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Student Name……………………………………

MATHEMATICAL METHODS (CAS) UNITS 3 & 4

TRIAL EXAMINATION 2

2012

Reading Time: 15 minutes
Writing time: 2 hours

Instructions to students

This exam consists of Section 1 and Section 2.
Section 1 consists of 22 multiple-choice questions, which should be answered on the detachable answer sheet which can be found on page 24 of this exam.
Section 2 consists of 4 extended-answer questions.
Section 1 begins on page 2 of this exam and is worth 22 marks.
Section 2 begins on page 11 of this exam and is worth 58 marks.
There is a total of 80 marks available.
All questions in Section 1 and Section 2 should be answered.
Diagrams in this exam are not to scale except where otherwise stated.
Where more than one mark is allocated to a question, appropriate working must be shown.
A formula sheet can be found on page 23 of this exam.
Students may bring one bound reference into the exam.
Students may bring a CAS calculator into this exam.

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SECTION 1

Question 1

A line passes through the points $(0,3)$ and $(-6,0)$. The gradient of the line that is perpendicular to this line is

A. -2
B. $-\frac{1}{2}$
C. 0
D. $\frac{1}{2}$
E. 2

Question 2

The maximal domain of the function $y=\frac{1}{\sqrt{4-x}}$ is

A. $x \in (-\infty, 4)$
B. $x \in (-\infty, 4]$
C. $x \in [0, \infty)$
D. $x \in (4, \infty)$
E. $x \in [4, \infty)$

Question 3

The functions f and g both have maximal domains.
If $f(x)=e^{2x}$ and $g(x)=\frac{2}{x}$ the domain and rule of $f(g(x))$ are respectively

A. R^+ and $f(g(x))=\frac{2}{e^{2x}}$
B. R^+ and $f(g(x))=e^{\frac{4}{x}}$
C. R and $f(g(x))=e^4$
D. $R \setminus \{0\}$ and $f(g(x))=e^{\frac{4}{x}}$
E. $R \setminus \{0\}$ and $f(g(x))=\frac{2}{e^{2x}}$

Question 4

The function $h:[1,\infty)\to R,\ h(x)=(x-1)^3+2$ has an inverse function given by

A. $h^{-1}:[0,\infty)\to R,\ h^{-1}(x)=1+\sqrt[3]{x-2}$
B. $h^{-1}:[1,\infty)\to R,\ h^{-1}(x)=\sqrt[3]{x-1}$
C. $h^{-1}:[1,\infty)\to R,\ h^{-1}(x)=1+\sqrt[3]{x+2}$
D. $h^{-1}:[2,\infty)\to R,\ h^{-1}(x)=\sqrt[3]{x-1}$
E. $h^{-1}:[2,\infty)\to R,\ h^{-1}(x)=1+\sqrt[3]{x-2}$

Question 5

Let $f(x)=e^{3g(x)}$.

$f'(x)$ is equal to

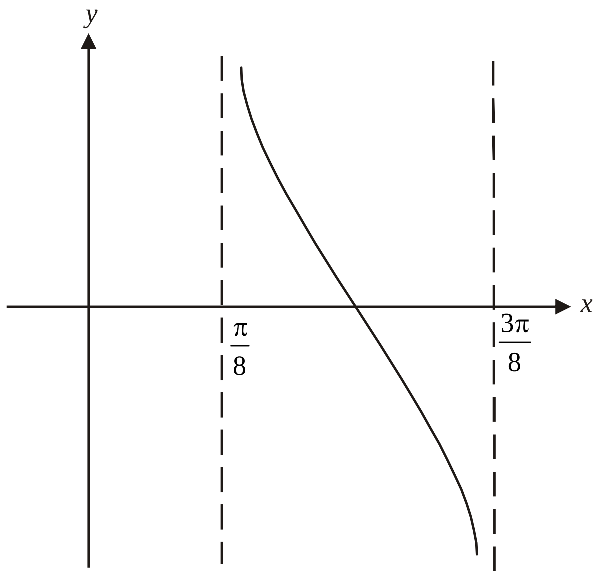
A. $e^{3g'(x)}$
B. $g'(x)e^{f(x)}$
C. $g'(x)e^{3g'(x)}$
D. $3g'(x)f(x)$
E. $3f'(x)g(x)$

Question 6

Using the linear approximation method $f(1+h)\approx f(1)+hf'(1)$ and considering the point $(1,e)$ which lies on the graph of $f(x)=e^x$, an approximation for $e^{1.3}$ is

A. $0.3e$
B. $1.3e$
C. $1.5e$
D. $1.7e$
E. $2e$

Question 7



The graph shown above could have the rule

A. $y = -\tan\left(\frac{x}{4}\right)$

B. $y = -\tan(4x)$

C. $y = \tan\left(-\frac{x}{8}\right)$

D. $y = \tan\left(\frac{x}{8}\right)$

E. $y = \tan(4x)$

Question 8

The simultaneous linear equations $(a+1)x + y = 1$ and $5x + (a-3)y = 2.5a$ have **no** solution for

A. $a \in \{-2\}$

B. $a \in \{4\}$

C. $a \in \{-2, 4\}$

D. $a \in R \setminus \{-2, 4\}$

E. $a \in R \setminus \{-2\}$

Question 9

George is a student and receives homework each school day. If George does his homework one day the probability he does it the next day is 0.6. If George doesn't do his homework one day the probability he does it the next day is 0.7.
Over the long term the probability that George does his homework is

A. 0.36
B. 0.43
C. 0.47
D. 0.57
E. 0.64

Question 10

The average value of the function with rule $y = e^{2x}(x^3 - 2)$ over the interval $[-1, 2]$ is

A. $\frac{e^{2x}}{3}(x^3 - 2)$

B. $6e^4 + \frac{3}{e^2}$

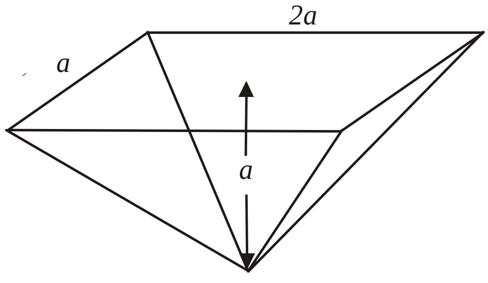
C. $2e^4 + \frac{1}{e^2}$

D. $\frac{9e^4}{8} + \frac{27}{8e^2}$

E. $\frac{3(e^6 + 3)}{8e^2}$

Question 11

A container is in the shape of an inverted pyramid with a rectangular top. The width of the rectangular top is *a* metres, the length is 2*a* metres and the height of the inverted pyramid is *a* metres.



Liquid is being poured into the container at the rate of $0.1\text{m}^3/\text{hr}$. The rate at which the height of the liquid is increasing when the height of the liquid in the container is 0.2m is

A. 0.008m/hr
B. 0.2m/hr
C. 0.25m/hr
D. 1.25m/hr
E. 3.75m/hr

Question 12

A continuous random variable *X* has a normal distribution with mean 15 and standard deviation 2.5.
A second continuous random variable *Z* has the standard normal distribution.
$\Pr(-1 < Z < 3)$ is equal to

A. $\Pr(7.5 < X < 15)$
B. $\Pr(7.5 < X < 17.5)$
C. $\Pr(10 < X < 20)$
D. $\Pr(10 < X < 22.5)$
E. $\Pr(12.5 < X < 20)$

Question 13

Let $g:[1,\infty)\to R,\ g(x)=\log_e(x)$ and $h:R\to R,\ h(x)=|x|$ and let $f = g\times h$.
The domain of the inverse function f^{-1} is

A. $(-\infty,1]$
B. $(-\infty,1)$
C. $[0,\infty)$
D. $[1,\infty)$
E. R

Question 14

The continuous random variable *X* has a probability density function given by

$$f(x)=\begin{cases}|2x|-2 & 1\le x\le 2\\ 0 & \text{elsewhere}\end{cases}$$

The median of *X* is closest to

A. 1.4
B. 1.5
C. 1.6
D. 1.7
E. 1.8

Question 15

The tangent to the curve with equation $y=\log_e\left(\sqrt{x}\right)$, at the point $(4,\log_e(2))$ is parallel to the line with equation

A. $y=\dfrac{x}{4}$
B. $y=8x$
C. $y-8x=1$
D. $8x-8y=1$
E. $x=8y$

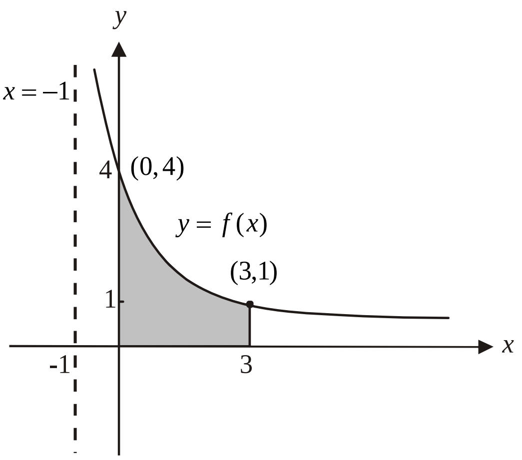
Question 16

At a biscuit factory, 5000 packets of biscuits are weighed and 1520 of them are found to weigh more than 375 grams. The weights of the biscuit packets are normally distributed with a mean of μ and a standard deviation of 2 grams.
The closest value of μ is

A. 372.41
B. 373.61
C. 373.97
D. 374.39
E. 376.03

Question 17

Let $f:(-1,\infty)\rightarrow R,\ f(x)=\dfrac{4}{x+1}$
The graph of $y=f(x)$ is shown below.



The expression which does **not** equal the area of the shaded region above is

A. $\displaystyle\int_0^3 \frac{4}{x+1}\,dx$

B. $\displaystyle\int_3^0 \frac{-4}{x+1}\,dx$

C. $\displaystyle 3+\int_0^3 \left(\frac{4}{x+1}-1\right)dx$

D. $\displaystyle -4\int_0^3 \frac{1}{x+1}\,dx+4$

E. $\displaystyle 4\int_0^2 \frac{1}{x+1}\,dx+\int_2^3 \frac{4}{x+1}dx$

Question 18

Let $f: R \to R,\ f(x) = \left|x^2 - 1\right| + 2$.

The graph of $y = f(x)$ is

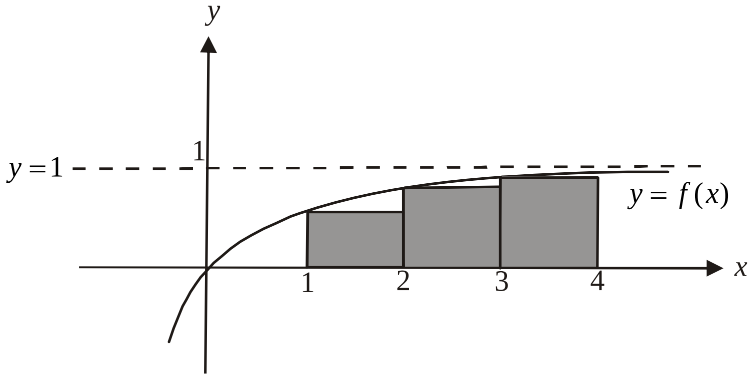
- dilated by a factor of 3 from the *x*-axis
- reflected in the *x*-axis

to become the graph of $y = g(x)$. The maximum value of *g* is

A. -9
B. -6
C. 3
D. 2
E. 9

Question 19

The approximate area enclosed by the graph of the function $f(x) = 1 - e^{-x}$, the *x*-axis and the line with equation $x = 4$ is found using the shaded rectangles below.



This approximate area underestimates the actual area by

A. 7%
B. 9%
C. 13%
D. 19%
E. 23%

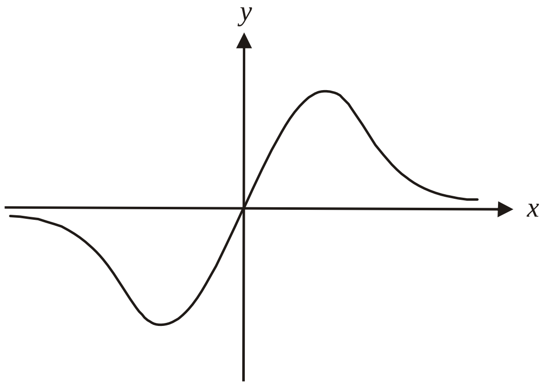
Question 20

$\log_a(b) \times \log_b(c)$ is equal to

A. $\log_c(a)$
B. $\log_b(a)$
C. $\log_a(c)$
D. $\log_c(b)$
E. $-\log_b(c)$

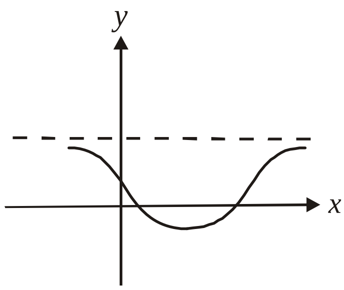
Question 21

The graph of the function g is shown below.

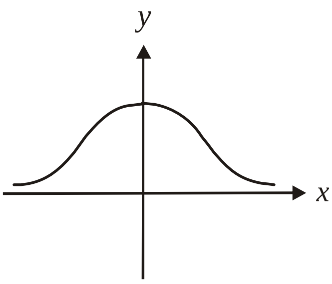


The graph of an **antiderivative** of g could be

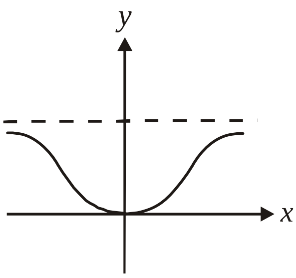
A.



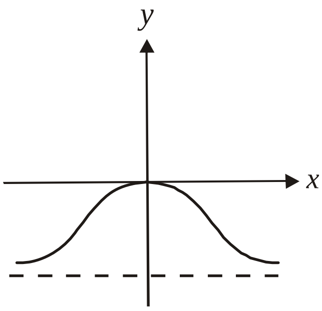
B.



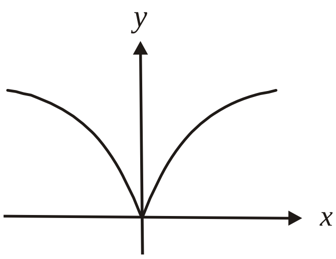
C.



D.



E.



Question 22

Nick plays a board game with his mate a number of times. Each time Nick plays, his chance of winning is 0.4. The probability that Nick wins at least two of the games is greater than 0.9. Nick must have played the game at least

A. 6 times
B. 8 times
C. 9 times
D. 11 times
E. 15 times

SECTION 2

Answer all questions in this section.

Question 1

Consider the function $f: R \to R,\ f(x) = e^{ax} - k$ where a and k are real constants and $a > 0$.

a. **i.** Find $f'(x)$.

ii. Explain why f has no stationary points.

$1+1=2$ marks

b. **i.** Find an expression in terms of a and k for the x-intercept of the graph of $y = f(x)$.

ii. Find the value(s) of k for which the graph of $y = f(x)$ has a positive x-intercept.

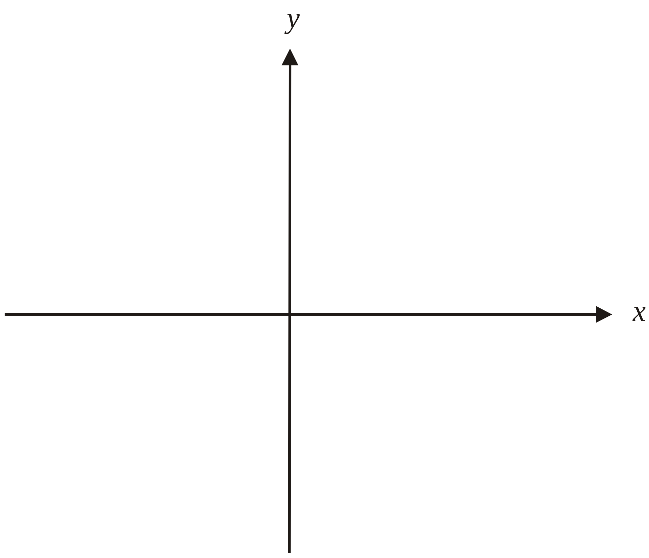
$1+1=2$ marks

c. Find the values of k for which the graph of $y = f(x)$ has a positive y-intercept.

2 marks

d. Let $k > 1$.

i. On the set of axes, below sketch the graph of $y = |f(x)|$. Label the *x* and *y* intercepts together with the equation of any asymptotes in terms of the parameters *a* and *k*.



ii. Write down an expression in terms of *a* and *k* that gives the area enclosed by the graph $y = |f(x)|$ and the *x* and *y* axes.

iii. Find the area described in part ii.

$3 + 2 + 1 = 6$ marks
Total 12 marks

Question 2

Brett and Dora each subscribe to a particular website.
The time X, in seconds, it takes Brett to log-in to this website has a probability density function given by

$$f(x)=\begin{cases}0 & x<0\\ 0.3125 & 0\le x\le 3\\ \dfrac{5}{16}e^{-5(x-3)} & x>3\end{cases}$$

The time Y, in seconds, it takes Dora to log-in to the same website is normally distributed with a mean of 2 and a standard deviation of 0.6.

a. Find $\Pr(Y\le 1)$. Express your answer correct to four decimal places.

1 mark

b. Sixty percent of the time, Dora can log-in to the website in under p seconds. Find the value of p correct to 4 decimal places.

1 mark

c. Find $\Pr(2\le X\le 4)$. Express your answer correct to four decimal places.

3 marks

d. Find the mode of Y.

1 mark

e. Find the variance of

i. Y

ii. X, given the mean of X is 1.60625. Express your answer correct to four decimal places.

$1+3=4$ marks

f. Six of Dora's log-in times are randomly selected to check the efficiency of her computer. What is the probability, correct to 4 decimal places that

i. exactly one of these log-in times took less than 2 seconds?

ii. at least one of these log-in times took less than 2 seconds?

$1+1=2$ marks

Brett and Dora's subscriptions to the website expire after a certain, large number of log-ins have been made. Both their subscriptions have just expired. The log-in times for both Brett and Dora during their subscriptions were analyzed and a randomly selected log-in time was found to be less than 1 second.

g. Find the probability correct to three decimal places that this log-in was one that Dora made.

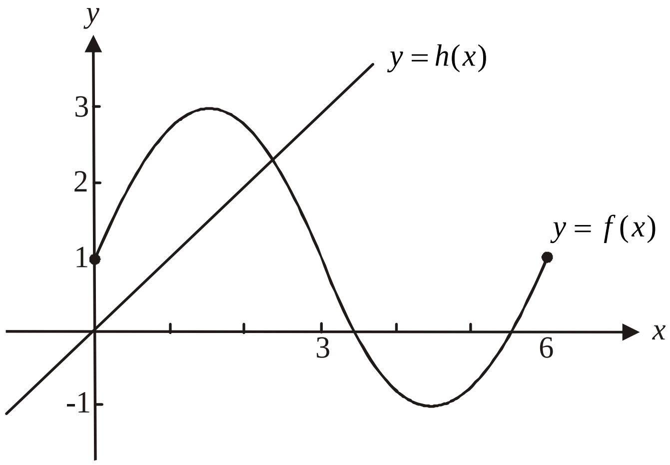
4 marks
Total 16 marks

Question 3

Let $f:[0,6]\to R,\ f(x)=2\sin\left(\frac{\pi x}{3}\right)+1$

and $h:R\to R,\ h(x)=x$

The graphs of f and h are shown below.



a. Show that the period of the graph of f is 6.

1 mark

b. For what values of x is the function f strictly decreasing?

1 mark

c. Explain why the inverse function f^{-1} does not exist.

1 mark

Let $g:[a,3]\to R,\ g(x)=2\sin\left(\frac{\pi x}{3}\right)+1$

d. **i.** Find the least value of *a* such that the inverse function g^{-1} exists.

ii. Sketch the graph of $y=g^{-1}(x)$ on the same set of axes as the graph of $y=f(x)$ shown above. Clearly mark the coordinates of the endpoints.

iii. Find the rule of the inverse function g^{-1}.

$1+2+1=4$ marks

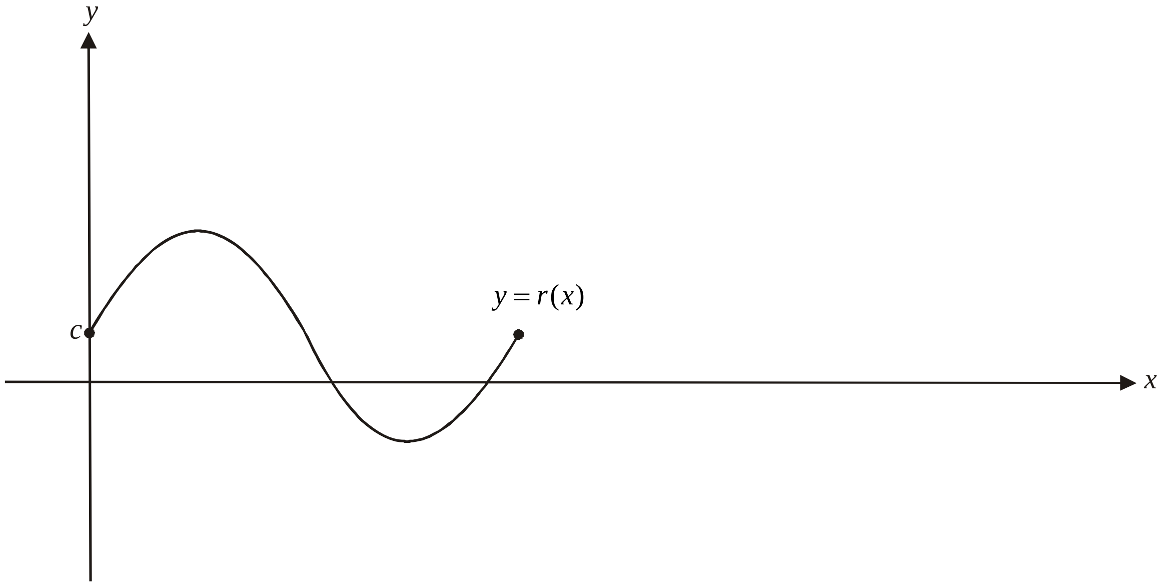
e. **i.** Find the value of x for which the graph of $y=g(x)$ intersects with the graph of $y=h(x)$. Express your answer correct to 3 decimal places.

ii. Hence write down a definite integral that gives the area enclosed by the graphs of $y=g(x)$ and $y=g^{-1}(x)$ together with the line with equation $x=3$.

$1+2=3$ marks

Let $r:[0,6]\rightarrow R,\ r(x)=a\sin\left(\frac{\pi x}{3}\right)+c$ where a and c are both positive constants and $a>c$.

The graph of $y=r(x)$ is shown below.



The graph of *r* undergoes the following transformations

- a dilation by a factor of 2 from the *y*-axis
- a reflection in the *x*-axis

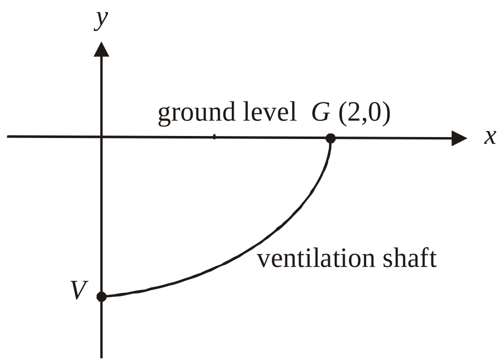
to become the graph of $y=t(x)$.

f. Sketch the graph of $y=t(x)$ on the set of axes above. Indicate clearly on your graph the coordinates of any endpoints and maximum and minimum points. It is not necessary to give the *x*-intercepts.

3 marks
Total 13 marks

Question 4

Victoria James is a spy.
She has become trapped in an underground mine at point *V* and her only means of escape is through a ventilation shaft with equation $y = -\sqrt{2-x},\ \ 0 \le x \le 2$ as shown below. The shaft ends at ground level at the point $G(2, 0)$ and all distances are measured in kilometres.



a. Find the coordinates of the point *V*, where Victoria is trapped.

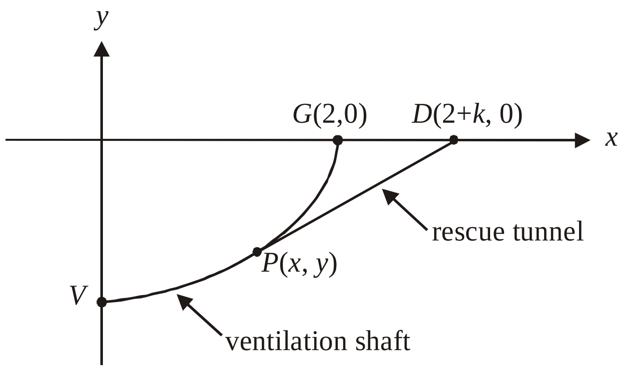
1 mark

Victoria can move along the ventilation shaft towards point *G* as long as the gradient of the shaft is two or less.

b. Show that the coordinates of the furthest point along the shaft that Victoria can reach are $\left(\frac{31}{16}, -\frac{1}{4}\right)$.

3 marks

In order to prevent the shaft caving in, Victoria's rescuers set up a drilling site at point $D\ (2+k, 0)$ where k is a constant and $k > 0$. From this point they will drill a straight tunnel that will intersect with the shaft as shown below. At this point of intersection $P(x, y)$, the gradient of the shaft will equal the gradient of the tunnel.



c. **i.** Find an expression in terms of x and k for the gradient of the rescue tunnel.

ii. Show that the coordinates of point P are $(2-k, -\sqrt{k})$.

$2+3=5$ marks

d. Find the possible values that k can take.

2 marks

The time taken by Victoria to move from point V through the shaft to point P is $6(\sqrt{2}-\sqrt{k})$ hours. From point P she can move through the rescue tunnel to point D at 0.5km/hr.

e. Show that the total time, in hours, taken by Victoria to escape from point V to point D is given by $T = 6\left(\sqrt{2}-\sqrt{k}\right) + 2\sqrt{k+4k^2}$.

2 marks

f. **i.** Find $\frac{dT}{dk}$.

ii. Hence find the coordinates of the point where the rescue tunnel should meet the ventilation shaft if Victoria is to escape in the minimum time. Express the coordinates correct to 2 decimal places.

iii. What is the minimum time that it takes Victoria to escape? Express your answer correct to 2 decimal places.

$1+2+1=4$ marks
Total 17 marks

Mathematical Methods (CAS) Formulas

Mensuration

area of a trapezium:	$\frac{1}{2}(a+b)h$	volume of a pyramid:	$\frac{1}{3}Ah$
curved surface area of a cylinder:	$2\pi rh$	volume of a sphere:	$\frac{4}{3}\pi r^3$
volume of a cylinder:	$\pi r^2 h$	area of a triangle:	$\frac{1}{2}bc\sin A$
volume of a cone:	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}\left(x^n\right)=nx^{n-1}$ | $\int x^n dx=\frac{1}{n+1}x^{n+1}+c,\ n\neq -1$

$\frac{d}{dx}\left(e^{ax}\right)=ae^{ax}$ | $\int e^{ax}dx=\frac{1}{a}e^{ax}+c$

$\frac{d}{dx}\left(\log_e(x)\right)=\frac{1}{x}$ | $\int \frac{1}{x}dx=\log_e|x|+c$

$\frac{d}{dx}\left(\sin(ax)\right)=a\cos(ax)$ | $\int \sin(ax)dx=-\frac{1}{a}\cos(ax)+c$

$\frac{d}{dx}\left(\cos(ax)\right)=-a\sin(ax)$ | $\int \cos(ax)\,dx=\frac{1}{a}\sin(ax)+c$

$\frac{d}{dx}\left(\tan(ax)\right)=\frac{a}{\cos^2(ax)}=a\sec^2(ax)$

product rule: $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$

quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

chain rule: $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$

approximation: $f(x+h)\approx f(x)+hf'(x)$

Probability

$\Pr(A)=1-\Pr(A')$

$\Pr(A\cup B)=\Pr(A)+\Pr(B)-\Pr(A\cap B)$

$\Pr(A|B)=\frac{\Pr(A\cap B)}{\Pr(B)}$

transition matrices: $S_n=T^n\times S_0$

mean: $\mu=\mathrm{E}(X)$

variance: $\mathrm{var}(X)=\sigma^2=\mathrm{E}((X-\mu)^2)=\mathrm{E}(X^2)-\mu^2$

probability distribution		mean	variance
discrete	$\Pr(X=x)=p(x)$	$\mu=\Sigma x\,p(x)$	$\sigma^2=\Sigma(x-\mu)^2\,p(x)$
continuous	$\Pr(a<X<b)=\int_a^b f(x)dx$	$\mu=\int_{-\infty}^{\infty}x\,f(x)dx$	$\sigma^2=\int_{-\infty}^{\infty}(x-\mu)^2\,f(x)\,dx$

MATHEMATICAL METHODS (CAS)

TRIAL EXAMINATION 2

MULTIPLE- CHOICE ANSWER SHEET

STUDENT NAME:..

INSTRUCTIONS

Fill in the letter that corresponds to your choice. Example: A ▬ C D E

The answer selected is B. Only one answer should be selected.

1.	A	B	C	D	E	12.	A	B	C	D	E
2.	A	B	C	D	E	13.	A	B	C	D	E
3.	A	B	C	D	E	14.	A	B	C	D	E
4.	A	B	C	D	E	15.	A	B	C	D	E
5.	A	B	C	D	E	16.	A	B	C	D	E
6.	A	B	C	D	E	17.	A	B	C	D	E
7.	A	B	C	D	E	18.	A	B	C	D	E
8.	A	B	C	D	E	19.	A	B	C	D	E
9.	A	B	C	D	E	20.	A	B	C	D	E
10.	A	B	C	D	E	21.	A	B	C	D	E
11.	A	B	C	D	E	22.	A	B	C	D	E