Neap:

Trial Examination 2012

VCE Mathematical Methods (CAS) Units 3 & 4

Written Examination 2

Suggested Solutions

SECTION 1

1	A	B	**C**	D	E
2	A	B	C	**D**	E
3	A	B	**C**	D	E
4	**A**	B	C	D	E
5	A	B	C	D	**E**
6	A	**B**	C	D	E
7	A	B	C	D	**E**
8	A	B	**C**	D	E
9	A	B	**C**	D	E
10	**A**	B	C	D	E
11	A	B	C	**D**	E
12	A	B	C	**D**	E
13	A	B	C	D	**E**
14	A	**B**	C	D	E
15	A	**B**	C	D	E
16	A	B	C	**D**	E
17	**A**	B	C	D	E
18	**A**	B	C	D	E
19	A	B	**C**	D	E
20	A	B	C	D	**E**
21	A	**B**	C	D	E
22	A	B	C	**D**	E

SECTION 1

Question 1 C

The amplitude is 2.

The period is $\left(\frac{2\pi}{\frac{1}{4}}\right) = 8\pi$.

Question 2 D

$g(x+y) = g(x)g(y)$ for $g(x) = e^x$

$$\begin{aligned} g(x+y) &= e^{x+y} \\ &= e^x \times e^y \\ &= g(x)g(y) \end{aligned}$$

Question 3 C

$-1 \le \sin(x) \le 1$ and so $-\frac{3}{2} \le \sin(x) - \frac{1}{2} \le \frac{1}{2}$

Given that $0 \le \left|\sin(x) - \frac{1}{2}\right| \le \frac{3}{2}$, the maximum value of g is $\frac{3}{2}$.

Question 4 A

$A = \pi r^2$ and so $\frac{dA}{dr} = 2\pi r$

$\frac{dA}{dt} = -2\pi r \frac{dr}{dt}$ (using the chain rule, i.e. $\frac{dA}{dt} = \frac{dA}{dr}\frac{dr}{dt}$)

We are given that $\frac{dA}{dt} = -2\frac{dr}{dt}$.

So $-2\pi r \frac{dr}{dt} = -2\frac{dr}{dt}$ and hence $r = \frac{1}{\pi}$.

Question 5 E

We require $x^2 - 8 > 0$.

Solving this inequality we obtain $x < -2\sqrt{2}$ or $x > 2\sqrt{2}$.

So $|x| > 2\sqrt{2}$.

Question 6 **B**

Multiplying both sides of the equation $px - 2y = 0$ by 3 we obtain $3px - 6y = 0$.

Multiplying both sides of the equation $3x - (p + 1)y = 0$ by p we obtain $3px - p(p + 1)y = 0$.

A unique solution occurs for values of p such that $p(p + 1) \neq 6$ (equating the coefficients of y).

Solving $p(p + 1) \neq 6$ for p gives $p \neq -3$ and $p \neq 2$.

So $p = R \backslash \{-3, 2\}$.

Question 7 **E**

Using definite integral properties:

$$\int_3^9 g(x)dx = -\int_9^3 g(x)dx$$

$$\begin{aligned}\int_1^3 g(x)dx &= \int_1^9 g(x)dx - \int_3^9 g(x)dx \\ &= 4 - (-6) \\ &= 10\end{aligned}$$

Question 8 **C**

$$\Pr(X > \mu + 3) = \Pr\left(Z > \frac{\mu + 3 - \mu}{2}\right)$$

Simplifying we obtain $\Pr\left(Z > \frac{3}{2}\right)$.

From symmetry, $\Pr\left(Z > \frac{3}{2}\right) = \Pr\left(Z < -\frac{3}{2}\right)$.

Question 9 **C**

The graphs of $y = h(x)$ and $y = h(|x|)$ must be symmetric about the line $x = 0$, i.e. about the y-axis.

Hence we can disregard options **A**, **B** and **D**.

The graphs of $y = h(x)$ and $y = h(|x|)$ must be identical for $x > 0$.

Question 10 **A**

A continuous random variable is one that can take any value in an interval of the real number line.
So ii, iii and v are continuous random variables.

Question 11 D

Given $p(x) = (x+2)^2(x-k)$

When $p(x)$ is divided by $x-1$ the remainder is 36, and so $p(1) = 36$.

Solving $p(1) = 36$ for k we obtain $k = -3$.

Question 12 D

Solving $e^{2x} = b$ for x gives $x = \frac{1}{2}\log_e(b)$ and $b > 0$.

This solution can be re-expressed as $x = \log_e(\sqrt{b})$.

So $\sqrt{b} = 3$ and hence $b = 9$.

Question 13 E

$y' = 1 - 4g(2x' + 3)$

Rearranging we obtain $\frac{y'-1}{-4} = g(2x'+3)$.

Hence $y = \frac{y'-1}{-4}$ and $y' = -4y + 1$.

Hence $x = 2x' + 3$ and $x' = \frac{x}{2} - \frac{3}{2}$.

So $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix}$.

Question 14 B

At the point(s) of intersection, $x^2 + 5x = kx - 1$, i.e. $x^2 + (5-k)x + 1 = 0$.

Two distinct intersection points will occur when $\Delta > 0$.

Solving $(5-k)^2 - 4 > 0$ for k gives $k < 3$ or $k > 7$.

Question 15 B

$h'(x) = f'(g(x))g'(x)$ (use of chain rule)

$h'(1) = f'(g(1))g'(1)$ (substituting $x = 1$ into the above derivative)

$= f'(4)g'(1)$ (as $g(1) = 4$)

$= -8 \times -6$ (as $f'(4) = -8$ and $g'(1) = -6$)

So $h'(1) = 48$

Question 16 D

$\frac{dy}{dx} = 3x^2 - 12x$

Solving $3x^2 - 12x = 0$ for x we obtain $x = 0, 4$.

The graph of $y = x^3 - 3x^2 + d$ has a local maximum at $(0, d)$ and a local minimum at $(4, d - 32)$.

The graph will have three distinct x-intercepts if the local maximum and the local minimum are located above and below the x-axis.

Hence $d - 32 < 0 < d$, i.e. $0 < d < 32$.

Question 17 A

$\mathrm{E}(X) = 8$ and $\mathrm{sd}(X) = 8$

$\mathrm{var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$

$8^2 = \mathrm{E}(X^2) - 8^2$, and rearranging we obtain $\mathrm{E}(X^2) = 8^2 + 8^2$.

So $\mathrm{E}(X^2) = 128$

Question 18 A

Let $T = \begin{bmatrix} 0.75 & 0.20 \\ 0.25 & 0.80 \end{bmatrix}$ and $S_0 = \begin{bmatrix} 800 \\ 800 \end{bmatrix}$.

On Wednesday, the number of students that eat an apple or an orange is given by $T^2 S_0$.

$$T^2 S_0 = \begin{bmatrix} 0.75 & 0.20 \\ 0.25 & 0.80 \end{bmatrix} \begin{bmatrix} 800 \\ 800 \end{bmatrix}$$

$$= \begin{bmatrix} 738 \\ 862 \end{bmatrix}$$

Hence, 738 students will eat an apple on Wednesday morning.

Question 19 C

$\Pr(X = 0) = \binom{n}{0} p^0 (1 - p)^{n-1}$

As $\binom{n}{0}$ and p^0 both equal 1, and given that $\Pr(X = 0) = k$, we obtain $k = (1 - p)^n$.

$\Pr(X = 1) = \binom{n}{1} p^1 (1 - p)^{n-1}$

Simplifying the RHS, we obtain $\Pr(X = 1) = \frac{np(1 - p)^n}{1 - p}$.

As $k = (1 - p)^n$, $\Pr(X = 1) = \frac{npk}{1 - p}$.

Question 20 E

If $x = -4$ is a vertical asymptote, then $p = 4$.

$y = \dfrac{mx + n}{x + 4}$ can be re-expressed as $y = m - \dfrac{4m - n}{x + 4}$.

If $y = 3$ is a horizontal asymptote, then $m = 3$.

So $m + p = 7$

Question 21 B

The graph of $y = x^2 - 9$ changes from positive to negative at $x = -3$, and from negative to positive at $x = 3$. Since $w(x) < 0$, v has a local minimum at $x = -3$ and a local maximum at $x = 3$.

Question 22 D

The graphs $y = g(x)$ and $y = h(x)$ do not need to intersect.

However, if the two graphs do intersect, they will only intersect once, as $g'(x) > h'(x)$.

SECTION 2

Question 1

a. Many approaches could be used.

The minimum value of $\cos\left(\frac{\pi t}{12}\right)$ is –1 and occurs when $t = 12$. A1

So the maximum temperature is $21 - 4(-1)$°C, i.e. 25°C, and occurs at midday. A1

b. 2 pm corresponds to $t = 14$, and 10 pm corresponds to $t = 22$.

$\frac{T(22) - T(14)}{22 - 14} = -\frac{\sqrt{3}}{2}$ (using average rate of change = $\frac{T(t_2) - T(t_1)}{t_2 - t_1}$) M1

So the exact average rate of decrease in temperature is $\frac{\sqrt{3}}{2}$ °C per hour. A1

c. The average temperature over the 24-hour period is given by $\frac{1}{2}(T_{\max} + T_{\min})$.

So the average temperature is $\frac{1}{2}(25 + 17)$ °C, i.e. 21°C. A1

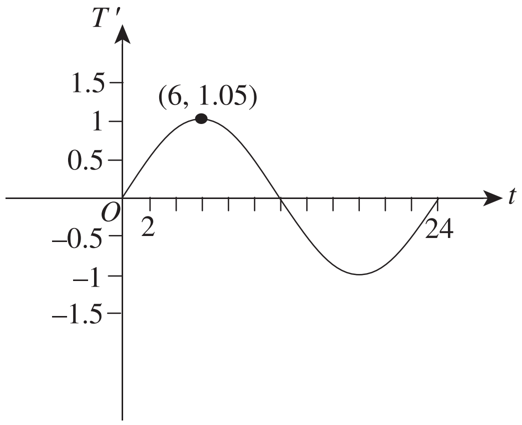
d. The average temperature between 8 am and 4 pm is given by

$\frac{1}{16-8}\int_8^{16}\left(21 - 4\cos\left(\frac{\pi t}{12}\right)\right)dt$ (using $\frac{1}{b-a}\int_a^b f(t)dt$) M1

$\frac{1}{16-8}\int_8^{16}\left(21 - 4\cos\left(\frac{\pi t}{12}\right)\right)dt = 24.3$ (°C) M1 A1

So the average temperature between 8 am and 4 pm, correct to one decimal place, is 24.3°C.

e. Attempting an appropriate graphical approach, e.g. graphing $y = T'(t)$. M1



$T'(t)$ is a maximum at $t = 6$, i.e. the temperature is increasing most rapidly at 6 am. A1

$T'(6) = 1.05$, i.e. the maximum rate of increase in temperature at 6 am is 1.05°C per hour, correct to two decimal places. A1

f. A calculus-based or a graphical approach could be used here.

Method 1: Calculus

$D(t) = W(t) - T(t)$

$D(t) = -\frac{1}{18}(t-12)^2 + 25 - \left(21 - 4\cos\left(\frac{\pi t}{12}\right)\right)$

$D'(t) = -\frac{\pi}{3}\sin\left(\frac{\pi t}{12}\right) - \frac{t}{9} + \frac{4}{3}$ A1

Solving $D'(t) = 0$ for *t*, i.e. solving $-\frac{\pi}{3}\sin\left(\frac{\pi t}{12}\right) - \frac{t}{9} + \frac{4}{3} = 0$ for *t* with $0 \le t \le 24$, gives

$t = 3.928...$, 12 and 20.071... (hours). M1

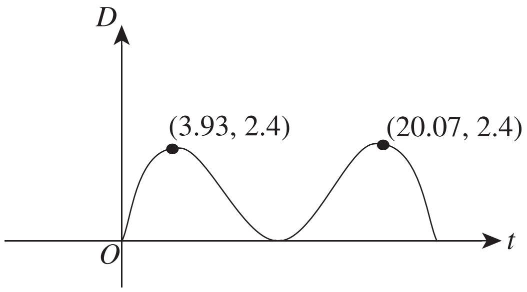
As $D(12) = 0$, we reject $t = 12$ and so $t = 3.93$ and $t = 20.07$ (hours) (correct to two decimal places). A1

Substituting these two values we obtain 2.4°C (correct to one decimal place). A1

Method 2: Graphical

Graph $y = D(t)$ for $0 \le t \le 24$.

A correct sketch. A1



So $t = 3.93$ and $t = 20.07$ (hours) (correct to two decimal places). M1 A1

Substituting these two values, we obtain 2.4°C (correct to one decimal place). A1

Question 2

a. The length of pipeline under water between *P* and *X* is $\sqrt{x^2 + 25}$ (km) and so the cost is $\sqrt{x^2 + 25}$ million dollars.

The length of pipeline over land between *X* and *R* is $8 - x$ (km) and so the cost is $\frac{3}{4}(8 - x)$ million dollars.

Hence, $C(x) = \sqrt{x^2 + 25} + \frac{3(8 - x)}{4}$. A1

b. $C(8) = \sqrt{89}$

So to the nearest ten thousand dollars, the cost of laying pipeline directly from *P* to *S* to *R* is $9 430 000. A1

c. $C(0) = 11$

So the cost of laying pipeline directly from *P* to *R* is $11 000 000. A1

d. $C'(x) = \dfrac{x}{\sqrt{x^2+25}} - \dfrac{3}{4}$ A1

e. Solving $\dfrac{x}{\sqrt{x^2+25}} - \dfrac{3}{4} = 0$ for x with $0 \le x \le 8$ gives $x = \dfrac{15}{\sqrt{7}}$ (km). M1 A1

So X must be located $\dfrac{15}{\sqrt{7}}$ km from S in the direction of R.

f. $C\left(\dfrac{15}{\sqrt{7}}\right) = \dfrac{5\sqrt{7}}{4} + 6$

So to the nearest ten thousand dollars, the minimum cost of laying pipeline from P to X to R is \$9 310 000. A1

g. If the cost over land is T million dollars per km, then the cost under water is kT million dollars per km.

$C(x) = T(k\sqrt{x^2+25} + (8-x))$ where k is the relative cost. A1

$C'(x) = T\left(\dfrac{kx}{\sqrt{x^2+25}} - 1\right)$

For a minimum we require x such that $C'(x) = 0$.

Solving $\dfrac{kx}{\sqrt{x^2+25}} - 1 = 0$ for x gives $x = \dfrac{5}{\sqrt{k^2-1}}$. (Note that $T \neq 0$.) M1

Solving $\dfrac{5}{\sqrt{k^2-1}} = 8$ for k gives $k = \dfrac{\sqrt{89}}{8}$. M1

Hence, the direct route from P to R is least expensive for $1 < k \le \dfrac{\sqrt{89}}{8}$. A1

Question 3

a. $X \sim \mathrm{N}(59.5, 3^2)$

i. $\Pr(59.5 \le X \le 63) = 0.3783$ (correct to four decimal places). A1

ii. $\Pr(X \ge 63 | X \ge 59.5) = \dfrac{\Pr(X \ge 63 \cap X \ge 59.5)}{\Pr(X \ge 59.5)}$ M1

$= \dfrac{\Pr(X \ge 63)}{\Pr(X \ge 59.5)}$ A1

$= 0.2433$ (correct to four decimal places) A1

b. $\Pr(X \le x) = 0.25$ M1

$x = 57.5$ (correct to the nearest tenth of a metre). A1

c. $(\Pr(59.5 \le X \le 63))^5 = 0.1483$ M1 A1

d. $X \sim N(59.5, 3^2)$ and $Y \sim N(60.5, 1.9^2)$

i. Petra: $\Pr(X > 63) = 0.1217$ (correct to four decimal places) A1

Louise: $\Pr(Y > 63) = 0.0941$ (correct to four decimal places) A1

Hence Petra is more likely to qualify on the first throw. A1

Note: Only award the final A1 if the first A1 has been awarded.

ii. Recognising binomial situation with $n = 5$ M1

Let P represent the number of Petra's throws that measure longer than 63 metres.

$P \sim \text{Bi}(5, 0.1216...)$

$\Pr(P \geq 1) = 0.4772...$ A1

Let L represent the number of Louise's throws that measure longer than 63 metres.

$L \sim \text{Bi}(5, 0.0941...)$

$\Pr(L \geq 1) = 0.3899...$ A1

Using $\Pr(P \geq 1) \times \Pr(L = 0) + \Pr(L \geq 1) \times \Pr(P = 0)$ M1

So Pr(only one of Petra or Louise qualify) $= (0.4772... \times 0.6100...) \times (0.3899... \times 0.5227...)$

$= 0.4950$ (correct to four decimal places) A1

Question 4

a. Method 1:

For f to be defined we require $(x-1)^2 - k > 0$. M1

So $x > \sqrt{k} + 1$ or $x < -\sqrt{k} + 1$. A1

For $x < -2$, the maximum value of k is 9. A1

Method 2:

For f to be defined we require $(x-1)^2 - k > 0$, i.e. $(x-1)^2 > k$. M1

For $x < -2$, $(x-1)^2 > 9$. A1

Hence the maximum value of k is 9. A1

b. $m = \dfrac{-2\log_e(2) - 3 - (-3)}{5\log_e(2) - 3\log_e(2)}$ (using $m = \dfrac{y_2 - y_1}{x_2 - x_1}$)

So $m = -1$. A1

Given that $k = 8$, $f'(x) = \dfrac{2(x-1)}{x^2 - 2x - 7}$. A1

Solving $\dfrac{2(x-1)}{x^2 - 2x - 7} = -1$ for x with $x < -2$ gives $x = -3$. M1

Substituting $m = -1$ and $x = 3\log_e(2)$ into $y = mx + c$, we obtain $y = 3\log_e(2)$.

So the exact coordinates of P are $(-3, 3\log_e(2))$. A1

c. Solving $(x-1)^2 - 8 = 1$ for x we obtain $x = -2$ or $x = 4$. M1

As $x < -2$, there are no solutions to the equation $f(x) = 0$. A1

Note: The A1 can only be awarded if both $x = -2$ or $x = 4$ are obtained.

d. Method 1: Direct CAS use

Solving $f(y) = x$ for y with $x < -2$ gives $y = 1 - \sqrt{e^x + 8}$. M1 A1

Method 2: Using algebra

$f(x) = \log_e((x-1)^2 - 8)$

Rearranging $e^y = (x-1)^2 - 8$ to make x the subject we obtain $x = 1 \pm \sqrt{e^y + 8}$. M1

Since $x < -2$, $x = 1 - \sqrt{e^y + 8}$ and so $f^{-1}(x) = 1 - \sqrt{e^x + 8}$. A1

e. The domain of f^{-1} is the range of f.

So the domain of f^{-1} is $x > 0$ (correct alternative notation accepted). A1

The range of f^{-1} is the domain of f.

So the range of f^{-1} is $y < -2$ (correct alternative notation accepted). A1

f. $g^{-1}(x) = \dfrac{x^2 + 4}{2}$ A1

$g^{-1}(f^{-1}(x)) = \dfrac{(1 - \sqrt{e^x + 8})^2 + 4}{2}$ M1

$= \dfrac{1 - 2\sqrt{e^x + 8} + e^x + 8 + 4}{2}$

$= \dfrac{e^x - 2\sqrt{e^x + 8} + 13}{2}$ A1