INSIGHT

YEAR 12 *Trial Exam Paper*

2012

MATHEMATICAL METHODS (CAS)

Written examination 1

Worked Solutions

This book presents:

- correct solutions with full working
- explanatory notes
- mark allocations

Question 1

Let $f(x)=\sqrt{3-2x}$. Write down the rule for $f(\cos x)$.

1 mark

Worked solution

Replace x with $\cos x$ to get $f(x)=\sqrt{3-2\cos(x)}$.

Mark allocation

- 1 mark for correct answer

End of Question 1

Question 2

For the function $f:(-1,\infty)\to R,\ f(x)=\frac{1}{3}\log_e(\frac{x+1}{2})$

a. Find the rule for the inverse function, f^{-1}.

2 marks

Worked solution

Swap x and y to get $x=\frac{1}{3}\log_e\left(\frac{y+1}{2}\right)$.

Now rearrange to make y the subject:

$$3x=\log_e\left(\frac{y+1}{2}\right)$$

$$e^{3x}=\frac{y+1}{2}$$

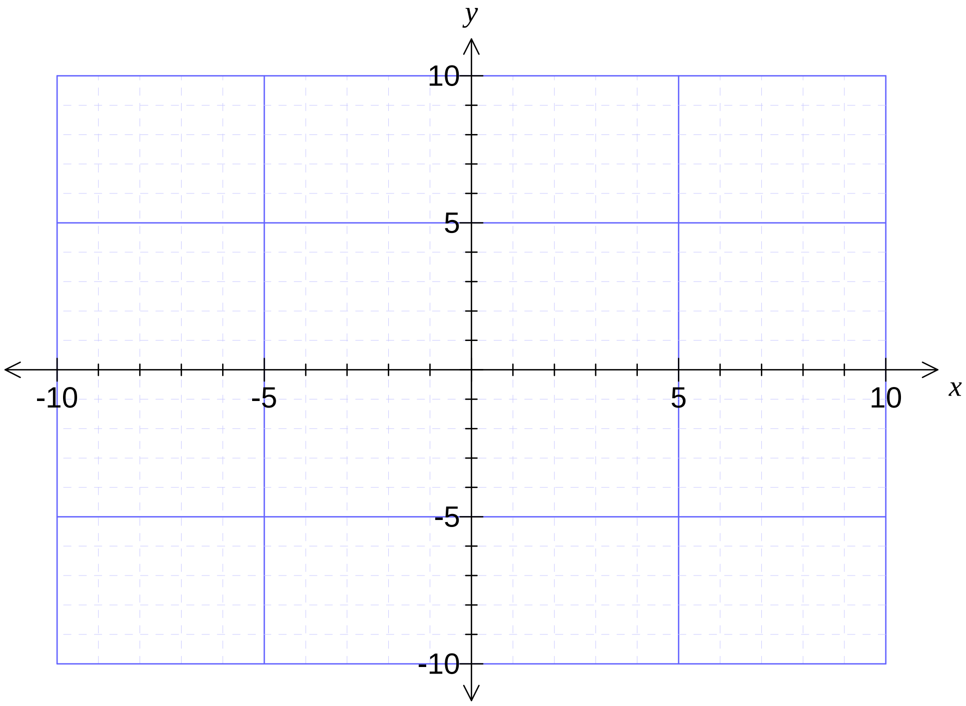
$$2e^{3x}=y+1$$

$$y=2e^{3x}-1$$

Mark allocation

- 1 mark for swapping x and y.
- 1 mark for the correct answer.

b. Sketch the graph of $y=f^{-1}(f(x))$ on the axes below.



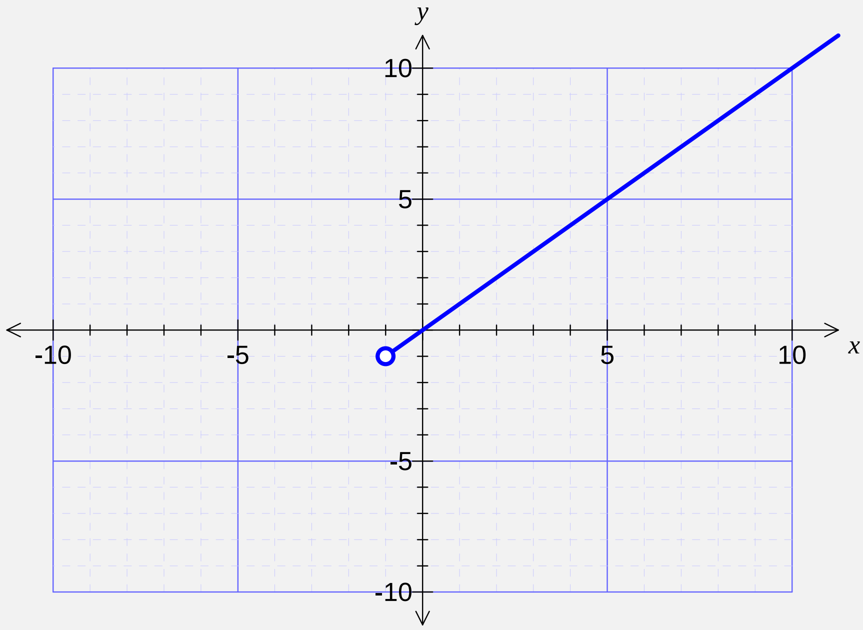
1 mark

Question 2 – continued
TURN OVER

Worked solution

The graph of $y = f^{-1}(f(x))$ will be the graph of $y = x$ for the domain of $f(x)$; i.e., $x \in (-1, \infty)$.

So the graph will be



Mark allocation

- 1 mark for correctly drawn graph. (It must have an open circle at $x = -1$).

c. The function $f(x)$ undergoes a transformation as defined by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

State the new equation.

1 mark

Worked solution

The matrix $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ describes a transformation that produces a dilation of factor 2 in the y direction (from the x-axis).

So the new equation is $y = \frac{2}{3}\log_e\left(\frac{x+1}{2}\right)$.

Mark allocation

- 1 mark for the correct answer.

End of Question 2

Question 3

a. Let $y = \frac{e^{2x}}{x}$. Find $\frac{dy}{dx}$.

2 marks

Worked solution

Use the quotient rule to differentiate.

$$y = \frac{e^{2x}}{x} = \frac{u(x)}{v(x)}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{2xe^{2x} - e^{2x}}{x^2}$$

Mark allocation

- 1 mark for using quotient rule.
- 1 mark for correct answer.

b. Let $f(x) = \sqrt{\sin(2x)}$. Find $f'\left(\frac{\pi}{4}\right)$.

3 marks

Worked solution

First, we must use the chain rule.

$$f'(x) = \frac{1}{2}\left(\sin(2x)\right)^{-\frac{1}{2}} 2\cos(2x)$$

$$= \frac{\cos(2x)}{\sqrt{\sin(2x)}}$$

$$f'\left(\frac{\pi}{4}\right) = \frac{\cos\left(\frac{\pi}{2}\right)}{\sqrt{\sin\left(\frac{\pi}{2}\right)}}$$

$$= \frac{0}{1} = 0$$

Mark allocation

- 1 mark for using chain rule.
- 1 mark for obtaining $f'(x)$.
- 1 mark for correct answer.

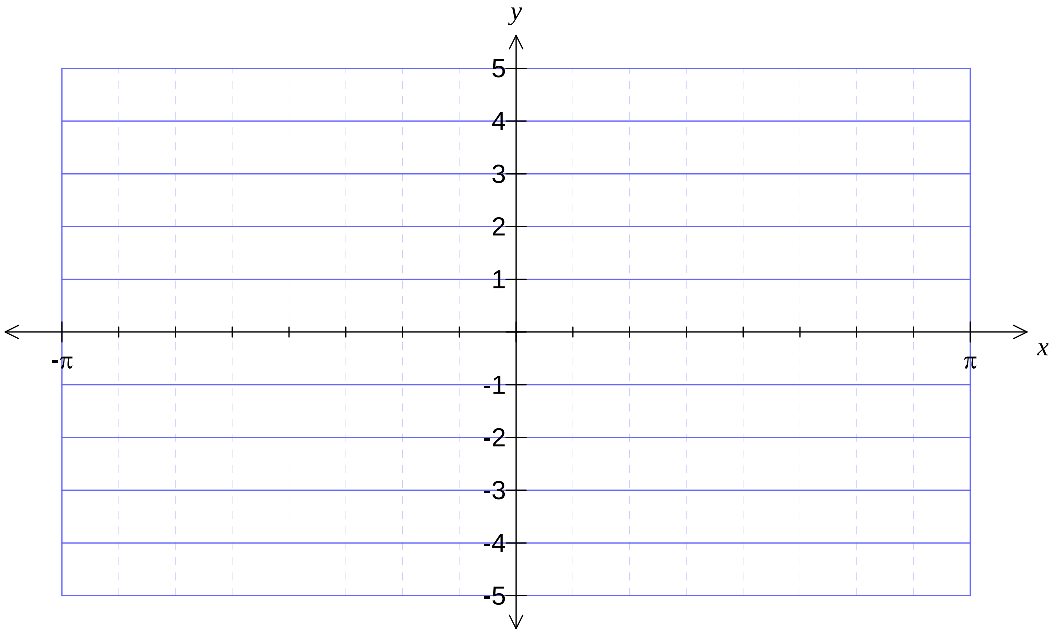
End of Question 3
TURN OVER

Question 4

The graph of $y = \cos(x)$ undergoes the following transformations:

- a dilation of factor $\frac{1}{2}$ from the *y*-axis
- a translation of $+3$ units up.

a. Sketch the transformed graph over the domain $[-\pi, \pi]$ on the axes below.
Label all intercepts and endpoints as co-ordinates.



2 marks

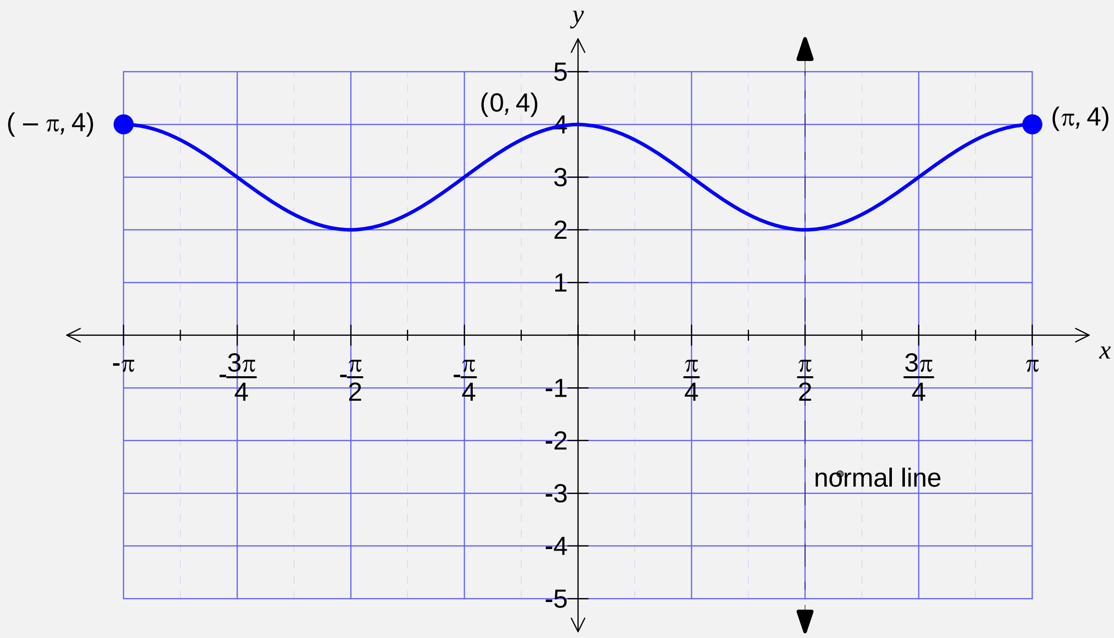
Question 4 – continued

Worked solution

The transformations

- a dilation of factor $\frac{1}{2}$ from the *y*-axis; and
- a translation of +3 units up

give a new equation of $y = \cos(2x) + 3$.



Mark allocation

- 1 mark for showing two cycles.
- 1 mark for shifting graph up 3 units and labelling points.

b. State the equation of the normal to the graph in part **a** at $x = \frac{\pi}{2}$ and sketch the normal on the axes above.

2 marks

Worked solution

The equation of the normal to the curve is $x = \frac{\pi}{2}$ and is shown on the graph.

Mark allocation

- 1 mark for equation.
- 1 mark for line.

End of Question 4
TURN OVER

Question 5

Voicefone, a telephone company, currently has 30% of the market for a new type of home phone system. There are no contracts and customers simply take out a plan for a month at a time. Of the current customers, 80% will still be customers in the next month.

Assume that, for this type of system, *a*% of the rest of the market switches to this telephone company from one month to the next.

a. **i.** Write a transition matrix for this situation.

ii. Find the value of *a* needed in order for Voicefone to maintain its market share.

3 marks

Worked solution

For the transition matrix, treat the value of *a* as a proportion rather than a percentage.

The transition matrix is $\begin{bmatrix} 0.8 & a \\ 0.2 & 1-a \end{bmatrix}$

and in order to maintain the market share of 30%, then $\dfrac{a}{a+0.2} = 0.3,$ giving:

$$a = 0.3(a + 0.2)$$

$$a = 0.06 + 0.3a$$

$$0.7a = 0.06$$

$$a = \frac{0.06}{0.7} = \frac{6}{70}$$

So a is $\dfrac{6}{70} \times 100\% = \dfrac{60}{7}\%$

Mark allocation

- 1 mark for matrix.
- 1 mark for relevant method.
- 1 mark for the correct answer.

Question 5 – continued

Voicefone runs an advertising campaign and hopes to eventually hold 60% of the total market, for this type of system.

b. Find the value of *a* for this situation.

1 mark

Worked solution

$\begin{bmatrix} 0.8 & a \\ 0.2 & 1-a \end{bmatrix}$ *and* $\dfrac{a}{a+0.2} = 0.6$; hence:

$$a = 0.6(a + 0.2)$$
$$a = 0.6a + 0.12$$
$$0.4a = 0.12$$
$$a = \frac{12}{40} = 0.3$$

So a is 30%.

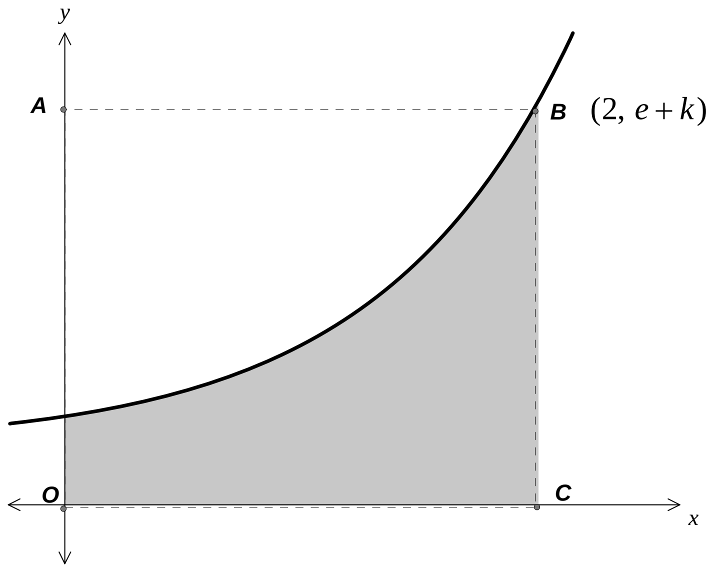
Mark allocation

- 1 mark for the correct answer.

End of Question 5
TURN OVER

Question 6

Consider the graph of $y = e^{\frac{x}{2}} + k$. *OABC* is a rectangle, as shown in the diagram below. If the shaded and unshaded regions are equal in area, find *k*.



3 marks

Worked solution

Area of the rectangle is $2(e+k)$.

Hence, the area under the curve is equal to $\frac{1}{2}(2(e+k)) = (e+k)$.

So, $\int_0^2 e^{\frac{x}{2}} + k\, dx = e + k$

$$\text{LHS} = [2e^{\frac{x}{2}} + kx]_0^2$$
$$= (2e + 2k) - (2 + 0)$$
$$= 2e + 2k - 2$$

$$\Rightarrow 2e + 2k - 2 = e + k$$
$$k = -e + 2$$

Mark allocation

- 1 mark for finding area of rectangle.
- 1 mark for setting up integral.
- 1 mark for correct answer.

End of Question 6

Question 7

a. Find the general solution of $\sqrt{2}\cos(3x) = -1$.

3 marks

Worked solution

$\cos(3x) = \frac{-1}{\sqrt{2}}$, 2nd and 3rd quadrants

$$3x = \frac{3\pi}{4}, \frac{5\pi}{4}$$

$$3x = \pm\frac{3\pi}{4} + 2k\pi$$

$$x = \pm\frac{\pi}{4} + \frac{2k\pi}{3}$$

Mark allocation

- 1 mark for using $\frac{\pi}{4}$ as basic angle.
- 1 mark for finding two quadrants.
- 1 mark for the correct answer.

b. Find the average value of the function $y = 2\cos(2x)$ over the interval $\left[0, \frac{\pi}{12}\right]$.

3 marks

Worked solution

The average value of a function is defined as $\frac{1}{b-a}\int_a^b f(x)\,dx$, so in this case

$$\frac{1}{\frac{\pi}{12} - 0}\int_0^{\frac{\pi}{12}} 2\cos(2x)\,dx$$

$$= \frac{12}{\pi}\left[\sin(2x)\right]_0^{\frac{\pi}{12}}$$

$$= \frac{12}{\pi}\left[\sin\left(\frac{\pi}{6}\right) - \sin(0)\right]$$

$$= \frac{12}{\pi}\left(\frac{1}{2} - 0\right)$$

$$= \frac{6}{\pi}$$

Mark allocation

- 1 mark for setting up calculation.
- 1 mark for antidifferentiating $2\cos(2x)$.
- 1 mark for the correct answer.

End of Question 7
TURN OVER

Question 8

A spherical balloon is being inflated. Its volume is increasing at the rate of 4 cm^3 per second. Find the rate, in cm s^{-1}, at which the radius of the balloon is increasing when the radius is 2 cm.

3 marks

Worked solution

The rate equation is $\frac{dV}{dt}=\frac{dr}{dt}\times\frac{dV}{dr}$. We must find $\frac{dV}{dr}$.

As object is a sphere, use

$$V=\frac{4}{3}\pi r^3$$

$$\therefore \frac{dV}{dr}=4\pi r^2$$

$$\frac{dV}{dt}=\frac{dr}{dt}\times\frac{dV}{dr}$$

$$4=\frac{dr}{dt}\times 4\pi r^2$$

$$\frac{dr}{dt}=\frac{4}{4\pi r^2}$$

At $r=2$:

$$\frac{dr}{dt}=\frac{1}{\pi(2)^2}=\frac{1}{4\pi}$$

Mark allocation

- 1 mark for writing rate equation.
- 1 mark for finding $\frac{dV}{dr}$.
- 1 mark for the correct answer.

End of Question 8

Question 9

Let X be a random variable with a normal distribution with mean 6 and variance 4, and let Z be a random variable with the standard normal distribution. If $\Pr(Z > 1) = 0.16$,

a. Find $\Pr(X > 8)$.

1 mark

Worked solution

$$\begin{aligned}\Pr(X > 8) &= \Pr\left(Z > \frac{8-6}{2}\right)\\ &= \Pr\left(Z > \frac{2}{2}\right)\\ &= \Pr(Z > 1)\\ &= 0.16\end{aligned}$$

Mark allocation

- 1 mark for the correct answer.

b. Find $\Pr(X > 8 | X > 6)$.

2 marks

Worked solution

$$\begin{aligned}\Pr(X > 8 | X > 6) &= \frac{\Pr(X > 8 \cap X > 6)}{\Pr(X > 6)}\\ &= \frac{\Pr(X > 8)}{\Pr(X > 6)} = \frac{0.16}{0.5} = 0.32\end{aligned}$$

Mark allocation

- 1 mark for using conditional probability.
- 1 mark for the correct answer.

c. Find a such that $\Pr(Z > a) = \Pr(X < 5)$.

2 marks

Worked solution

$$\Pr(X < 5) = \Pr\left(Z < \frac{5-6}{2}\right) = \Pr\left(Z < -\frac{1}{2}\right)$$

and using symmetry $\Pr\left(Z < -\frac{1}{2}\right) = \Pr\left(Z > \frac{1}{2}\right)$, so $a = \frac{1}{2}$.

Mark allocation

- 1 mark for method.
- 1 mark for the correct answer.

End of Question 9
TURN OVER

Question 10

a. Using the linear approximation $f(x+h) \approx f(x) + hf'(x)$, where h is 0.03, and $f(x) = \sqrt{x}$, find an approximate value of $\sqrt{16.03}$.

2 marks

Worked solution

$f(x+h) \approx f(x) + hf'(x)$ and $f(x) = \sqrt{x}$, $f'(x) = \dfrac{1}{2\sqrt{x}}$

So, $\sqrt{16.03} \approx \sqrt{16} + 0.03\dfrac{1}{2\sqrt{16}} = 4 + 0.03\dfrac{1}{8}$

$$= 4 + \frac{3}{800}$$

$$= 4\frac{3}{800}$$

Mark allocation

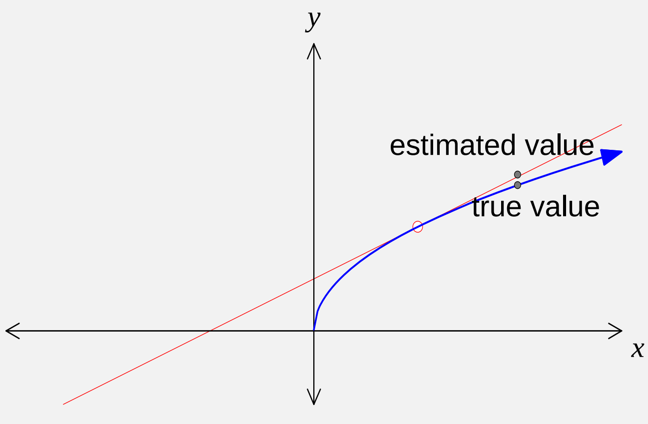
- 1 mark for finding $f'(x) = \dfrac{1}{2\sqrt{x}}$.
- 1 mark for the correct answer.

b. Explain why your answer to part **a** overestimates the value of $\sqrt{16.03}$.

1 mark

Worked solution

The approximate calculation is an overestimation, as it is calculated from a tangent line drawn to the curve $y = \sqrt{x}$ and the tangent line projects above the graph.



Mark allocation

- 1 mark for written explanation or using graph to explain.

End of Question 10

Question 11

If $f(x) = 3x^2$, show that $f(u+v) + f(u-v) = 2(f(u) + f(v))$.

2 marks

Worked solution

$f(u+v) + f(u-v) = 2(f(u) + f(v))$

$$
\begin{aligned}
\text{LHS} &= 3(u+v)^2 + 3(u-v)^2 \\
&= 3u^2 + 6uv + 3v^2 + 3u^2 - 6uv + 3v^2 \\
&= 6u^2 + 6v^2 \\
&= 2(3u^2 + 3v^2) \\
&= 2(f(u) + f(v)) \\
&= \text{RHS}
\end{aligned}
$$

Mark allocation

- 1 mark for expanding $3(u+v)^2 + 3(u-v)^2$.
- 1 mark for the correct answer.

END OF SOLUTIONS BOOK