

2011 Mathematical Methods (CAS) 1: GA 2: Examination

GENERAL COMMENTS

In 2011, 15 494 students sat Mathematical Methods (CAS) examination 1.

While the quality of responses was varied, it was pleasing to see that students paid attention to the clarity and legibility of their written logic/reasoning. Students who were astute in their use of the conventions and notation of mathematics were less likely to lose marks for careless errors.

Several questions had a variety of solution pathways, some more efficient than others. Questions 5, 6, 8 and 9 have alternatives discussed below.

Students need to make good use of the 15 minutes of reading time. Not only should they read to make sense of the questions, but they should try to identify those that have familiar concepts and routines.

Time management appeared to be an issue for some students this year and Question 10 was either rushed or not attempted. Students are reminded that they do not need to answer questions sequentially. They should think about completing as much as they can of the straightforward questions before they embark on the longer, more involved questions.

In 2011, students would have drawn on knowledge and skills related to: algebraic manipulation; use of trigonometric ratios; simplification of fractions, ratios and surds; collecting like terms; factorisation of quadratics; perimeter of irregular polygons; area of triangles; basic probability, Venn diagrams and the null factor law.

Students should be aware that when solving equations involving factored expressions if $A \times B = 0$ then either $A = 0$, $B = 0$ or both could be 0. Similarly, students should know that if $f(x) \times g(x) = 0$ then they need to test both cases. In addition to this, when solving an equation of the form $f(x) \times g(x) = f(x) \times k(x)$ they should not divide through by $f(x)$ unless they have considered whether $f(x)$ could be equal to zero. The explicit elimination of impossible solutions indicates students who are demonstrating that they understand what is happening. A recommended approach in such situations is to collect terms, factorise, use the null factor law and eliminate or accept solutions; for example, $f(x) \times g(x) = f(x) \times k(x) \Rightarrow f(x) \times (g(x) - k(x)) = 0$ so $f(x) = 0$ and/or $g(x) = k(x)$.

Students should be able to use the chain rule for differentiation in combination with the product rule; find and use exact values in trigonometry; antidifferentiate expressions of the form $\frac{1}{ax+l}$; recognise a simple composite quadratic form; determine features of circular function graphs; determine a maximal domain; utilise the properties of probability distribution functions; determine when no solutions, infinitely many solutions or unique solutions occur for systems of simultaneous linear equations; formulate and calculate the area between two curves; and answer a 'show that' question.

When answering a 'show that' question there are several approaches students can take, depending on the question. One approach is to start with the left side of a given equation and manipulate it to obtain the right side (but not assume it equal to the right-hand side). A variation of this is to start with the left side and simplify it as much as possible, then work separately on the right side to see if the same point of simplification can be reached – thus showing that they both have the same result. The wording of the question is often a guide to the approach that can be used. Whatever the nature of the question, the student must provide clear working so it is obvious where decisions are made and so the process used is clear.

Question 10c. required students to show that the derivative was zero when $BD = 2CD$. This implies that after having worked out the rule for the derivative the student would use the fact that $BD = 2CD$ in their derivative and (if all was well) they would get 0 as the answer. The implication was not to start with the derivative equal to 0 and prove that $BD = 2CD$.

The necessity to write 'dx' at the end of an integral was understood by the majority of students. Fewer students were penalised for omitting this than in previous years.

It should be understood that answers are only accepted if they are a result of correct working. When a correct answer is presented without working, the maximum score that can be awarded is the one mark for the answer. The marks awarded

are an indication of working involved. Students attempting Question 2b. and Question 9 could end up with the seemingly correct answer from incorrect working and so have a zero score for these questions.

In general, students struggled with Questions 4b. and 10d. Question 4b. required them not only to understand that $(x+8)(x+2) \geq 0$ but also that this means $x+8 \leq 0$ or $x+2 \geq 0$. This could be readily observed by looking at a sketch of the quadratic graph. Question 10d. relied on the student using the knowledge from 10c. to deduce that the right triangle BCD had sides in the ratio of $1:2:\sqrt{5}$ (by using Pythagoras' theorem).

SPECIFIC INFORMATION

Question 1a.

Marks	0	1	Average
%	40	60	0.6

$$\frac{d(\sqrt{4-x})}{dx} = \frac{d((4-x)^{\frac{1}{2}})}{dx} = \frac{1}{2}(4-x)^{-\frac{1}{2}} \times -1 = \frac{-1}{2\sqrt{4-x}}$$

This was a differentiation question involving the use of the chain rule. Some students differentiated correctly but transcribed their final answer incorrectly, omitting the negative sign. Some had difficulty with the index, showing it as $\frac{3}{2}$.

Question 1b.

Marks	0	1	2	Average
%	14	42	44	1.3

$$g'(x) = 2x \sin(2x) + 2x^2 \cos(2x)$$

$$g'\left(\frac{\pi}{6}\right) = 2 \times \frac{\pi}{6} \times \sin\left(\frac{\pi}{3}\right) + 2 \times \frac{\pi^2}{36} \times \cos\left(\frac{\pi}{3}\right)$$

$$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \text{ and } \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

$$\text{Answer: } g'\left(\frac{\pi}{6}\right) = \frac{\pi\sqrt{3}}{6} + \frac{\pi^2}{36}$$

The product rule was generally used well by students. However, the determination of exact values was poor. Many students with the correct derivative did not get the second mark because they left their answers containing $\sin\left(\frac{\pi}{3}\right)$ and $\cos\left(\frac{\pi}{3}\right)$.

Question 2a.

Marks	0	1	Average
%	57	43	0.4

$$\frac{1}{3} \log_e(|3x-4|) + c \text{ or } \frac{1}{3} \log_e\left|x - \frac{4}{3}\right| + c$$

(The constant c is not necessary. It can be any real number, including zero.)

Common errors included missing the $\frac{1}{3}$ and omitting the modulus (absolute value) signs. Students should be aware that the general anti-derivative, $\int f(x)dx$, requires an arbitrary constant to be included.

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Question 2b.

Marks	0	1	2	3	Average
%	61	6	5	28	1

$$4^x - 15 \times 2^x = 16$$

$$\Rightarrow 2^{2x} - 15 \times 2^x - 16 = 0$$

$$\text{let } a = 2^x$$

$$\Rightarrow a^2 - 15a - 16 = 0$$

$$\Rightarrow (a - 16)(a + 1) = 0$$

$$\Rightarrow 2^x = 16 \text{ since } 2^x + 1 > 0$$

$$\Rightarrow x = 4$$

This question was poorly done. Many students started by writing $a = 2^x \Rightarrow 4^x = 2a$ rather than $a = 2^x \Rightarrow 4^x = a^2$. Several students resorted to guessing and checking that $x = 4$. Attempts involving logarithms rarely helped.

Question 3a.

Marks	0	1	2	Average
%	20	29	51	1.3

range: $[1, 7]$

period = 4

Common incorrect responses were a period of 4π or $\frac{4\pi}{\pi}$ and a range of $[-1, 7]$ or $[7, 1]$.

Question 3b.

Marks	0	1	2	Average
%	26	43	31	1.1

$$\sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2}, \quad \frac{\pi}{6} \text{ is base angle}$$

$$(2x + \frac{\pi}{3}) = \dots \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6} \dots$$

$$x = \frac{\pi}{4}, \frac{11\pi}{12} \text{ are the only solutions within domain}$$

Most students could determine the base angle. With some preferring to work in degrees, the final answer was required in radians. The majority of students who did get the base angle did not get both correct answers. This was either a result of poor algebra, poor arithmetic, choosing a value outside the required domain or not looking for multiple solutions.

Question 4a.

Marks	0	1	2	Average
%	8	17	75	1.7

$$f(g(x)) = \sqrt{(x+5)^2 - 9}$$

$$= \sqrt{(x+2)(x+8)}$$

$$\text{so } c = 2 \text{ and } d = 8 \quad \text{or} \quad c = 8 \text{ and } d = 2$$

The majority of students gained full marks for this question. However, many did not state a value for c and d , and others factorised incorrectly or did not expand $(x + 5)^2$ correctly.

Question 4b.

Marks	0	1	2	Average
%	84	6	10	0.3

maximal domain when $(x+c)(x+d) \geq 0$

$x \in (-\infty, -8] \cup [-2, \infty)$ or $R \setminus (-8, -2)$

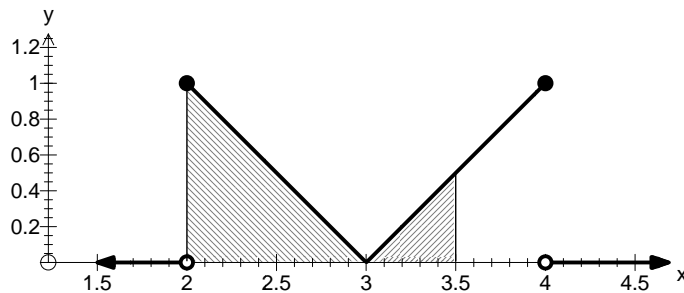
This question was very poorly done. Common incorrect responses included $[-3, 3]$, (the domain of $f(x)$); $x \geq -2$ (as the ‘intersection’ of $x \geq -8$ with $x \geq -2$); or $x \geq -8$ (as the ‘union’ of $x \geq -8$ with $x \geq -2$). Those who attempted to use the properties of composite functions tended to get confused. Students needed to look for a domain that would make the square root function work.

Question 5a.

Marks	0	1	2	Average
%	71	11	18	0.5

$\Pr(X < 3.5) = \Pr(X < 3) + \Pr(3 < X < 3.5)$

Method 1. using triangles: $(0.5 \times 1 \times 1) + (0.5 \times 0.5 \times 0.5) = 0.5 + 0.125 = 0.625$



Method 2. By integration: $\int_2^{3.5} |3-x| dx$

$$= \int_2^3 (3-x) dx + \int_3^{3.5} (x-3) dx = \left[3x - \frac{x^2}{2} \right]_2^3 + \left[\frac{x^2}{2} - 3x \right]_3^{3.5}$$

$$= 0.625$$

This question was very poorly done. Few students understood what the graph of the function looked like or that the probability (given by the area under the graph) had to be between 0 and 1. Many students gave answers with negative values and values greater than 1.

The following incorrect determination of the area was common.

$$\int_2^{3.5} |3-x| dx = \left[3x - \frac{x^2}{2} \right]_2^{3.5} = 0.375$$

Question 5b.

Marks	0	1	2	Average
%	54	32	14	0.6

$$\begin{aligned} & \Pr(X < 2.5 | X < 3.5) \\ &= \frac{\Pr(X < 2.5 \cap X < 3.5)}{\Pr(X < 3.5)} = \frac{\Pr(X < 2.5)}{\Pr(X < 3.5)} \\ &= \frac{0.375}{0.625} = \frac{3}{5} = 0.6 \end{aligned}$$

The majority of students were aware that $\Pr(X < 2.5 \cap X < 3.5)$ was $\Pr(X < 2.5)$. However, a few expressed it as $\Pr(X < 2.5) \cap \Pr(X < 3.5)$, which is not correct. More correct expressions for the numerator were evident than for the denominator.

Question 6a.

Marks	0	1	2	3	Average
%	43	22	10	25	1.2

The following are three different methods that can be used to identify and justify $k = -4$.

Method 1: Using gradient, m , and y -intercept, c .

$$m_1 = \frac{k}{3}, c_1 = -\left(\frac{k+3}{3}\right)$$

$$m_2 = \frac{-4}{k+7}, c_2 = \frac{1}{k+7}$$

from the gradient $k(k+7) = -12$ so

$$k = -4 \text{ or } k = -3$$

from the intercepts $k = -3 \Rightarrow c_1 = 0$ and $c_2 = \frac{1}{4}$ so there are no solutions in this case

$$k = -4 \Rightarrow c_1 = c_2 = \frac{1}{3} \text{ so there are infinitely many solutions in this case}$$

Answer: $k = -4$

Method 2: Using ratios

Comparison of coefficients as common ratios (all ratios would require the same value of k to have infinite solutions).

$$\frac{k}{4} = \frac{-3}{k+7} = \frac{k+3}{1}$$

$$\text{Using } \frac{k}{4} = \frac{-3}{k+7}$$

$$k^2 + 7k = -12 \Rightarrow k = -4 \text{ or } k = -3$$

$$\text{Using } \frac{-3}{k+7} = \frac{k+3}{1}$$

$$k^2 + 10k + 21 = -3 \Rightarrow k = -4 \text{ or } k = -6$$

$$\text{Using } \frac{k}{4} = \frac{k+3}{1} \Rightarrow k = -4$$

$k = -4$ satisfies all 3 ratios and the lines will be identical.

Answer: $k = -4$

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Method 3: Using determinant

$$\text{determinant} = k(k+7)+12$$

determinant = 0 for infinitely many solutions or no solutions

$$k^2 + 7k + 12 = (k+4)(k+3) = 0$$

$$\Rightarrow k = -4 \text{ or } k = -3$$

when $k = -3$ the two equations become

$$-3x - 3y = 0 \text{ and } 4x + 4y = 1 \text{ so there are no solutions in this case}$$

when $k = -4$ the two equations become

$$-4x - 3y = -1 \text{ and } 4x + 3y = 1 \text{ so there are infinitely many solutions in this case}$$

Answer: $k = -4$

Many students who tried to solve simultaneous equations rarely succeeded. Other variations of these methods were seen. Whichever method was chosen, students had to justify their selection of k .

Question 6b.

Marks	0	1	Average
%	67	33	0.4

For unique solutions

$$k^2 - 7k + 12 \neq 0$$

$$\therefore k \in R \setminus \{-4, -3\}$$

Most students were aware of the two values of k that led to either no or infinite solutions. The most successful approach was using the determinant. Many students chose a value of k that was not selected as the answer to **part a**.

Question 7ai.

Marks	0	1	Average
%	29	71	0.7

$$p^3$$

This question was mostly well done, although $p \times p \times p = 3p$ was a common incorrect response. There were also a number of responses where students incorrectly assigned a numerical value to p .

Question 7a.ii.

Marks	0	1	Average
%	59	41	0.4

$$3p^2(1-p)$$

The majority of incorrect responses neglected the 3 (for the number of possible arrangements).

Question 7b.

Marks	0	1	2	Average
%	24	58	18	1

$$p^3 = 3p^2 - 3p^3$$

$$4p^3 - 3p^2 = p^2(4p - 3) = 0$$

$$p = 0 \text{ and } p = \frac{3}{4}$$

The cancelling out of p was rarely supported. Many students incorrectly assumed p could not be zero.

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Question 8a.

Marks	0	1	2	Average
%	34	29	38	1.1

$$\Pr(A' \cap B) = \Pr(A \cup B) - \Pr(A) = \frac{3}{4} - \frac{3}{5} = \frac{3}{20}$$

Alternatively:

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

$$\Rightarrow \Pr(A \cap B) = \Pr(A) + \Pr(B) - \Pr(A \cup B) = \frac{3}{5} + \frac{1}{4} - \frac{3}{4} = \frac{1}{10}$$

	A	A'	
B	$\frac{1}{10}$	$\Pr(A' \cap B)$	$\frac{1}{4}$
B'			
	$\frac{3}{5}$		1.0

From the table

$$\Pr(A \cap B') = \frac{1}{4} - \frac{1}{10} = \frac{3}{20}$$

A simple Venn diagram would have assisted many students in answering this question. Most students tried, with varying measures of success, to utilise the relation $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$. There were a number of students who used this but wrote $\Pr(A' \cap B)$ when rearranging it. Arithmetic was a weakness for some students.

Question 8b.

Marks	0	1	Average
%	56	44	0.5

$$\Pr(A \cap B) = 0 \Rightarrow \Pr(A' \cap B) = \Pr(B) = \frac{1}{4}$$

The most common incorrect response was due to students interpreting 'mutually exclusive' as 'independent'.

Question 9

Marks	0	1	2	3	4	Average
%	27	21	27	8	16	1.7

$$\text{Point of intersection } (m, am) \Rightarrow am = m^3 - am, m > 0 \Rightarrow m^2 = 2a$$

$$\text{Area} = \int_0^m (ax - (x^3 - ax)) dx = 64$$

$$\text{or Area} = \int_0^{\sqrt{2a}} (2ax - x^3) dx = 64$$

$$\Rightarrow \left[ax^2 - \frac{x^4}{4} \right]_0^{\sqrt{2a}} = (2a^2 - \frac{4a^2}{4}) = a^2 = 64$$

$$\Rightarrow a = 8 \text{ as } m = \sqrt{16} = 4 \text{ since both } a \text{ and } m \text{ are greater than zero}$$

The majority of students could set up the correct integral equation for the area, but only a few realised the significance of the point of intersection to achieving the final answers. Many students spent time determining the 'area between two curves' by relating all areas back to the x -axis. Instead of a single definite integral to determine the area, they had three definite integrals or the area of a triangle and two definite integrals. Poor use of brackets led to a number of students incorrectly cancelling ax in the integral and then quickly getting the answer $m = 4$. Some students incorrectly left the point of intersection in terms of x with $ax = x^3 - ax$, $\Rightarrow x^2 = 2a \Rightarrow a = \frac{x^2}{2}$ and substituted this directly into the integral.

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Question 10a.

Marks	0	1	2	Average
%	34	4	62	1.3

$$BD = a \cos(\theta) \text{ and } CD = a \sin(\theta)$$

This question required a straightforward application of trigonometry for a right-angled triangle. However, some students tried to use Pythagoras' theorem, reciprocal or inverse functions. Others mixed variables as well as units. Very few students had only one of the correct answers. Some students did not answer this question.

Question 10b.

Marks	0	1	Average
%	42	58	0.6

$$L = 2 + 2 + a + BD + CD + AE$$

$$L = 4 + a + 2BD + CD = 4 + a + 2a \cos(\theta) + a \sin(\theta)$$

The second horizontal length of wire (AE or BD) was missing from some responses, despite particular mention in the question.

Question 10c.

Marks	0	1	2	Average
%	58	14	28	0.7

$$\frac{dL}{d\theta} = -2a \sin(\theta) + a \cos(\theta) \quad (1)$$

$$\text{When } BD = 2CD \Rightarrow a \cos(\theta) = 2a \sin(\theta)$$

substitute in (1)

$$\Rightarrow \frac{dL}{d\theta} = -2a \sin(\theta) + 2a \sin(\theta) = 0 \text{ as required}$$

Most students who had L in terms of θ could differentiate it correctly. Some incorrect responses were a consequence of differentiating with respect to a , attempting to differentiate with respect to a and θ , or differentiating $a \sin(\theta)$ as $a \theta \cos(\theta)$. The 'show that' part of the question was rarely done well, with many students making a direct substitution, but making no mention of either BD or CD .

Question 10d.

Marks	0	1	Average
%	95	5	0.1

Using trigonometry:

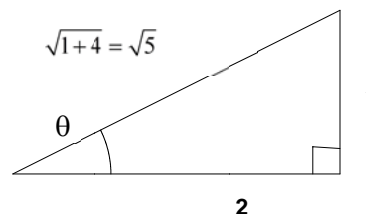
$$\tan(\theta) = \frac{1}{2} \Rightarrow \sin(\theta) = \frac{1}{\sqrt{5}}, \cos(\theta) = \frac{2}{\sqrt{5}}$$

$$BD = a \cos(\theta) = 3\sqrt{5} \times \frac{2}{\sqrt{5}} = 6$$

$$CD = a \sin(\theta) = 3\sqrt{5} \times \frac{1}{\sqrt{5}} = 3$$

$$L = 4 + a + 2BD + CD = 4 + 3\sqrt{5} + 2 \times 6 + 3$$

$$L = 19 + 3\sqrt{5}$$



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Alternatively:

Maximum L when $BD = 2CD$,

$$a^2 = (2CD)^2 + (CD)^2 = 5(CD)^2$$

$$\text{If } a = 3\sqrt{5} \Rightarrow a^2 = 45 \Rightarrow CD = 3 \Rightarrow BD = 6$$

$$L = 4 + a + 2BD + CD = 4 + 3\sqrt{5} + 12 + 3 = 19 + 3\sqrt{5}$$

$$L = 19 + 3\sqrt{5}$$

Few students obtained the correct answer. It was not a simple one-step problem and most students did not make the connection that the condition stated in Question 10c. provided enough information to use Pythagoras' theorem and calculate the lengths of all the sides necessary to determine the maximum L .