The Mathematical Association of Victoria

MATHEMATICAL METHODS (CAS) 2012

Trial Written Examination 2 SOLUTIONS

SECTION 1

Answers:

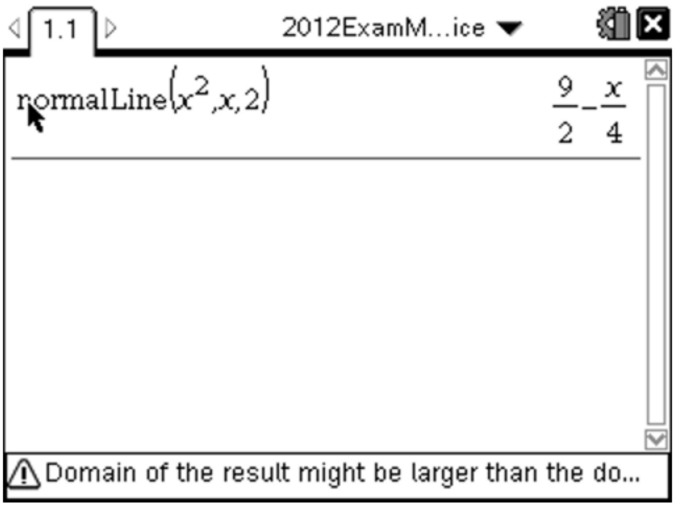
1. C 2. B 3. A 4. E 5. B 6. D 7. D 8. C 9. D 10. C 11. A

12. B 13. A 14. E 15. C 16. E 17. B 18. D 19. B 20. E 21. C 22. D

Question 1

$y=-\frac{x}{4}+\frac{9}{2}$

$x+4y=18$ **C**



OR

$g'(x)=2x$, $m_t=g'(2)=4$

The gradient of the normal $m_n=-\frac{1}{4}$, $g(2)=4$

$y=-\frac{x}{4}+c$, at (2, 4), $4=-\frac{1}{2}+c$, $c=\frac{9}{2}$

$y=-\frac{x}{4}+\frac{9}{2}$

$x+4y=18$ **C**

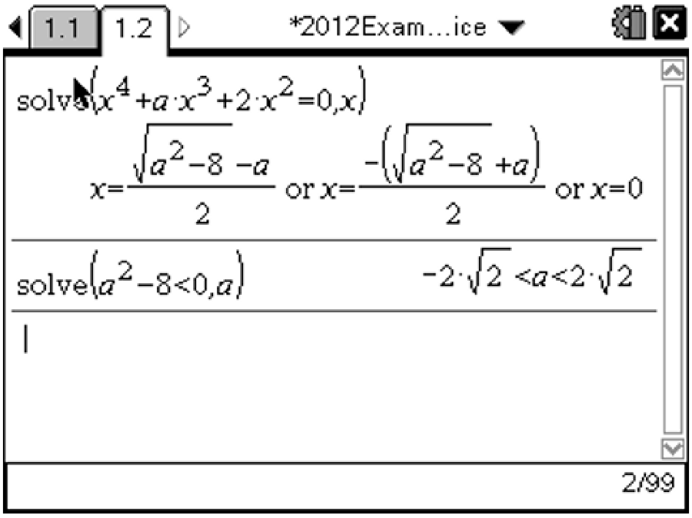
Question 2

$x^4 + ax^3 + 2x^2 = 0$

$x = 0$ or $x = \dfrac{\sqrt{a^2 - 8} - a}{2}$ or $x = \dfrac{-\sqrt{a^2 - 8} - a}{2}$

Solve $a^2 - 8 < 0$ so that $x = 0$ is the only solution

$-2\sqrt{2} < a < 2\sqrt{2}$ **B**

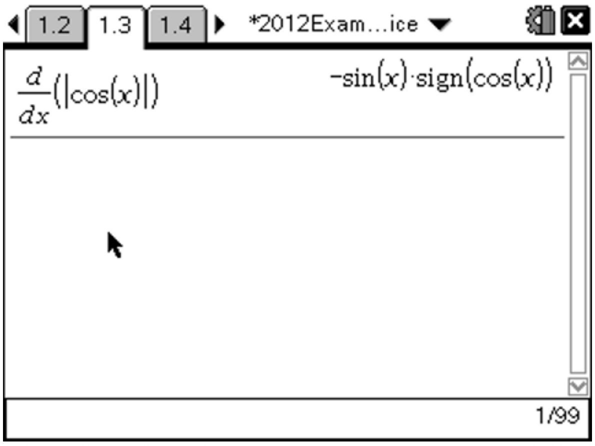


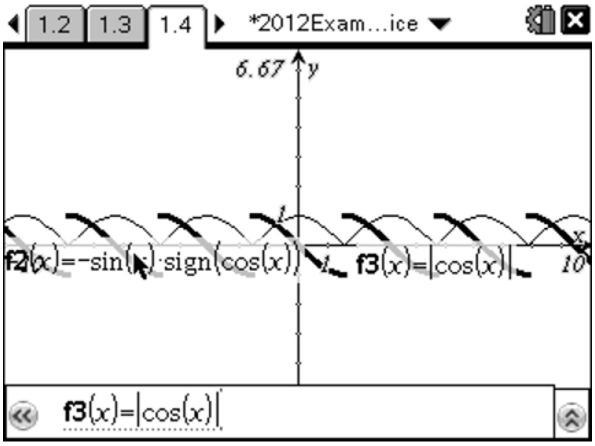
Question 3

$f(x) = |\cos(x)|$

The graph of f has a sharp point when $\cos(x) = 0$.

$$f'(x) = \begin{cases} -\sin(x) \text{ when } \cos(x) > 0 \\ \text{undefined when } \cos(x) = 0 \\ \sin(x) \text{ when } \cos(x) < 0 \end{cases}$$ **A**





Question 4

Option A $f(x) = x^3 - 4x$, $f'(x) = 3x^2 - 4$, two stationary points

Option B $f(x) = x^3 - 4x + 2$, $f'(x) = 3x^2 - 4$, two stationary points

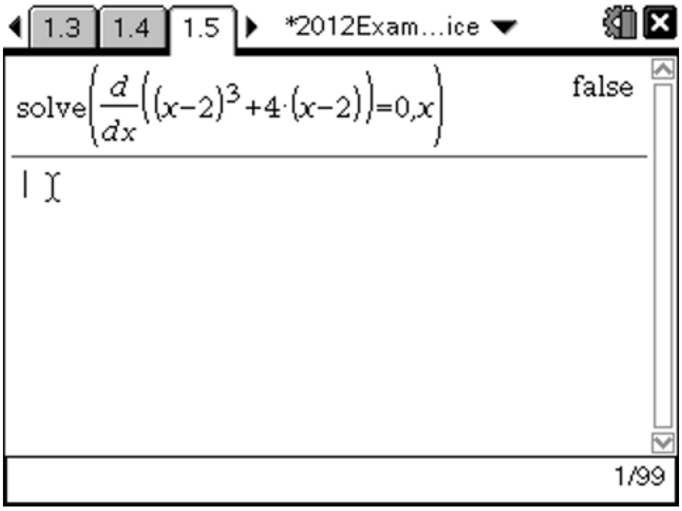
The curve in Option B is a translation of the curve in Option A 2 units up.

Option C $f(x) = (x-2)^3 - 4(x-2)$, $f'(x) = 3(x-2)^2 - 4$, two stationary points

The curve in Option C is a translation of the curve in Option A 2 units to the right.

Option D $f(x) = x^4 + 4x$, $f'(x) = 4x^3 + 4$, one stationary point

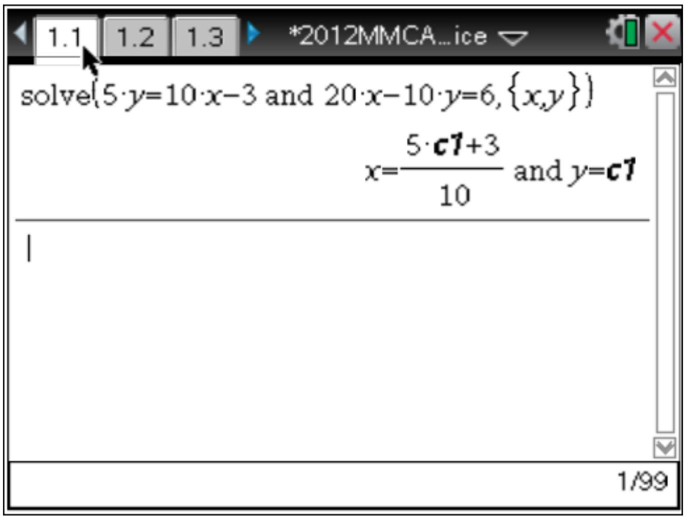
Option E $f(x) = (x-2)^3 + 4(x-2)$, $f'(x) = 3(x-2)^2 + 4$, no stationary points **E**



Question 5

The simultaneous equations $5y = 10x - 3$ and $20x - 10y = 6$ have infinitely many solutions. The equations represent the same straight line. Let $y = \lambda$, $5\lambda = 10x - 3$, $x = \dfrac{5\lambda + 3}{10}$

$\left\{\left(\dfrac{5\lambda + 3}{10}, \lambda\right) : \lambda \in R\right\}$ **B**



Question 6

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$x' = -3y$, $y = -\dfrac{x'}{3}$

$y' = 2x$, $x = \dfrac{y'}{2}$

$y = e^{2x+3}, -\dfrac{x'}{3} = e^{y'+3}$

$y = \log_e\left(-\frac{x}{3}\right) - 3$, $x < 0$ **D**

Question 7

$g(x) = 1 + 3\log_e(1-2x)$

$1 - 2x > 0$

$x < \frac{1}{2}$

The equation of the asymptote is $x = \frac{1}{2}$

The *x*-axis intercept is $\frac{1}{2}\left(1 - e^{-\frac{1}{3}}\right)$

$0 = 1 + 3\log_e(1-2x)$

$-\frac{1}{3} = \log_e(1-2x)$

$e^{-\frac{1}{3}} = 1 - 2x$

$x = \frac{1}{2}\left(1 - e^{-\frac{1}{3}}\right)$

Hence the inverse of *g* has an asymptote with equation $y = \frac{1}{2}$

and a *y*-axis intercept at $\frac{1}{2}\left(1 - e^{-\frac{1}{3}}\right)$ **D**

Question 8

The domain of $f(x) = \sqrt{ax+b}$ is $\left[-\frac{b}{a}, \infty\right)$

The domain of $g(x) = \sqrt{b-ax}$ is $\left(-\infty, \frac{b}{a}\right]$

The domain of $f + g$ is $\left[-\frac{b}{a}, \frac{b}{a}\right]$

The domain of the derivative of $f + g$ is $\left(-\frac{b}{a}, \frac{b}{a}\right)$ **C**

Question 9

Range: $5 -^{-} 1 = 6$ $\Rightarrow$ amplitude is 3

Graph is reflected in *x*-axis $\Rightarrow a = -3$

Period is 10: $\frac{2\pi}{n} = 10$

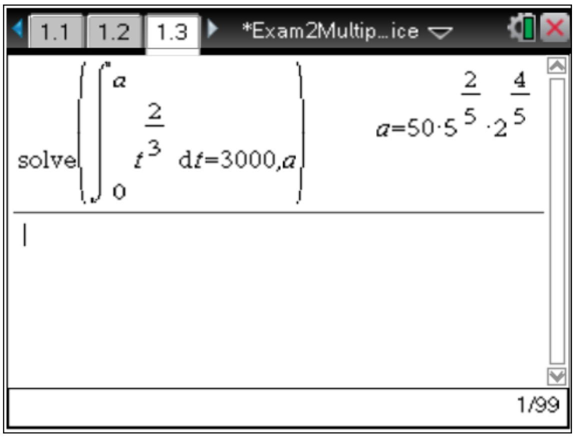
$n = \frac{\pi}{5}$

There is a vertical translation of 2 units $\Rightarrow b = 2$ **D**

Question 10

Solve $\int_0^a \left(t^{\frac{2}{3}}\right) dt = 3000$ for a

$a = 50 \times 20^{\frac{2}{5}}$ minutes **C**



OR

$$\frac{dv}{dt} = -t^{\frac{2}{3}}$$

$$v = -\int \left(t^{\frac{2}{3}}\right) dt$$

$$v = -\frac{3}{5} t^{\frac{5}{3}} + c$$

(0, 3000), $c = 3000$

$$v = -\frac{3}{5} t^{\frac{5}{3}} + 3000$$

$$0 = -\frac{3}{5} t^{\frac{5}{3}} + 3000$$

$t = (5000)^{\frac{3}{5}} = 50 \times 20^{\frac{2}{5}}$ minutes **C**

Question 11

$$f(x+h) \approx f(x) + hf'(x)$$

$x = 121$, $h = -0.1$, $f(x) = \frac{1}{\sqrt{x}}$, $f'(x) = -\frac{1}{2} x^{-\frac{3}{2}}$

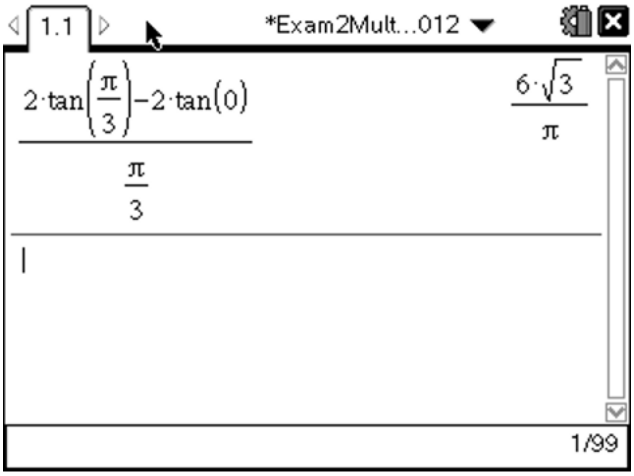
$f(x+h) \approx \frac{1}{\sqrt{121}} - \frac{1}{10} \times \frac{-1}{2(121)^{\frac{3}{2}}} = \frac{1}{\sqrt{121}} + \frac{1}{20(121)^{\frac{3}{2}}}$ **A**

Question 12

$$\text{Average rate of change } = \frac{2\tan\left(\frac{\pi}{3}\right) - 2\tan(0)}{\left(\frac{\pi}{3}\right)}$$

$$= \frac{2\sqrt{3}}{\left(\frac{\pi}{3}\right)}$$

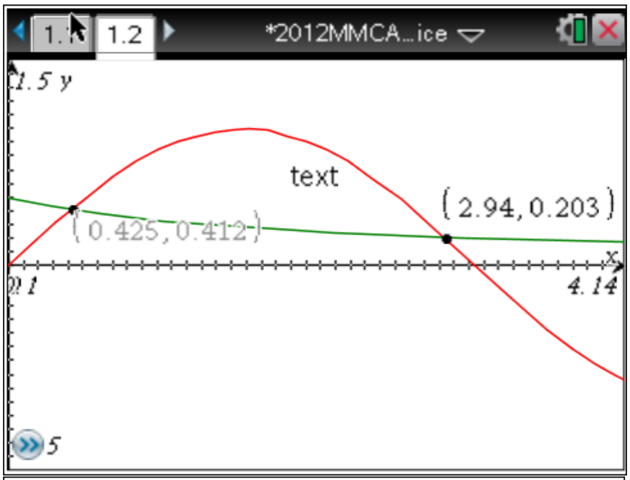
$$= \frac{6\sqrt{3}}{\pi}$$ **B**

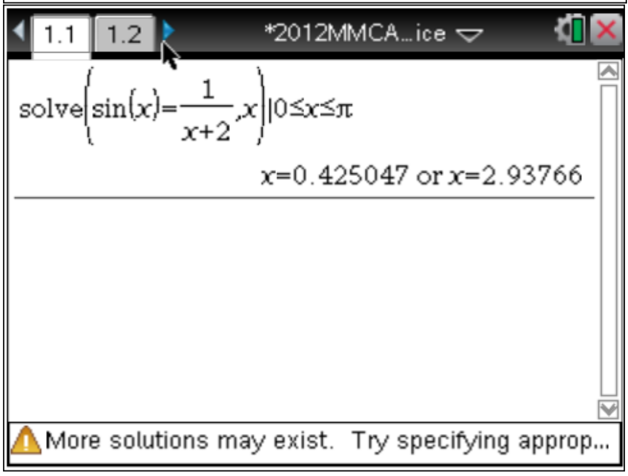


Question 13

The graphs intersect when $x = 0.425047$ and 2.93766.

Area between curves is given by $\int_{0.425}^{2.938}\left(\sin(x) - \frac{1}{x+2}\right)dx$ **A**





Question 14

$$2\int_{1}^{5}\left(f(x)+3\right)dx$$

$$=2\int_{1}^{5}\left(f(x)\right)dx+2\int_{1}^{5}(3)dx$$

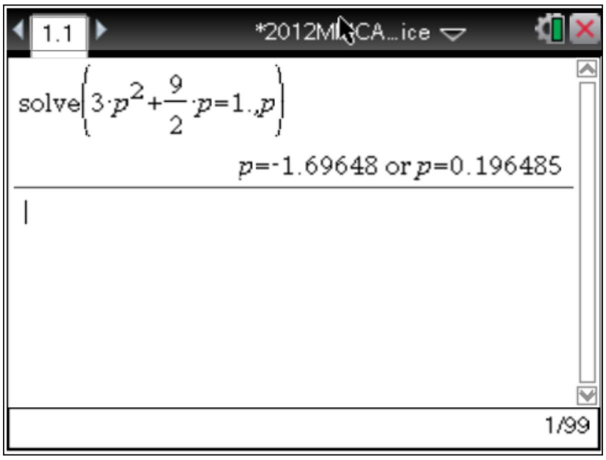
$$=2\times 6+2(15-3)$$

$$=36$$ **E**

Question 15

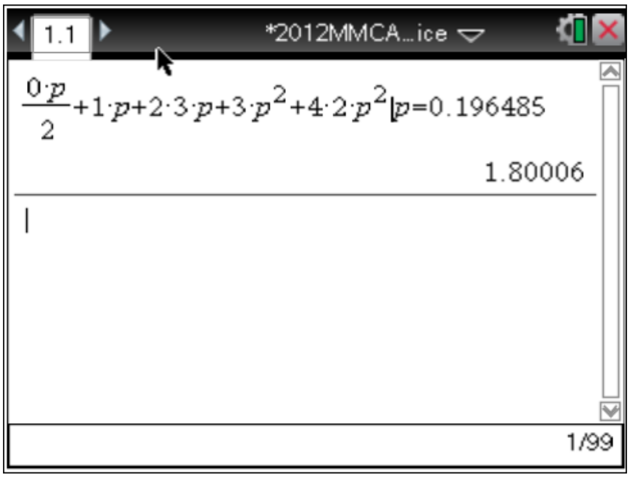
$$\frac{p}{2}+p+3p+p^2+2p^2=1$$

$$3p^2+4\frac{1}{2}p=1$$



$p = 0.196485$ (as $p \in [0,1]$)

$E(X) = 0 \times \frac{p}{2} + 1 \times p + 2 \times 3p + 3 \times p^2 + 4 \times 2p^2 = 11p^2 + 7p$, and $p = 0.196485$



$E(X) = 1.80006$

$E(X) = 1.80$ **C**

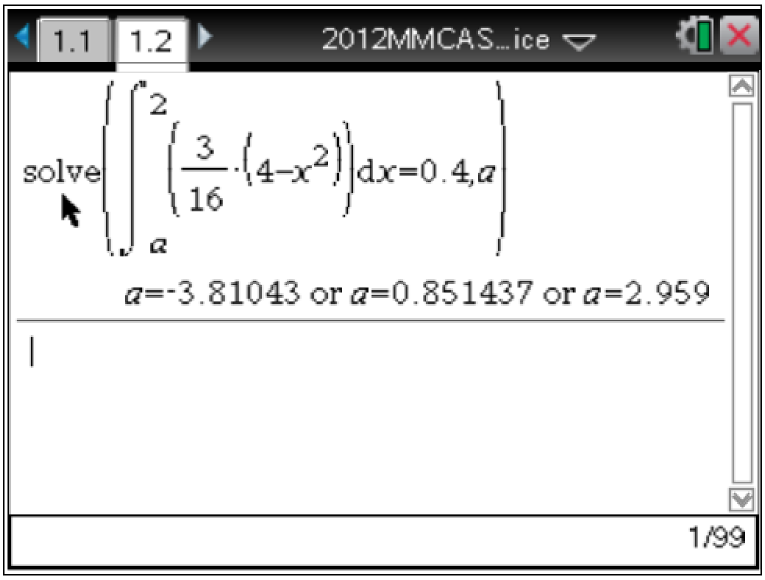
Question 16

Solve $\int_a^2 \left(\frac{3}{16}(4 - x^2)\right) dx = 0.4$ for a

$a = -3.81043, 0.851437, 2.959$

Since $0 \le a \le 2$

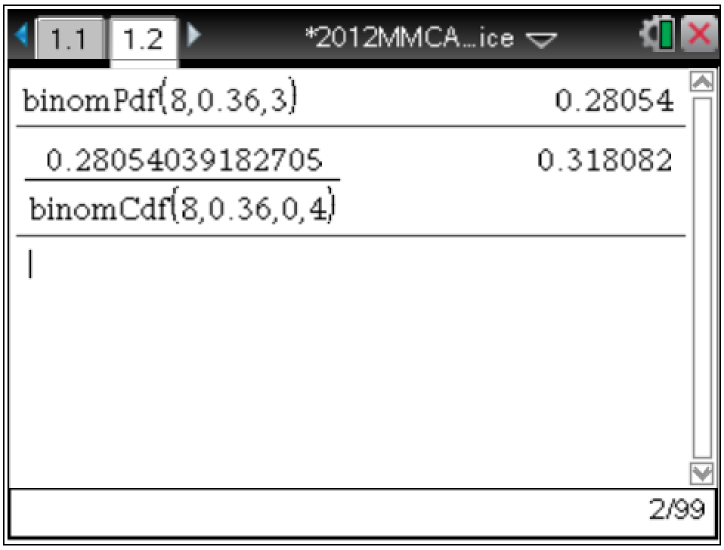
$a = 0.8514$ correct to 4 decimal places **E**



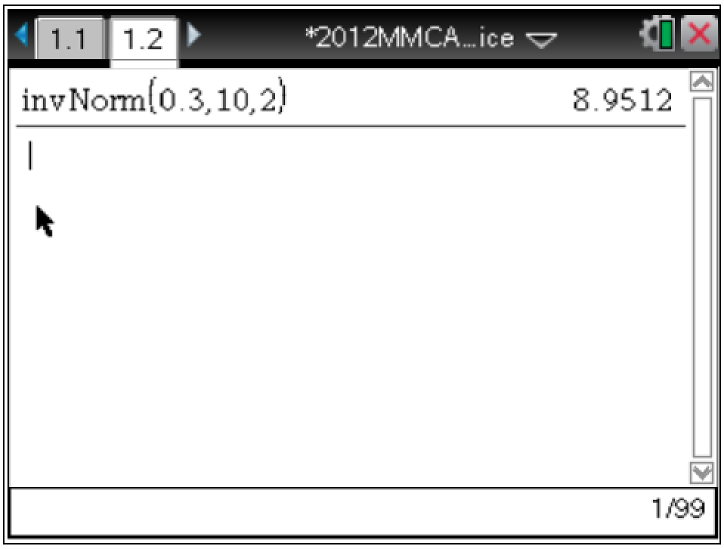
Question 17
Let X be the number of people with blue eyes out of 8.
$X \sim \text{Bi}(8, 0.36)$

$$\begin{aligned}\Pr(X = 3 \mid X < 5) &= \frac{\Pr(X = 3 \cap X < 5)}{\Pr(X < 5)} \\ &= \frac{\Pr(X = 3)}{\Pr(X < 5)} \\ &= \frac{0.28054}{0.970741} \\ &= 0.3181\end{aligned}$$

B



Question 18
$X \sim \text{N}(10, 4)$
The area under the curve to the left of a is 0.3.
A = 8.951 correct to 3 decimal places **D**



Question 19

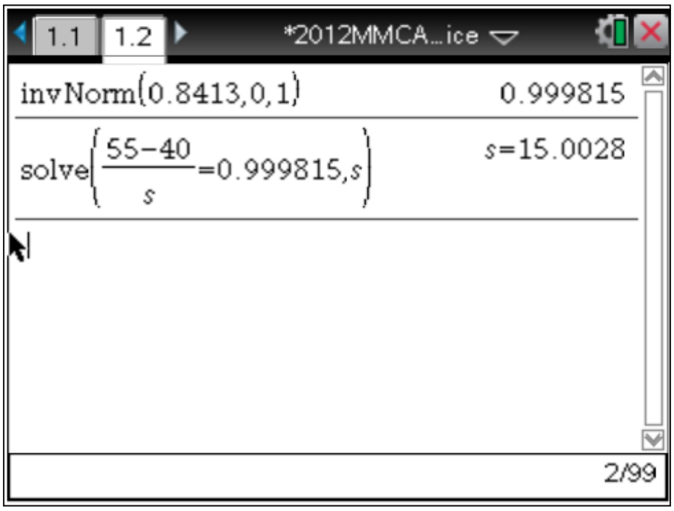
From Standard Normal Curve: $\Pr(Z < a) = 0.8413$

$a = 0.999815$

Thus $\frac{x-\mu}{\sigma} = 0.999815$

$\frac{55-40}{\sigma} = 0.999815$

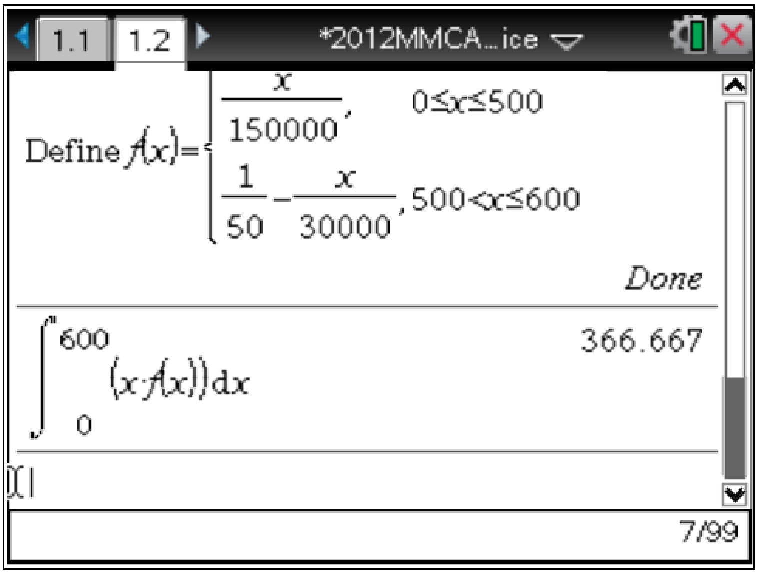
$\sigma = 15.0028 \approx 15$ **B**



Question 20

$$-\int_{600}^{0} (xf(x))dx = \int_{0}^{600} (xf(x))dx$$ **E**

Note The hybrid function can be defined on the calculator if you need to evaluate the expression.

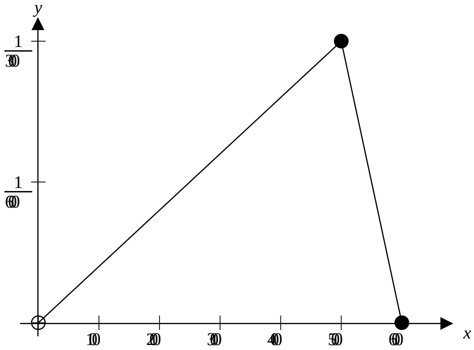


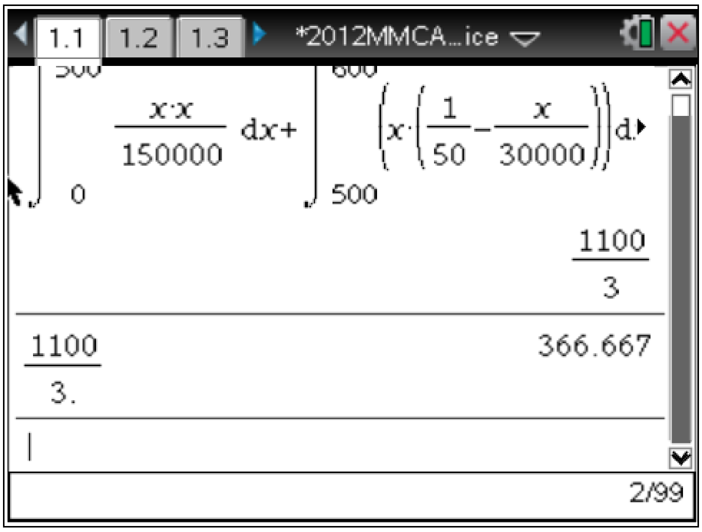
OR

$$E(X) = \int_0^{600} (x \times f(x))dx$$

$$= \int_0^{500} \left(x \times \frac{1}{150000} x \right) dx + \int_{500}^{600} \left(x \times \left(\frac{1}{50} - \frac{1}{30000} x \right) \right) dx$$

$$E(X) = \frac{1100}{3} = 366\frac{2}{3} \approx 367$$





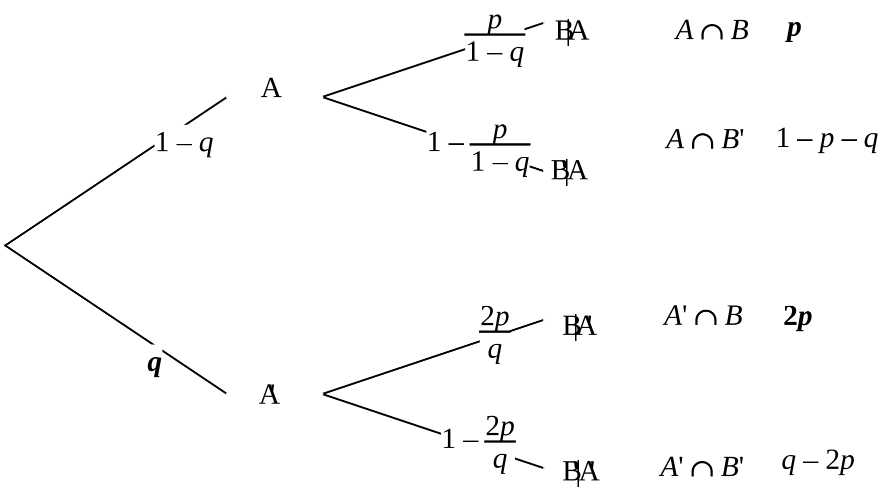
Question 21

$\Pr(A \cap B') + \Pr(A \cap B) + \Pr(A') = 1$

$\Pr(A \cap B') + p + q = 1$
$\Pr(A \cap B') = 1 - p - q$ **C**

OR

Using a tree diagram



From tree diagram: $\Pr(A \cap B') = 1 - p - q$ **C**

OR

Using a Karnaugh Map: $\Pr(A \cap B') = 1 - p - q$ **C**

	A	A'	
B	p	$2p$	$3p$
B'	$1-p-q$	$q-2p$	$1-3p$
	$1-q$	Q	1

Question 22

$V = \frac{1}{3}\pi r^2 h$

Given $2r = h \Rightarrow r = \frac{h}{2}$

$V = \frac{1}{3}\pi \frac{h^3}{4}$

$= \frac{\pi h^3}{12}$

$\frac{dV}{dh} = \frac{\pi h^2}{4}$ **D**

SECTION 2
Solutions to the Extended Answer

Question 1

a.i. $\sin(x) = \dfrac{h}{PQ}$

$PQ = \dfrac{h}{\sin(x)}$ **1A**

ii. $QR + 2 \times PQ = 20 \quad (PQ = RS)$

$QR = 20 - 2 \times \dfrac{h}{\sin(x)}$

$= 2\left(10 - \dfrac{h}{\sin(x)}\right)$ as required **1M**

b. i. $PS = QR + 2 \times \dfrac{h}{\tan(x)}$ **1M**

$= 20 - \dfrac{2h}{\sin(x)} + \dfrac{2h}{\tan(x)}$ **1A**

$= 2\left(10 - \dfrac{h}{\sin(x)} + \dfrac{h}{\tan(x)}\right)$

ii. Area $= \dfrac{PS + QR}{2} \times h$ **1M**

$= \dfrac{2\left(10 - \dfrac{h}{\sin(x)} + \dfrac{h}{\tan(x)} + 10 - \dfrac{h}{\sin(x)}\right)}{2} \times h$

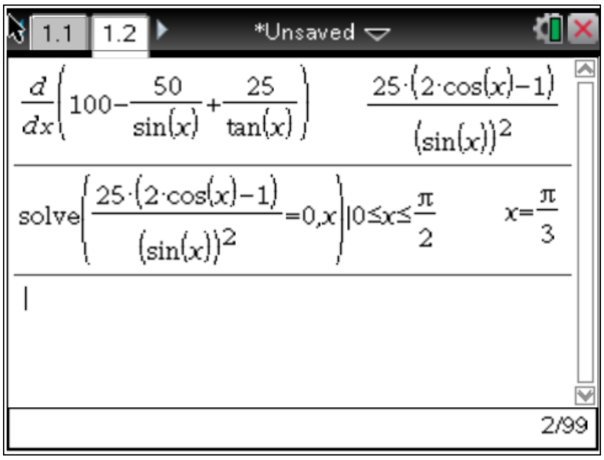
$= 20h - \dfrac{2h^2}{\sin(x)} + \dfrac{h^2}{\tan(x)}$ as required **1M**

iii. $A : \left(0, \dfrac{\pi}{2}\right) \to R, A(x) = 100 - \dfrac{50}{\sin(x)} + \dfrac{25}{\tan(x)}$ **1A**

c. $A'(x) = \dfrac{50\cos(x) - 25}{\sin^2(x)}$ **1A**

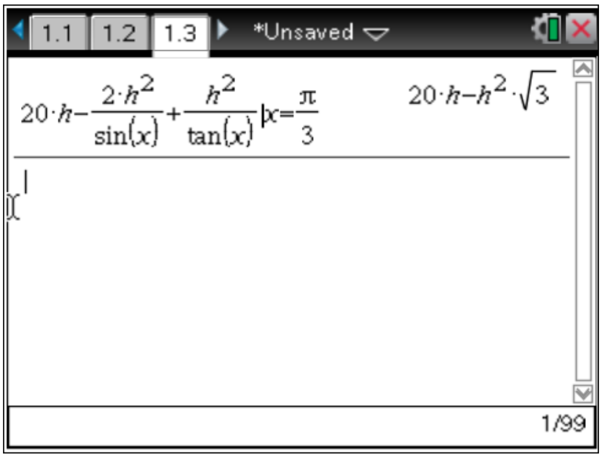
Maximum when $A'(x) = 0$

$x = \dfrac{\pi}{3}$ as required **1M**



d. i. $A\left(\dfrac{\pi}{3}\right) = 20h - 2h^2 \times \dfrac{2}{\sqrt{3}} + h^2 \times \dfrac{1}{\sqrt{3}}$

$= 20h - \sqrt{3}h^2$ **1A**

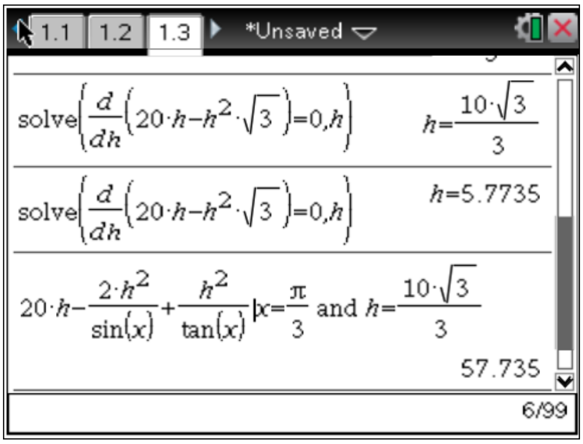


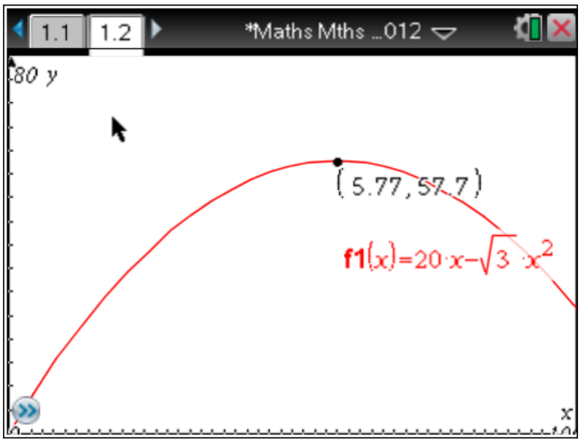
ii. Solve $\dfrac{dA}{dh} = 0$ **1M** (accept other methods)

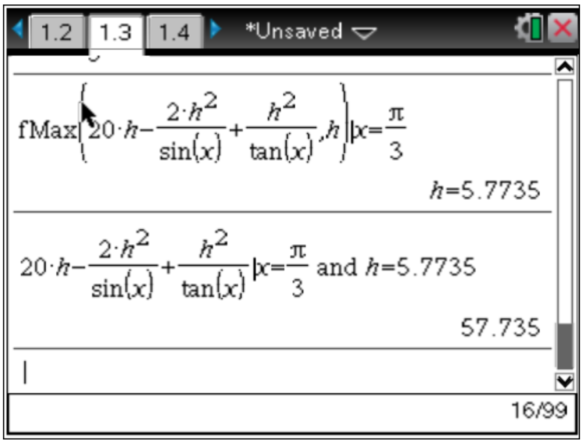
$h = 5.8$ metres **1A**

$A(5.7735) \approx 57.7$

Maximum area is 57.7 square metres **1A**


1.1 1.2 1.3 *Unsaved
solve(d/dh(20·h−h^2·√3)=0,h) h=10·√3/3
solve(d/dh(20·h−h^2·√3)=0,h) h=5.7735
20·h−2·h^2/sin(x)+h^2/tan(x)|x=π/3 and h=10·√3/3
57.735
6/99

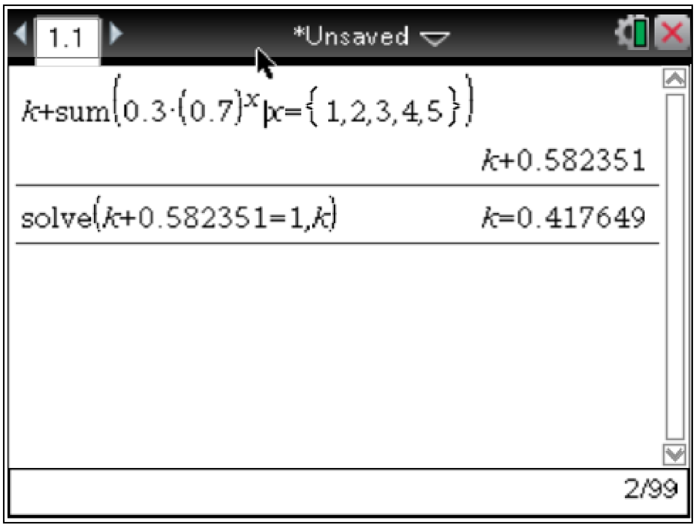

1.1 1.2 *Maths Mths ...012
80 y
(5.77, 57.7)
f1(x)=20·x−√3·x^2
x


1.2 1.3 1.4 *Unsaved
fMax(20·h−2·h^2/sin(x)+h^2/tan(x),h)|x=π/3
h=5.7735
20·h−2·h^2/sin(x)+h^2/tan(x)|x=π/3 and h=5.7735
57.735
16/99

Question 2

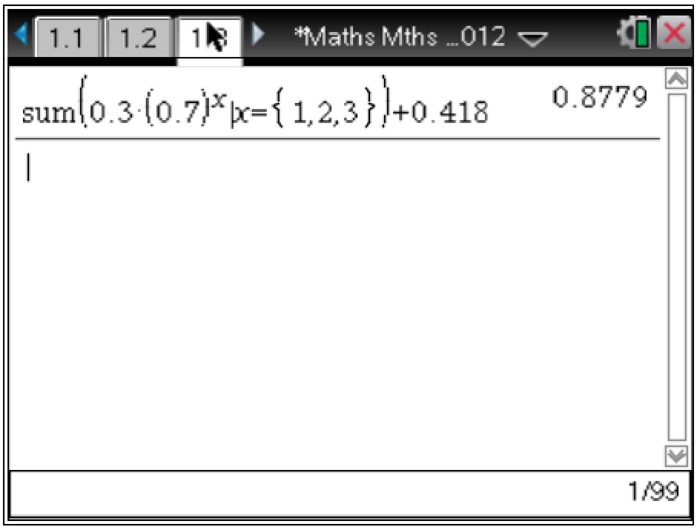
a. i. Solve $k + 0.3(0.7 + 0.7^2 + 0.7^3 + 0.7^4 + 0.7^5) = 1$ for k **1M**

$k = 0.417649 = 0.418$ correct to 3 decimal places as required **1M**



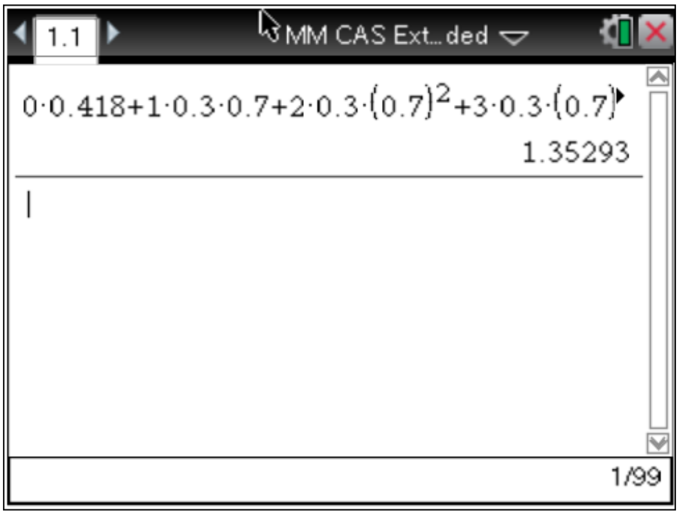
ii. $\Pr(x \le 3) \approx 0.418 + 0.3(0.7 + 0.7^2 + 0.7^3)$ **1M**

$= 0.878$ correct to 3 decimal places **1A**



b. $E(X) = 0 \times 0.418 + 1 \times 0.3 \times 0.7 + 2 \times 0.3 \times 0.7^2 + 3 \times 0.3 \times 0.7^3 + 4 \times 0.3 \times 0.7^4 + 5 \times 0.3 \times 0.7^5$

$= 1.35$ correct to 2 decimal places **1A**

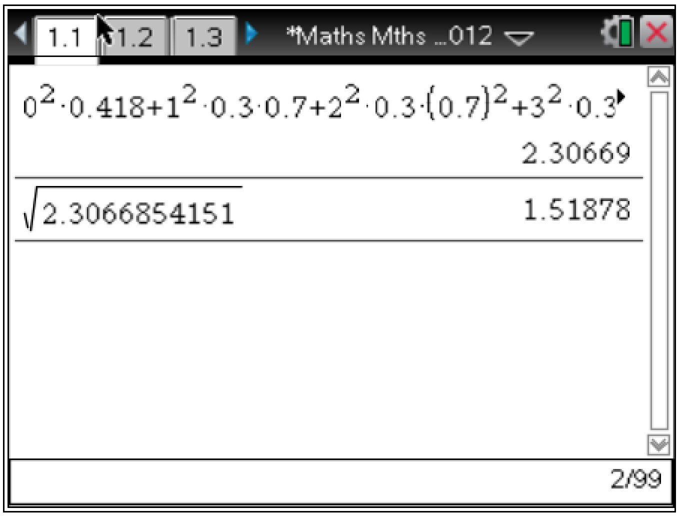


ii.

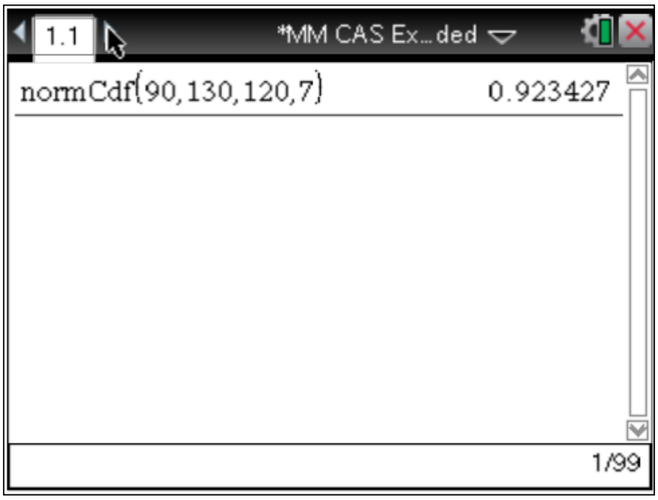
$\text{Var}(X) = (0^2 \times 0.418 + 1^2 \times 0.3 \times 0.7 + 2^2 \times 0.3 \times 0.7^2 + 3^2 \times 0.3 \times 0.7^3 + 4^2 \times 0.3 \times 0.7^4 + 5^2 \times 0.3 \times 0.7^5) - (1.3529...)^2$ **1M**

$\text{Var}(X) \approx 2.3066$

$\text{SD}(X) = \sqrt{\text{Var}(X)} \approx \sqrt{2.3067} \approx 1.5187 = 1.52$ correct to 2 decimal places **1A**

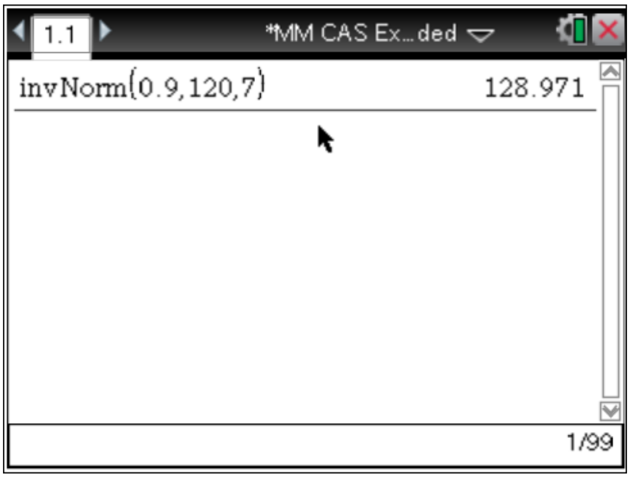


c. i. $\Pr(90 < X < 130) = 0.9234$ correct to 4 decimal places **1A**



ii. $\Pr(X > a) = 0.1$

$a \approx 128.97 = 129$ minutes to the nearest minute **1A**

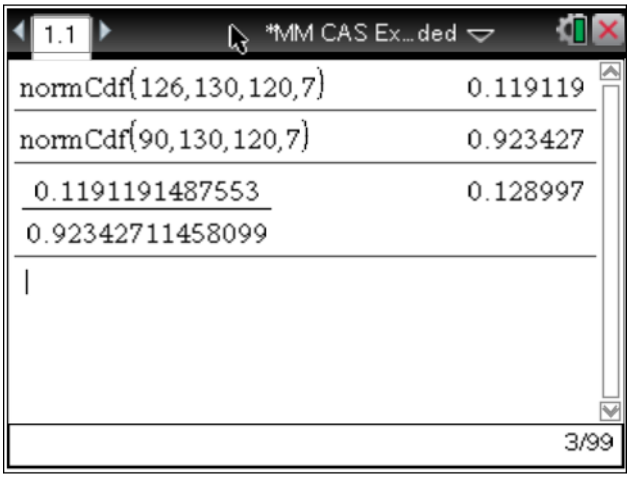


iii. $\Pr(X > 126 \mid 90 < X < 130)$

$= \dfrac{\Pr(126 < X < 130)}{\Pr(90 < X < 130)}$ **1M**

$\approx \dfrac{0.119119...}{0.923427...}$

$= 0.1290$ correct to 4 decimal places as required **1M**

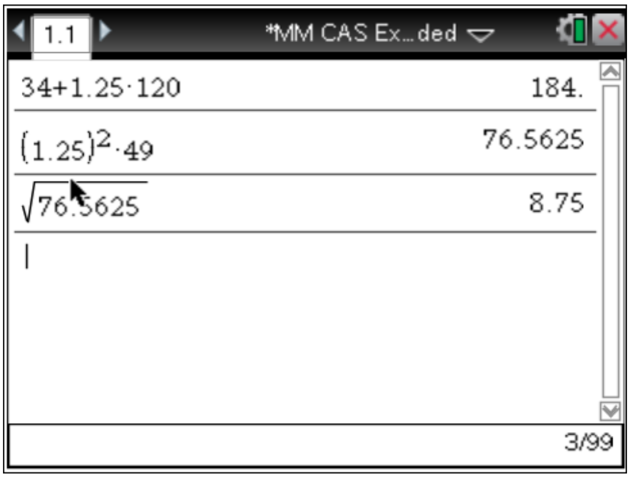


d. i. $\mathrm{E}(C) = 34 + 1.25 \times 120 = \$184$ **1A**

ii. $\mathrm{Var}(C) = 1.25^2 \times 49$

$= 76.5625$ **1A**

$\mathrm{SD}(C) = \sqrt{\mathrm{Var}(C)} = \$8.75$ **1A**



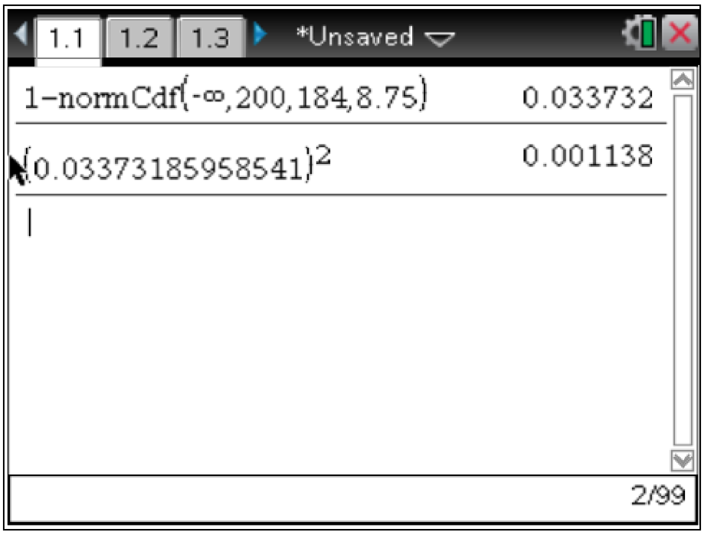
iii. $184 - 2 \times 8.75 \le C \le 184 + 2 \times 8.75$ **1M**

$\$166.50 \le C \le \$201.50$ **1A**

iv. $\Pr(C > 200) \approx 0.03373$ **OR**

$\Pr(C > 200) = 1 - \Pr(C < 200) \approx 1 - 0.966268 \approx 0.03373$ **1M**

For two months $\approx 0.033732 \times 0.033732 = 0.0011$ correct to 4 decimal places **1A**



Question 3

a. $f:(-\infty, 2] \to R$, where $f(x) = (x-2)^2 + 1$

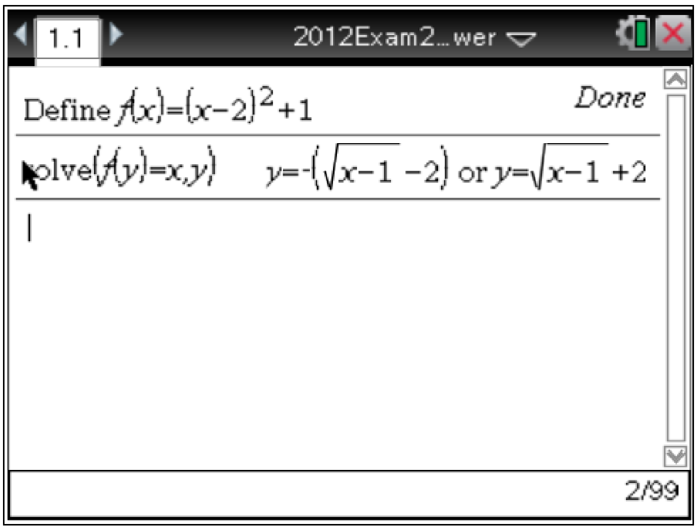
Let $y = (x-2)^2 + 1$.

Inverse: swap x and y **1M**

$x = (y-2)^2 + 1$

$y = -\sqrt{(x-1)} + 2$ due to the domain of f

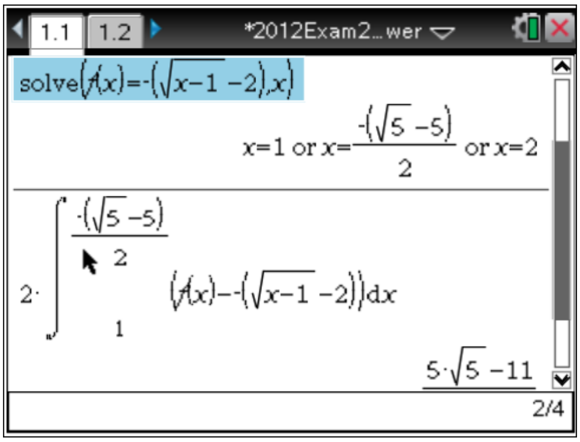
$f^{-1}:[1, \infty) \to R$, where $f^{-1}(x) = -\sqrt{(x-1)} + 2$ **1A** Domain **1A** Rule

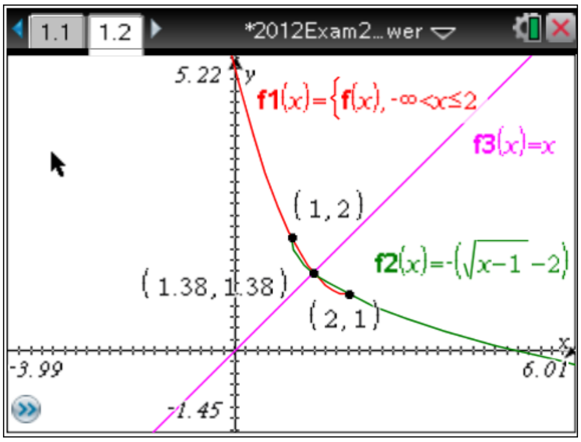


b. Solve $f(x) = f^{-1}(x)$

$x = 1$ or $x = \dfrac{-\sqrt{5}+5}{2}$ or $x = 2$ **1M**

Area $2 \times \int_{1}^{\frac{-\sqrt{5}+5}{2}} \left(f(x) - f^{-1}(x) \right) dx$ **1A**





c. 0, 1 or 3 solutions

Any 2 correct **1A**

All correct **2A**

d. Cross sectional area $= \dfrac{5\sqrt{5}-11}{3} \div 2$

$= \dfrac{5\sqrt{5}-11}{6}$

Volume $= \dfrac{5\sqrt{5}-11}{6} \times 2$

$= \dfrac{5\sqrt{5}-11}{3}$ m^3 **1A**

e. $\dfrac{dV}{dt} = -2\log_e(t+1)$ cm^3/min **1M**

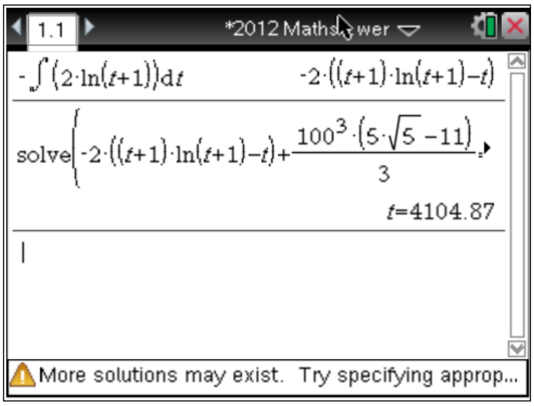
$V = -\int \left(2\log_e(t+1)\right)dt$

$= -2(t+1)\log_e(t+1) + 2(t) + c$, $\left(0, \dfrac{100^3\left(5\sqrt{5}-11\right)}{3}\right)$ **1M**

$V = -2(t+1)\log_e(t+1) + 2(t) + \dfrac{100^3\left(5\sqrt{5}-11\right)}{3}$

Solve $0 = -2(t+1)\log_e(t+1) + 2(t) + \dfrac{100^3\left(5\sqrt{5}-11\right)}{3}$ for t **1M**

t = 4105 minutes to the nearest minute **1A**

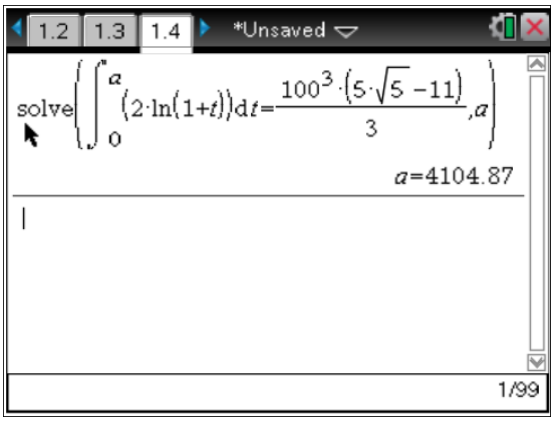


OR

$\dfrac{dV}{dt} = -2\log_e(t+1)$ cm^3/min **1M**

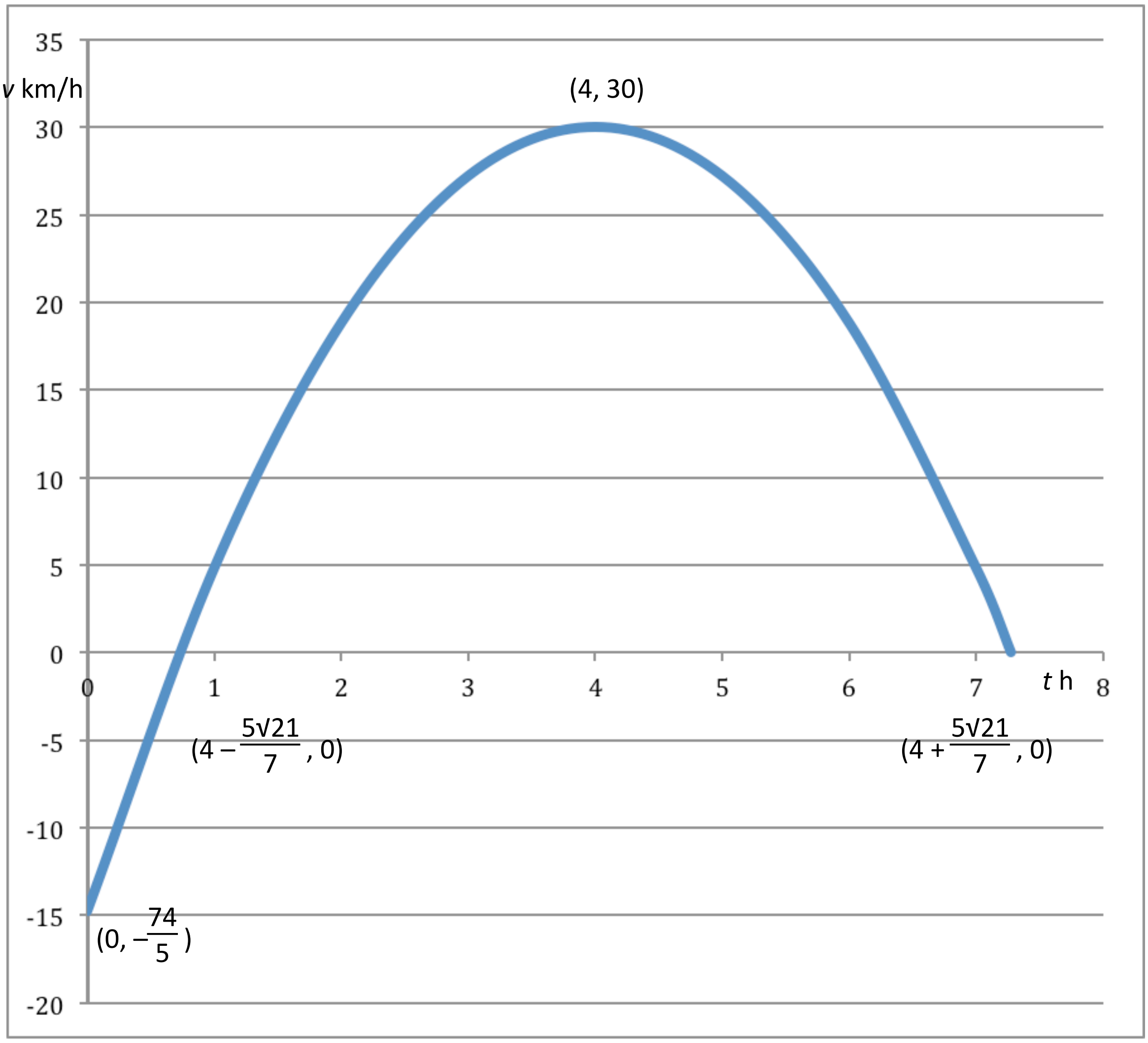
Solve $\displaystyle\int_0^a 2\left(\log_e(1+t)\right)dt = \dfrac{100^3(5\sqrt{5}-11)}{3}$ for a. **1A** terminals**, 1A** equation

t = 4105 minutes to the nearest minute **1A**



Question 4

a.



Shape **1A**

Correct coordinates **1A**

Turning point (4, 30)

x-axis intercepts $\left(-\frac{5}{7}\sqrt{21}+4, 0\right)$ $\left(\frac{5}{7}\sqrt{21}+4, 0\right)$

y-axis intercept $(0, -\frac{74}{5})$

b. $v(t) = -\frac{14}{5}(t-4)^2 + 30$

$x(t)=\int\left(-\frac{14}{5}(t-4)^2+30\right)dt$ **1M**

$=-\frac{14(t-4)^3}{15}+30t+c$**,**

when $t=0$, $x=0$

$c=-\frac{896}{15}$

$x(t)=-\frac{14(t-4)^3}{15}+30t-\frac{896}{15}$ **1A**

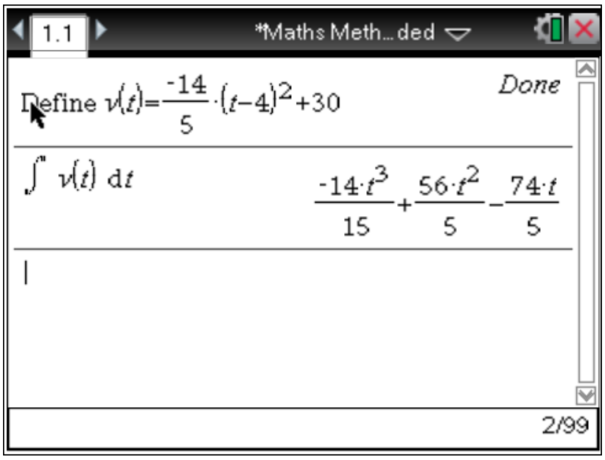
OR

$x(t)=\int\left(-\frac{14}{5}(t-4)^2+30\right)dt$ **1M**

$x(t)=-\frac{14}{15}t^3+\frac{56}{5}t^2-\frac{74}{5}t+c$

when $t=0$, $x=0$, hence $c=0$

$x(t)=-\frac{14}{15}t^3+\frac{56}{5}t^2-\frac{74}{5}t$ **1A**



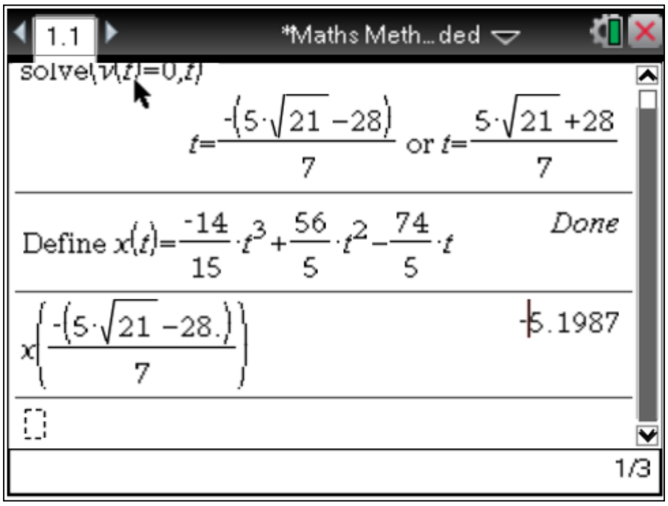
c.i. $v(t)=-\frac{14}{5}(t-4)^2+30$

Solve $v(t)=0$ for t

$t=-\frac{5}{7}\sqrt{21}+4$ **1M**

$x(-\frac{5}{7}\sqrt{21}+4) \approx -5.1987 \text{ km}$

Tasmania rides 5199 m to the nearest metre **1A**



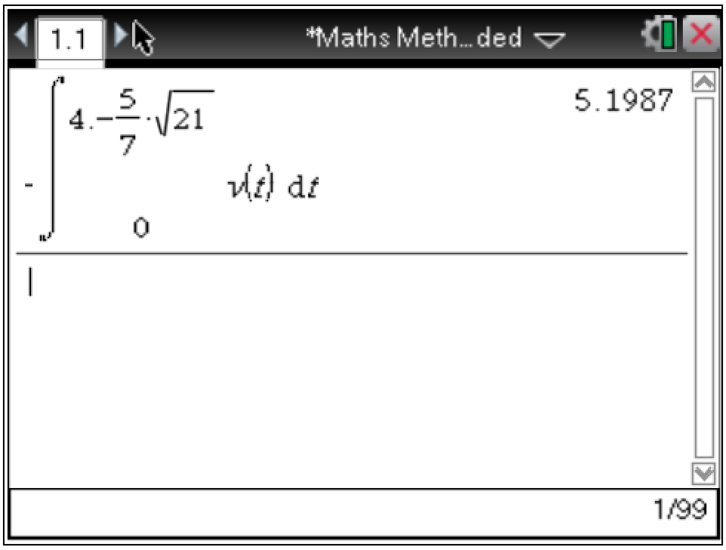
OR

Solve $v(t)=0$ for t

$t=-\frac{5}{7}\sqrt{21}+4$ **1M**

$$\text{distance} = -\int_0^{-\frac{5}{7}\sqrt{21}+4} (v(t))\,dt \approx 5.1987$$

Tasmania rides 5199 m to the nearest metre **1A**

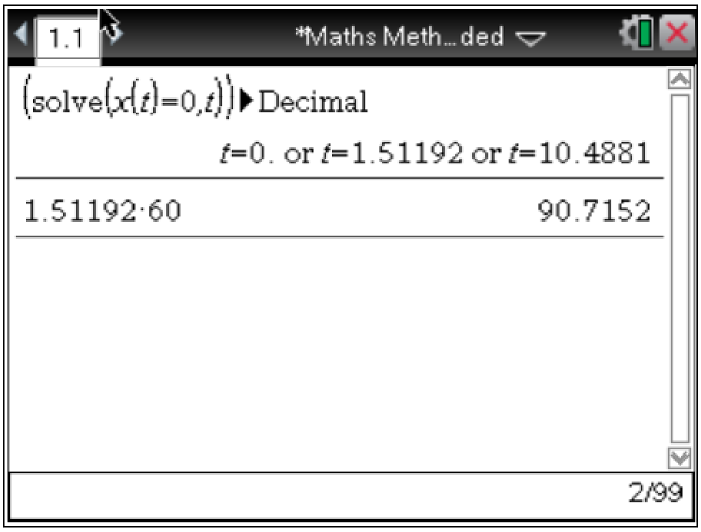


ii. Solve $x(t)=0$

$t \approx 1.51192$ h

≈ 91 min

10:31 am **1A**



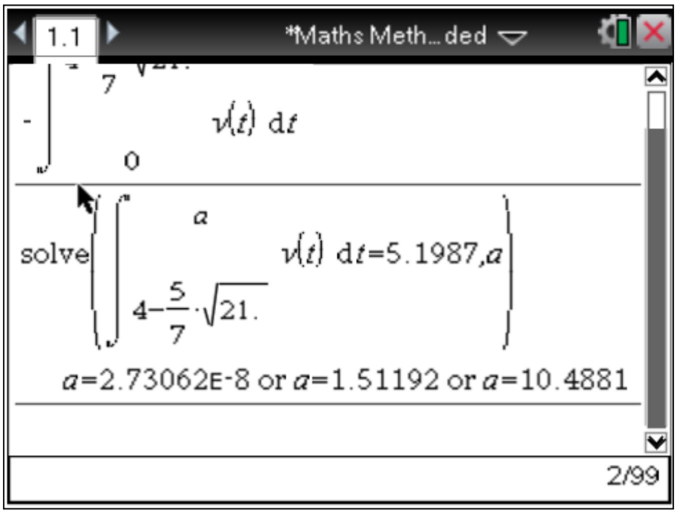
OR

Solve $\int_{-5/7\sqrt{21}+4}^{a} (v(t))dt = 1.51192...$ for a

$t \approx 1.51192$ h

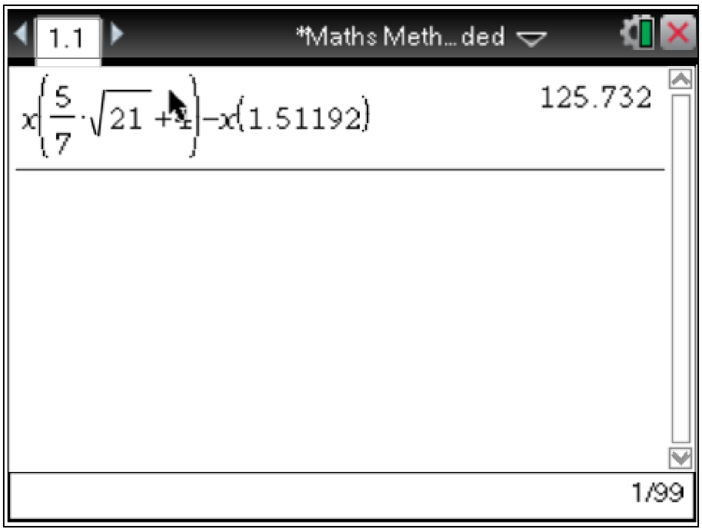
≈ 91 min

10:31 am **1A**



iii. Distance $\approx x\left(\frac{5}{7}\sqrt{21}+4\right)-x(1.51192...)$ **1M**

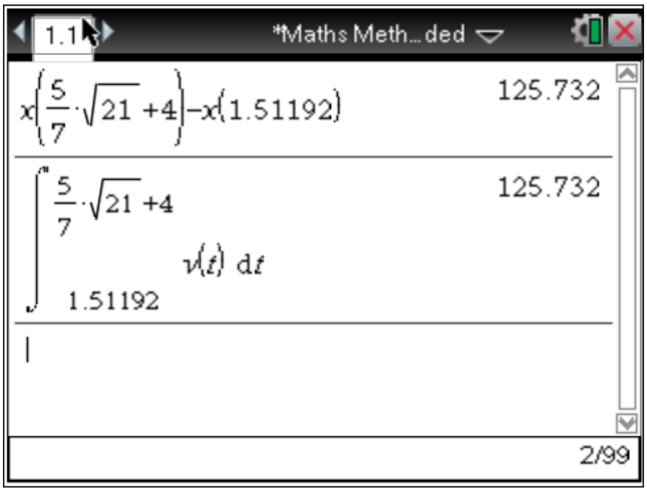
$= 126$ km to the nearest kilometre **1A**



OR

Distance $\approx \displaystyle\int_{1.51192...}^{\frac{5}{7}\sqrt{21}+4} (v(t))dt$ **1M**

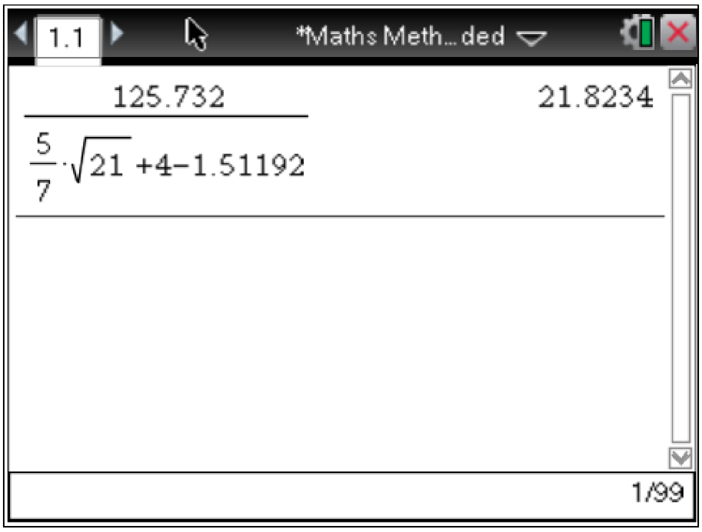
$= 126$ km to the nearest kilometre **1A**



iv. Average velocity $= \dfrac{\text{displacement}}{\text{change in time}}$ **1M**

$$\approx \frac{125.732...}{\left(\frac{5}{7}\sqrt{21}+4\right)-1.51192...}$$

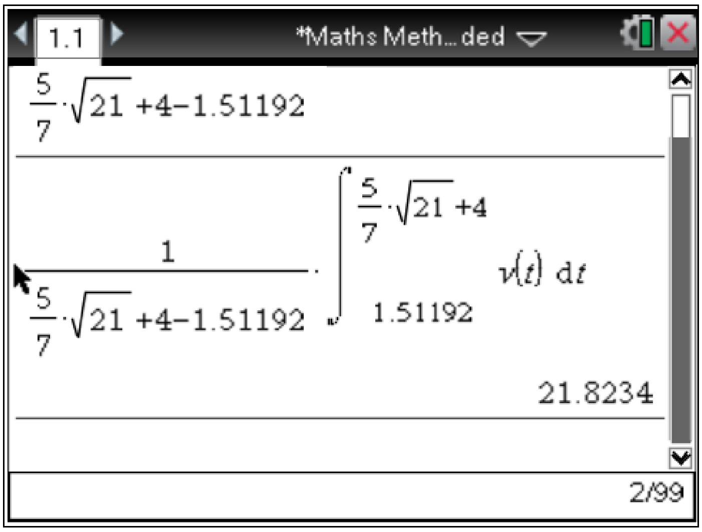
$= 21.8$ km/h correct to one decimal place **1A**



OR

Average value of $v \approx \dfrac{1}{\left(\frac{5}{7}\sqrt{21}+4\right)-1.51192...}\displaystyle\int_{1.51192...}^{\frac{5}{7}\sqrt{21}+4}\left(v(t)\right)dt$ **1M**

$= 21.8$ km/h correct to one decimal place **1A**



d. Distance between Strathton and Coram is approximately 125.732… km.

First checkpoint is at approximately $\dfrac{125.732...}{3} = 41.9107...$ km from Strathton. **1M**

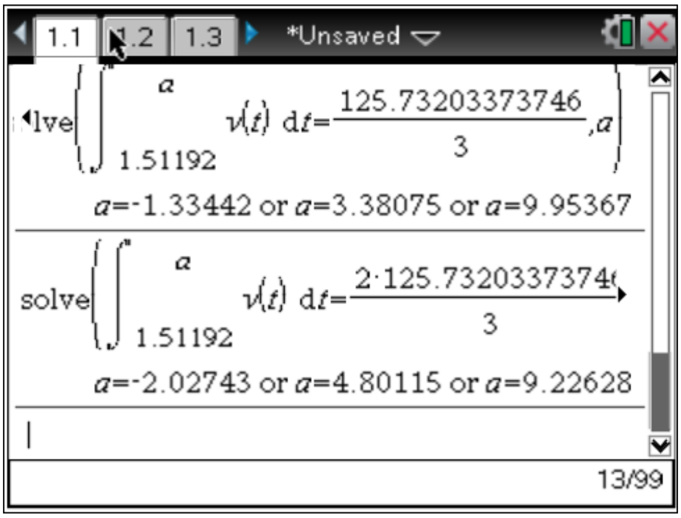
Second checkpoint is at approximately $\dfrac{2\times 125.732...}{3} = 83.8213...$ km from Strathton.

Time to reach first checkpoint: solve for *a*: $\displaystyle\int_{1.5119...}^{a} v(t)\,dt = \frac{125.732...}{3}$. **1M**

$a \approx 3.3807...$ hrs $\Rightarrow 12.23$ pm **1A**

Second checkpoint: $\displaystyle\int_{1.5119...}^{a} v(t)\,dt = \frac{2\times 125.732...}{3}$

$a \approx 4.8011...$ hrs $\Rightarrow 1.48$ pm **1A**



OR

Distance between Strathton and Coram is approximately 125.732… km.

First checkpoint is at approximately $\dfrac{125.732...}{3} = 41.9107...$ km from Strathton. **1M**

Second checkpoint is at approximately $\dfrac{2\times 125.732...}{3} = 83.8213...$ km from Strathton.

Time to reach first checkpoint: solve for *a*: $x(a) \approx \dfrac{125.732...}{3}$. **1M**

$a \approx 3.3807...$ hrs $\Rightarrow 12.23$ pm **1A**

Second checkpoint: $x(a) \approx \dfrac{2 \times 125.732...}{3}$

$a \approx 4.8011...$ hrs $\Rightarrow 1.48$ pm **1A**