The Mathematical Association of Victoria

MATHEMATICAL METHODS (CAS) 2012

Trial Written Examination 1 - SOLUTIONS

Question 1

a. The maximal domain of $f(x)=\log_e(2x-2)$ is

$2x-2>0$

$x>1$ **1A**

b. $g(f(x))=\cos(\log_e(2x-2))$ **1A**

c. $g(f(x))=\cos(\log_e(2x-2))$

$$g'(f(x))f'(x)=-\sin(\log_e(2x-2))\times\frac{2}{2x-2}$$ **1M** Chain rule

$$=\frac{-\sin(\log_e(2x-2))}{x-1}=\frac{\sin(\log_e(2x-2))}{1-x}$$ **1A**

Question 2

$$y=\left(\frac{1}{4}\right)^{x-1}-1$$

x-intercept

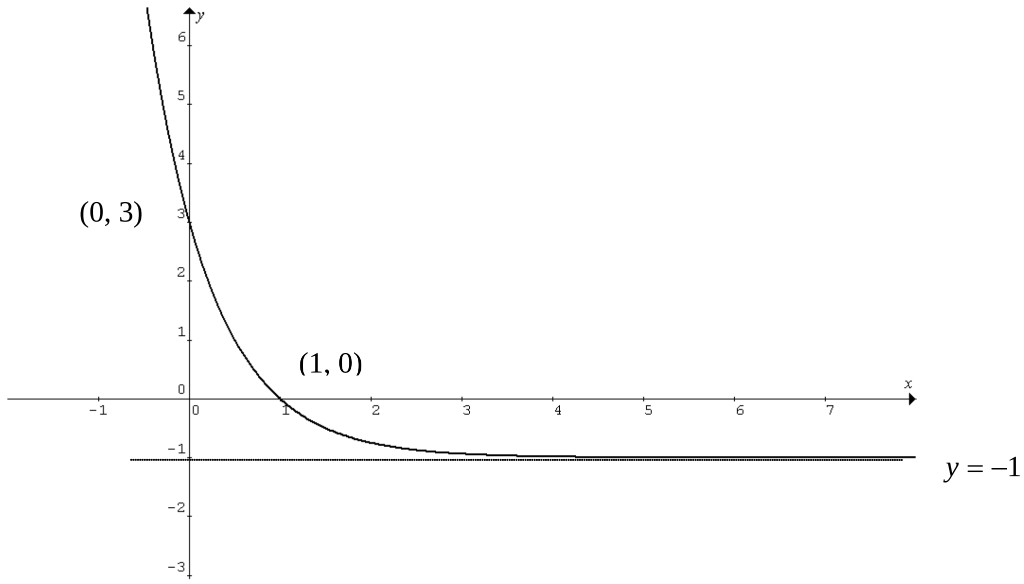
Let $y=0$

$$0=\left(\frac{1}{4}\right)^{x-1}-1,\ 1=\left(\frac{1}{4}\right)^{x-1},\ 0=x-1,\ x=1,\ (1, 0)$$

y-intercept

Let $x=0$

$$y=\left(\frac{1}{4}\right)^{-1}-1=4-1=3,\ (0, 3)$$



Shape **1A**

Equation of the asymptote and asymptotic behaviour **1A**

Coordinates of axes intercepts (must be scaled correctly) **1A**

Question 3

$2\log_e(x-2)-\log_e(x+1)=\log_e(2)$

$\log_e\left(\dfrac{(x-2)^2}{x+1}\right)=\log_e(2)$ **1M**

$\dfrac{(x-2)^2}{x+1}=2$

$x^2-4x+4=2x+2$

$x^2-6x+2=0$

$x=\dfrac{6+\sqrt{36-8}}{2}$ or $x=\dfrac{6-\sqrt{36-8}}{2}$ **1M**

$x=\dfrac{6+\sqrt{28}}{2}$ or $x=\dfrac{6-\sqrt{28}}{2}$

$x=3+\sqrt{7}$ or $x=3-\sqrt{7}$

Since $x > 2$

$x=3+\sqrt{7}$ **1A**

Question 4

a. $\dfrac{d}{dx}\left((x+2)\sqrt{(x-1)}\right)$

$=\sqrt{(x-1)}+\dfrac{1}{2}(x-1)^{-\frac{1}{2}}(x+2)$ **1M** Product rule, **1A**

$=\sqrt{(x-1)}+\dfrac{x+2}{2\sqrt{(x-1)}}$

$=\dfrac{2(x-1)+x+2}{2\sqrt{(x-1)}}$

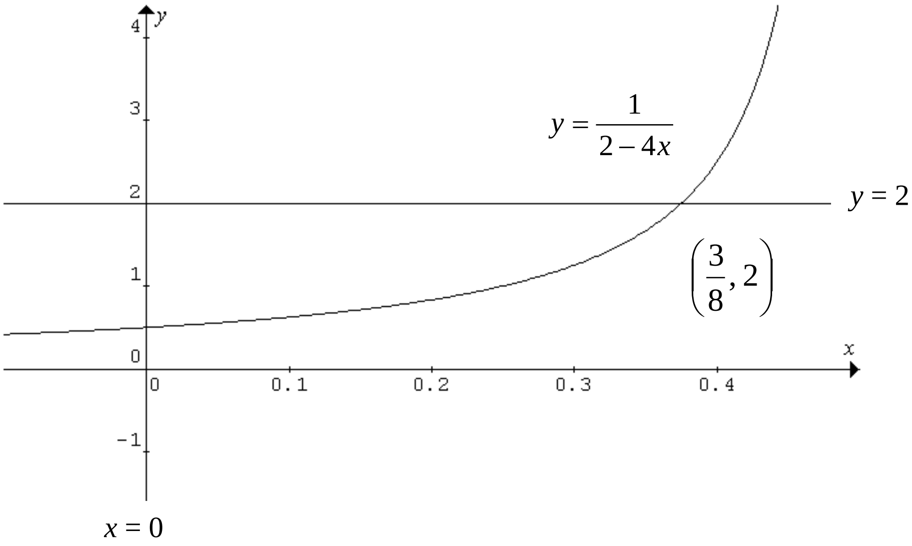
$=\dfrac{3x}{2\sqrt{(x-1)}}$ **1A**

b. Hence $\int\left(\dfrac{3x}{2\sqrt{(x-1)}}\right)dx=(x+2)\sqrt{(x-1)}+c$ **1M**

An antiderivative is

$\int\left(\dfrac{x}{\sqrt{(x-1)}}\right)dx=\dfrac{2(x+2)\sqrt{(x-1)}}{3}$ **1A**

Question 5



Solve $2 = \frac{1}{2-4x}$ for x

$4 - 8x = 1$

$x = \frac{3}{8}$ **1A**

Area = Area of the rectangle − Area under the curve

$= 2 \times \frac{3}{8} - \int_0^{\frac{3}{8}} \left(\frac{1}{2-4x} \right) dx$ **1H**

$= \frac{3}{4} + \left[\frac{1}{4} \log_e(2-4x) \right]_0^{\frac{3}{8}}$ **1H**

$= \frac{3}{4} + \frac{1}{4}\left(\log_e\left(\frac{1}{2}\right) - \log_e(2) \right)$ **1H**

$= \frac{3 + \log_e\left(\frac{1}{4}\right)}{4}$

$= \frac{3 - \log_e(4)}{4}$ **1A**

OR

Find the inverse function

$x = \frac{1}{2-4y}$ **1M**

$y = -\frac{1}{4x} + \frac{1}{2}$

$\int_{\frac{1}{2}}^{2} \left(-\frac{1}{4x} + \frac{1}{2} \right) dx$ **1A**

$= \left[-\frac{\log_e(x)}{4} + \frac{x}{2} \right]_{\frac{1}{2}}^{2}$ **1H**

$= -\frac{\log_e(2)}{4} + 1 + \frac{\log_e\left(\frac{1}{2}\right)}{4} - \frac{1}{4}$ **1H**

$= \frac{3 - \log_e(4)}{4}$ **1M**

Question 6

$h(x) = e^{(x-1)}$

$\text{LHS} = h(x+y) \times h(x-y)$

$= e^{x+y-1} \times e^{x-y-1}$ **1A**

$= e^{2x-2}$

$= \left(e^{x-1}\right)^2$

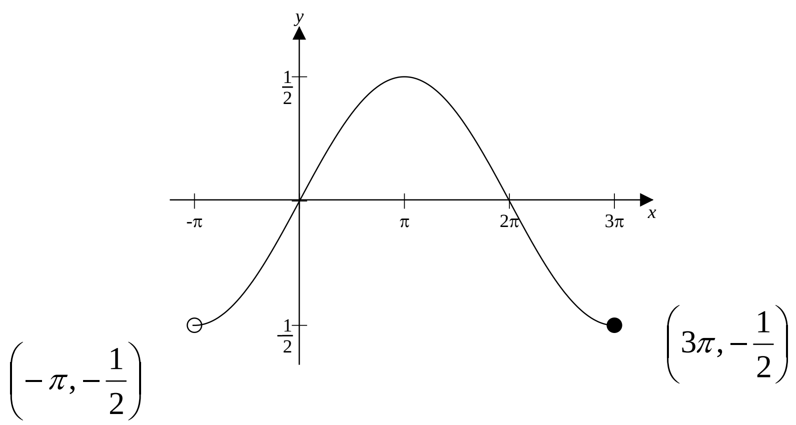
$= \left(h(x)\right)^2 = \text{RHS}$ **1M**

Question 7

$f : [-\pi, 3\pi] \to R$ where $f(x) = \frac{1}{2}\sin\left(\frac{x}{2}\right)$

Amplitude is $\frac{1}{2}$

Period = $\frac{2\pi}{\left(\frac{1}{2}\right)} = 4\pi$



Shape **1A**

Intercepts and end points **1A**

Question 8

$2\sin\left(3\pi(x-a)\right) = \sqrt{3}$

$\sin\left(3\pi(x-a)\right) = \frac{\sqrt{3}}{2}$

$3\pi(x-a) = \ldots -\frac{4\pi}{3}, \frac{\pi}{3}, \ldots$ **1M**

$x - a = \ldots -\frac{4}{9}, \frac{1}{9}, \ldots.$

$\frac{1}{6} - a = \ldots -\frac{4}{9}, \frac{1}{9}, \ldots$

$a = \frac{1}{18}$ or $a = \frac{11}{18}$ **2A**

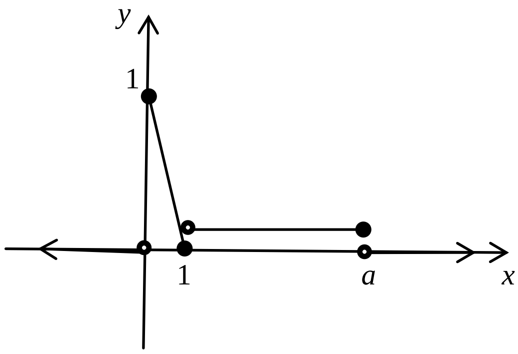
Question 9

$X \sim \text{Bi}(5, 0.1)$

$\Pr(X = 3) = \binom{5}{3}(0.1)^3(0.9)^2$ **1M**

$= 0.0081$ **1A**

Question 10



a. $0.5 \times 1 \times 1 + 0.1 \times (a-1) = 1$

$0.1(a-1) = 0.5$

$a - 1 = 5$

$a = 6$ **1A**

b. $\frac{1}{2} \times 0.5 \times 0.5 + 5 \times 0.1$

$= 0.5^3 + 0.5$ **1A**

$= 0.625$

c. $\Pr(X > 2 \mid X > 0.5)$

$= \frac{\Pr(X > 2)}{\Pr(X > 0.5)}$ **1M**

$= \frac{4 \times 0.1}{0.625}$

$= \frac{0.4}{0.625} = \frac{16}{25}$ **1A**

Question 11

a. $BC = a\sin(\theta)$ **1A**

$$AD = AB + BD = a\cos(\theta) + \frac{1}{2}a\cos(\theta) = \frac{3a}{2}\cos(\theta)$$ **1A**

b. $T = \frac{1}{2} \times AD \times BC$

$$= \frac{1}{2} \times \frac{3}{2}a\cos(\theta) \times a\sin(\theta)$$

$$= \frac{3}{4}a^2 \sin(\theta)\cos(\theta)$$ **1A**

c. i. $T = \frac{3}{4}a^2 \sin(\theta)\cos(\theta)$

$$\frac{dT}{d\theta} = \frac{3}{4}a^2\left[\cos(\theta) \times \cos(\theta) + (-\sin(\theta) \times \sin(\theta))\right]$$ Product Rule

$$= \frac{3}{4}a^2\left(\cos^2(\theta) - \sin^2(\theta)\right)$$ **1A**

ii. For maximum total sail area**:** $\cos^2(\theta) - \sin^2(\theta) = 0$

$$\sin^2(\theta) = \cos^2(\theta)$$

$$\frac{\sin(\theta)}{\cos(\theta)} = 1$$

$$\tan(\theta) = 1$$

$$\theta = \frac{\pi}{4}$$ **1M**

$$T_{\max} = \frac{3}{4}a^2 \times \cos\left(\frac{\pi}{4}\right) \times \sin\left(\frac{\pi}{4}\right)$$

$$= \frac{3}{4}a^2 \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}$$ **1M**

$$= \frac{3}{8}a^2$$

When $a = 4$**,** $T_{\max} = \frac{3}{8} \times 4^2 = 6\text{ m}^2$ **1A**