MATHEMATICAL METHODS (CAS)

Unit 3
Targeted Evaluation Task for School-assessed Coursework 2

2012 Test (multiple choice, short answer, extended response) on Differentiation for Outcomes 1 & 3

SOLUTIONS & RESPONSE GUIDE

SECTION 1- Short-answer Questions

Question 1

a. $f'(x) = 6x^2 + 2x - 4$

(1 mark)

b. Let $f'(x) = 0$

(1 mark)

$$2(3x^2 + x - 2) = 0$$
$$2(3x - 2)(x + 1) = 0$$
$$x = \frac{2}{3}, \ -1$$

(1 mark)

c. $\{x : x < -1\} \cup \{x : x > \frac{2}{3}\}$

(1 mark)

Question 2

a. Gradient, $m_T = \frac{dy}{dx} = e^{-2x} - 2xe^{-2x}$

(1 mark)

When $x = 1$, $m_T = e^{-2} - 2e^{-2} = -e^{-2} = \frac{-1}{e^2}$

(1 mark)

b. Gradient of normal, $m_N = \frac{-1}{m_T} = e^2$

(1 mark)

c. When $x = 1$, $\mathrm{y} = e^{-2}$

(1 mark)

$$y - e^{-2} = e^2(x - 1)$$
$$y = e^2 x - e^2 + e^{-2}$$

(1 mark)

Question 3

a. $f'(t) = \frac{2x}{3} + \frac{1}{8}\cos\left(\frac{t}{8}\right)$

(1 mark)

$$f'(5) = \frac{10}{3} + \frac{1}{8}\cos(0.625) = 3.435\,\text{ms}^{-1}$$

(1 mark)

b. Average velocity $= \frac{f(10) - f(5)}{10 - 5}$

(1 mark)

Average velocity $= \frac{\left(\frac{100}{3} + \sin 1.25\right) - \left(\frac{25}{3} + \sin 0.625\right)}{5} = 5.073\,\text{ms}^{-1}$

(1 mark)

Question 4

$\frac{dy}{dx} = 2ax - 4a$

y is a minimum when $\frac{dy}{dx} = 0,$ that is when $2ax - 4a = 0$

$\therefore 2a(x-2) = 0 \Rightarrow x = 2$

1 mark

When $x = 2,\ \text{y} = 4a - 8a + 20 = 20 - 4a$

1 mark

SECTION 2 Multiple-choice questions (1 mark each)

Question 1

Answer: C

Explanation

The gradient of the tangent of a curve at a particular point gives the gradient of the curve at that point. In function terminology the gradient at $x = a$ is given by $f'(a)$ or as a limit it is given by $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

Question 2

Answer: A

Explanation

$$\lim_{x \to 1} \frac{(3x+1)(x-1)}{x^2 e^x - xe^x} = \lim_{x \to 1} \frac{(3x+1)(x-1)}{xe^x (x-1)} = \lim_{x \to 1} \frac{3x+1}{xe^x} = \frac{4}{e} = 4e^{-1}$$

Question 3

Answer: B

Explanation

$\frac{d(\log_e(3x))}{dx} = \frac{3}{3x} = \frac{1}{x}$ using the chain rule.

Question 4

Answer: E

Explanation

Using the quotient rule with $u = e^{3x}$ and $v = x^2$

$$\frac{dy}{dx} = \frac{3x^2 e^{3x} - 2xe^{3x}}{x^4} = \frac{xe^{3x}(3x-2)}{x^4} = \frac{e^{3x}(3x-2)}{x^3}$$

Question 5

Answer: D

Explanation

Using the chain rule with $f(x) = 6u^{\frac{1}{2}}$ and $u = 3 - x^2 \Rightarrow \frac{du}{dx} = -2x$

$$f'(x) = 6 \times -2x \times \frac{1}{2} u^{-\frac{1}{2}} = \frac{-6x}{u^{\frac{1}{2}}} = \frac{-6x}{\sqrt{3 - x^2}}$$

Question 6

Answer: D

Explanation

$\frac{dy}{dx} = 10x - 8$

Gradient at $x = 1$ is $10 \times 1 - 8 = 2$

When $x = 1$, $y = 5 \times 1 - 8 \times 1 + 2 = -1$

$y - -1 = 2(x - 1)$

$y = 2x - 3$

Question 7

Answer: C

Explanation

$f'(x) = 0$ at the maximum turning point, $x = -2$ **and** the stationary point of inflection, $x = 3$. Because $x = -2$ is a maximum, $f(x)$ will be increasing to the left of $x = -2$ and decreasing to the right of $x = -2$ up to $x = 3$. Also the change in $f(x)$ will be the same (that is, decreasing) on both sides of the stationary point of inflection.

Question 8

Answer: B

Explanation

Using the product rule with $u = x^2$ and $v = \cos 4x$

$$\frac{d}{dx}(x^2 \cos 4x) = 2x \cos 4x + x^2 \times -4 \sin 4x$$

$$= 2x(\cos 4x - 2x \sin 4x)$$

Question 9

*Answer:*D

Explanation

Graph the function on a graphics calculator and use the gradient calculation function.

Question 10

*Answer:*E

Explanation

Graph the function on a graphics calculator and use the minimum calculation function.

SECTION 3- Analysis Questions

Question 1

a.

i. $V = \dfrac{\pi r^2 l}{2} = 62500\pi$

$l = \dfrac{125000}{r^2}$

(1 mark)

ii. $A = \dfrac{2\pi r^2 + 2\pi rl}{2} = \pi r^2 + \pi rl$

$A = \pi r^2 + \dfrac{\pi r \times 125000}{r^2}$

(1 mark)

$A = \pi r^2 + \dfrac{125000\pi}{r}$

(1 mark)

iii. $\dfrac{dA}{dr} = 2\pi r - \dfrac{125000\pi}{r^2} = 0$ for minimum area

(1 mark)

$2\pi r = \dfrac{125000\pi}{r^2}$

$r^3 = 62500$

$r = \sqrt[3]{62500}$

(1 mark)

iv. Substituting $r = \sqrt[3]{62500} \approx 39.685$ into the area function gives $A = 14843$ cm^2.

(1 mark)

v. $r = \sqrt{\dfrac{125000}{l}}$

When $l = 75,\ r = \sqrt{\dfrac{125000}{75}} = 40.8248$

(1 mark)

$\Rightarrow \dfrac{dA}{dr} = 2\pi \times 40.8248 - \dfrac{125000\pi}{40.8248^2} = 20.89$

(1 mark)

b.

i. $A = \pi r^2 + \pi r l = 20000$

$$l = \frac{20000 - \pi r^2}{\pi r}$$

(1 mark)

ii. $V = \dfrac{\pi r^2 l}{2} = \dfrac{\pi r^2 (20000 - \pi r^2)}{2\pi r}$

(1 mark)

$$V = \frac{r(20000 - \pi r^2)}{2} = \frac{20000r - \pi r^3}{2}$$

(1 mark)

iii. From the expression for l in part **i**, $20000 - \pi r^2 \geq 0$ as l cannot be negative.

$\pi r^2 \leq 20000$

$$r \leq \sqrt{\frac{20000}{\pi}}$$

$r \leq 79.79$ cm

(1 mark)

iv. $\dfrac{dV}{dr} = \dfrac{20000 - 3\pi r^2}{2} = 0$ for maximum volume

(1 mark)

$3\pi r^2 = 20000$

$$r = \sqrt{\frac{20000}{3\pi}}$$

(1 mark)

v. Substituting $r = \sqrt{\dfrac{20000}{3\pi}} \approx 46.0659$ into the volume function gives $V = 307106$ cm^3.

(1 mark)