t}{25}\right)+5=8.2$

A1

$\frac{4\pi t}{25}=0.7911,\ 2.3505,\ 7.0743,\ 8.6337$

$t=1.5738,\ 4.6762,\ 14.0739,\ 17.1762$

A1

Coordinates on graph; (1.5738, 8.2) (4.6762, 8.2) (14.0739, 8.2) (17.1762, 8.2)

B1

The depth will be 8.2 m at 10.34 am, 1.41 pm, and 11.04 pm on Sept 2 and 2.11 am on Sept 3.

B2

b. Graph $y = d\ (t)$ and $y = 3.5$ on a graphics calculator and find the t-values of the intersection points of the 2 graphs.

$d_1(t) \geq 3.5$ from $t = 0$ to the first intersection point at $t = 6.9261$, that is for a total of 6.9261 hours.

C1

$d_1(t) \geq 3.5$ again between the second and third intersection points at $t = 11.8239$ and $t = 19.4261$ for a total of $19.4261 - 11.8239 = 7.6022$ hours.

C1

The cyclical nature of this trigonometric function means that there will be 2 more periods of 7.6022 hours in the first 48 hours for a total time of $6.9261 + 3 \times 7.6022 = 29.7327$ hours.

B1

This is a percentage time of $\dfrac{29.7327 \times 100}{48} = 61.9\%$

A1

Question 3

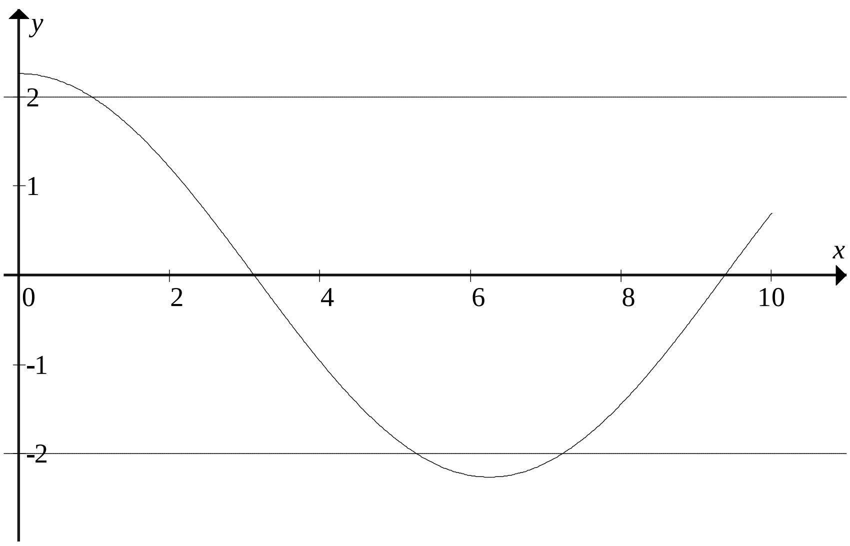
a. $d_1{}'(t) = \dfrac{18\pi}{25}\cos\left[\dfrac{4\pi t}{25}\right]$

A1

b. $\dfrac{18\pi}{25} = 2.26$ metres per hour

A1

c. Graph $y = d_1{}'(t)$, $y = 2$ and $y = -2$ as shown in the graph below and find the intersection points of the horizontal lines with $d_1{}'(t)$.



Intersection of $y = d_1{}'(t)$ and $y = 2$ at $t = 0.9669$

Intersection of $y = d_1{}'(t)$ and $y = -2$ at $t = 5.2831$ and $t = 7.2169$

C2

Therefore the ship can enter during the interval $t = 0.9669$ to $t = 5.2831$ which is a total of $5.2831 - 0.9669 = 4.3162$ hours as the depth is greater than 3.5 m during this time. Ship cannot enter between $t = 7.2169$ and $t = 10$ as depth is too shallow as found in **Question 2 b**.
Thus the ship can enter the port during a time interval of 4 hours 19 minutes.

B2

Question 4

a. The difference between successive high and low tides starts off quite large (almost 9 m) then decreases to almost zero and increases to about 9 m again before starting to decrease again.

C1, B1

b. The period of this variation is determined by the cosine term.

Period of $\cos\left(\frac{\pi t}{168}\right)$ is $\frac{2\pi}{\frac{\pi}{168}} = 2 \times 168 = 336$ hours

But the period of the variation in the difference between the heights of successive high and low tides is actually half of this value, that is 168 hours.

B2

c. Using the maximum and minimum calculation functions of a graphics calculator and looking in the region between, say, $t = 80$ and $t = 90$ gives:
Lowest high tide of 5.12 metres at $t = 85.95$ hours and
Highest low tide of 4.93 metres at $t = 82.52$ hours.

C2

Question 5

a. $d_2(t) = 4.5\cos\left(\frac{\pi t}{168}\right)\sin\left(\frac{4\pi t}{25}\right) + 5$ and using the product rule

$$d_2'(t) = 4.5\left[-\frac{\pi}{168}\sin\left(\frac{\pi t}{168}\right)\sin\left(\frac{4\pi t}{25}\right) + \cos\left(\frac{\pi t}{168}\right) \times \frac{4\pi}{25}\cos\left(\frac{4\pi t}{25}\right)\right]$$

$$d_2'(t) = 4.5\left[\frac{4\pi}{25}\cos\left(\frac{\pi t}{168}\right)\cos\left(\frac{4\pi t}{25}\right) - \frac{\pi}{168}\sin\left(\frac{\pi t}{168}\right)\sin\left(\frac{4\pi t}{25}\right)\right]$$

B2

b. i (6.233, -2.247)

C1

ii Find largest of the intersection points between $y = d_2'(t)$, $y = 2$ and $y = -2$.
T = 25.2098 which gives a time of 10.13 am Sept 3.
Care must be taken to use the correct region of the graph and to find the largest t-value as there is another intersection point (at t = 24.6420) which is quite close to the correct one.

C2

c. $d_2(t)$ will have turning points when $d_2'(t)=0$ that is when

$$\frac{4\pi}{25}\cos\left(\frac{\pi t}{168}\right)\cos\left(\frac{4\pi t}{25}\right)-\frac{\pi}{168}\sin\left(\frac{\pi t}{168}\right)\sin\left(\frac{4\pi t}{25}\right)=0$$

A1

$$\frac{4\pi}{25}-\frac{\pi}{168}\frac{\sin\left(\frac{\pi t}{168}\right)\sin\left(\frac{4\pi t}{25}\right)}{\cos\left(\frac{\pi t}{168}\right)\cos\left(\frac{4\pi t}{25}\right)}=0$$

$$\frac{4\pi}{25}-\frac{\pi}{168}\tan\left(\frac{\pi t}{168}\right)\tan\left(\frac{4\pi t}{25}\right)=0$$

$$\tan\left(\frac{\pi t}{168}\right)\tan\left(\frac{4\pi t}{25}\right)=\frac{4\pi}{25}\times\frac{168}{\pi}$$

$$\tan\left(\frac{\pi t}{168}\right)\tan\left(\frac{4\pi t}{25}\right)=\frac{672}{25}$$

B2

Summary of mark allocation per Outcome

Question	Part	Outcome 1	Outcome 2	Outcome 3
1	a	3		
	b	2		
	c	1	1	
	d	2	2	
2	a	2	3	
	b	1	1	2
3	a	1		
	b	1		
	c		2	2
4	a		1	1
	b		2	
	c			2
5	a		2	
	b i			1
	b ii			2
	c	1	2	
Raw Marks		14	16	10
Adjusted Marks		7	8	5

