QATs
Quality Assessment Tasks

UNIT 4 ■ OUTCOMES 1, 2 & 3

VCE Mathematical Methods (CAS)

SCHOOL-ASSESSED COURSEWORK

Introduction

Outcome 1
Define and explain key concepts as specified in the content from the areas of study, and apply a range of related mathematical routines and procedures.

Outcome 2
Apply mathematical processes in non-routine contexts, and analyse and discuss these applications of mathematics.

Outcome 3
Select and appropriately use technology to develop mathematical ideas, produce results and carry out analysis in situations requiring problem-solving, modeling or investigative techniques or approaches.

Task

Analysis Task

The task has been designed to allow achievement up to and including the highest level in the Performance Descriptors. It covers a broad range of **key knowledge** and **key skills** over the three outcomes for Unit 4.

It will contribute 20 out of the total (40) marks allocated for SAC in Unit 4.

This task will be marked out of 80 and then will be converted to a proportion of the contribution of this task to SAC in this unit. The marks for each question are indicated in brackets.
You have 120 minutes over no more than two days. Answer in space provided or as directed.

You can access your logbook and an approved CAS calculator.

CAS TECHNOLOGY Indicates where use of the technology is specifically required in order to answer the question.
Your teacher will advise you of any variation to these conditions.

Marking scheme

Question	1	2	3	4	5	6
Marks	15	14	11	13	12	15

NAME:

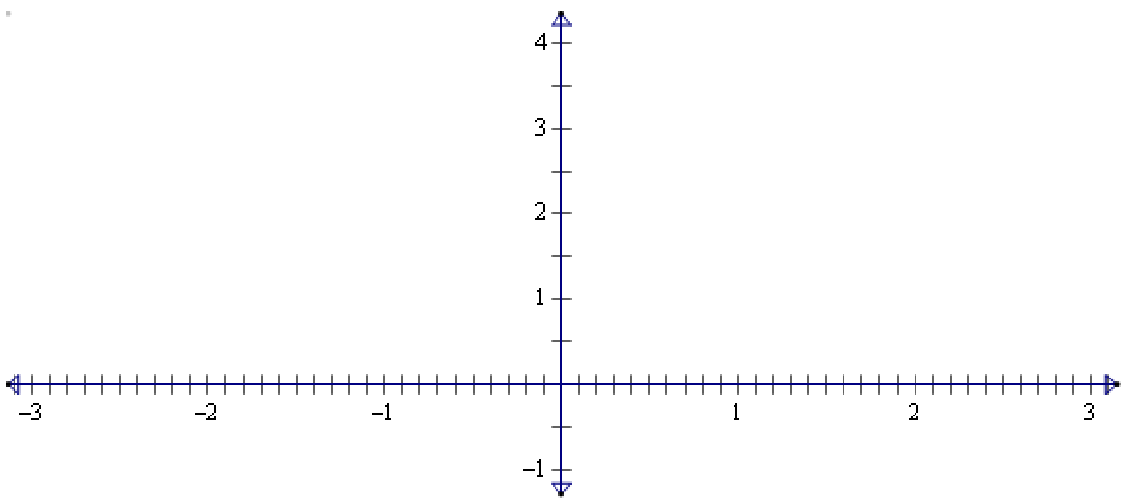
Task

PART 1

Question 1 (15 marks)

Reflecting telescopes often use parabolic mirrors to focus the light from distant objects onto a single point. This question will explore the property of parabolas that makes this possible.

a) Sketch the graph of $y = \frac{x^2}{4}$ on the axes provided.



(2 marks)

b) Using differentiation, find the equation for the gradient at any point on the parabola.

(2 marks)

Task

c) Find the equation of the normal to the curve $y = \frac{x^2}{4}$ where $x = 2$. Draw this line onto your graph.

(3 marks)

d) Now draw in the line $x = 2$ onto your graph.

(1 mark)

e) Determine the size of the acute angle between the line $x = 2$ and the normal to the parabola.

(2 marks)

Task

f) If the line $x = 2$ is reflected in the normal, the image has an equation of y = 1. Explain why.

(2 marks)

g) For the point $\left(1, \frac{1}{4}\right)$ on the parabola, the equation of the normal is $y = -2x + \frac{9}{4}$. A vertical beam of light along $x = 1$ would be reflected in $y = -2x + \frac{9}{4}$ to follow a line with a gradient of $\frac{-3}{4}$. Determine the value of the y-intercept for this light beam.

(2 marks)

h) Consider your answers for questions f) and g). What is significant about them?

(1 mark)

Task

Question 2 (15 marks)

Air pressure decreases exponentially with altitude.

Given that air pressure at sea level is 101.325 kPa and at a height of 2000m is 76.143 kPa:

$$P = P_0 e^{kh}$$

a) Write down the value of P_o.

(1 mark)

b) Find an algebraic expression for k and then write down the value of k to six decimal places.

(2 marks)

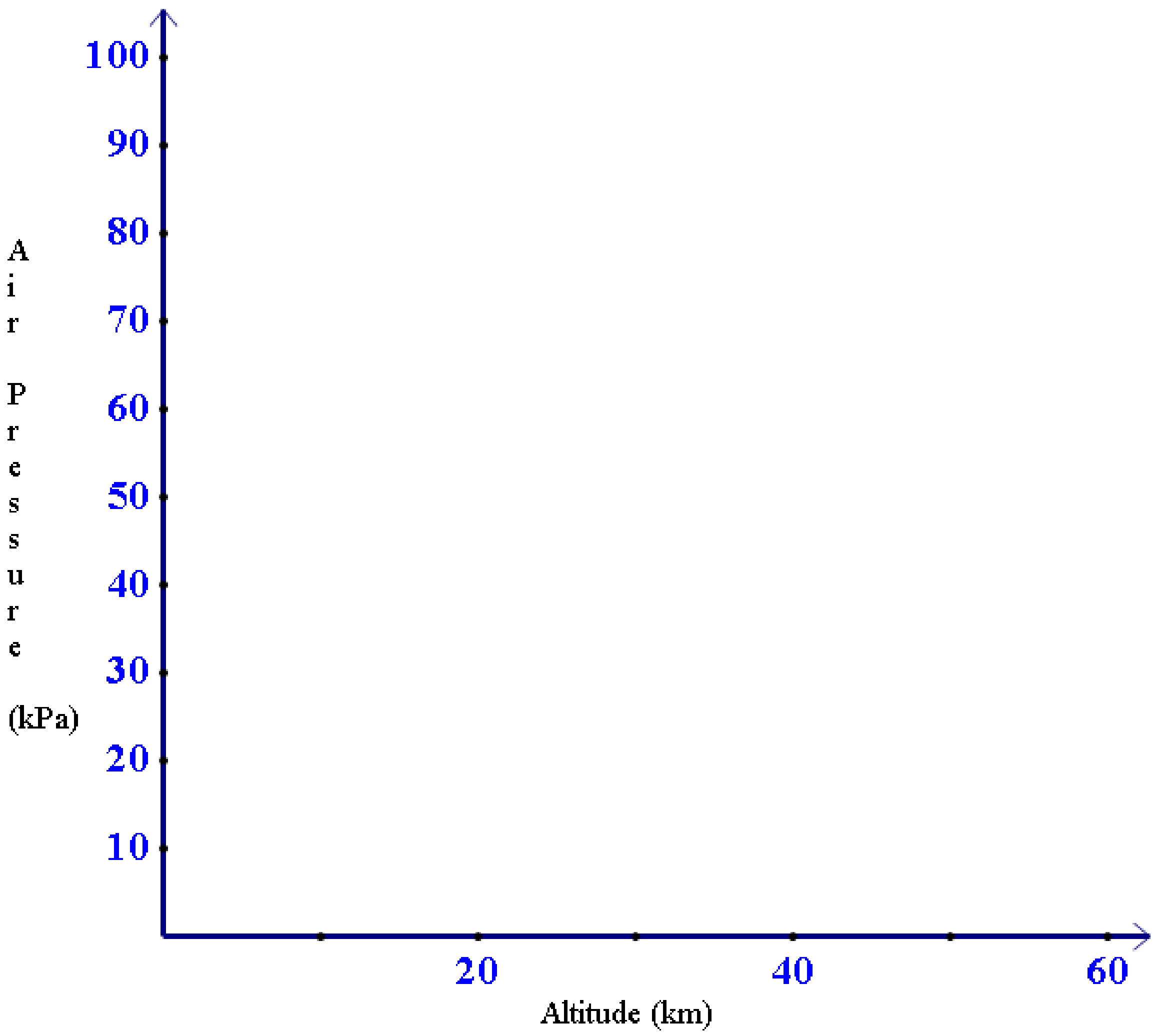
c) Use your formula to complete the table below.

Mountain	Altitude at Summit	Air Pressure at Summit
Mt Kosciusko	2228 m	
Mt Everest		28.6256 kPa

(1 mark)

Task

d) Sketch a graph of Pressure vs Altitude *in kilometres* on the axes provided. Clearly label all intercepts and asymptotes.



(3 marks)

1 kilopascal is the equivalent of 1000 kg of mass resting on an area o 1 m^2

e) Air pressure at sea level, then is equivalent to ______________________ kg resting on an area of 1 m^2.

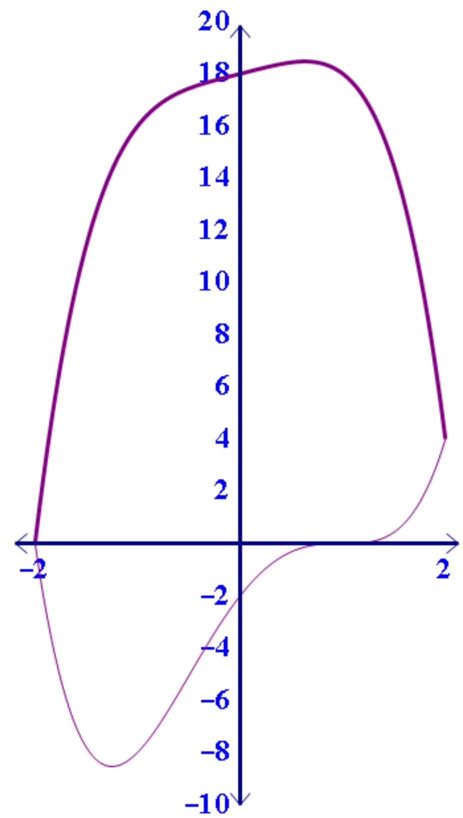
(1 mark)

Task

A swimming pool has a shape that can be modelled by the functions:

$$f(x) = -x^4 + x + 18 \quad -2 \le x \le 2$$
$$g(x) = (x+2)(x-1)^3 \quad -2 \le x \le 2$$

All measurements are in metres.



f) Use calculus to determine the integral of *f(x)*.

(2 marks)

g) Use calculus to determine the integral of *g(x)*.

(3 marks)

h) Now find the area enclosed by the two curves.

(1 mark)

i) At sea level, what is the equivalent mass of air above the pool?

(1 mark)

Task

Question 3 (11 marks)

CAS TECHNOLOGY

a) Use your calculator to collect and simplify the expression $\left|a\sin(x)\cos(x)-2a\sin^3(x)\cos(x)\right|$, where a is a constant.

(1 mark)

b) Use calculus to differentiate $f(x)=\left|\sin(4x)\right|$.

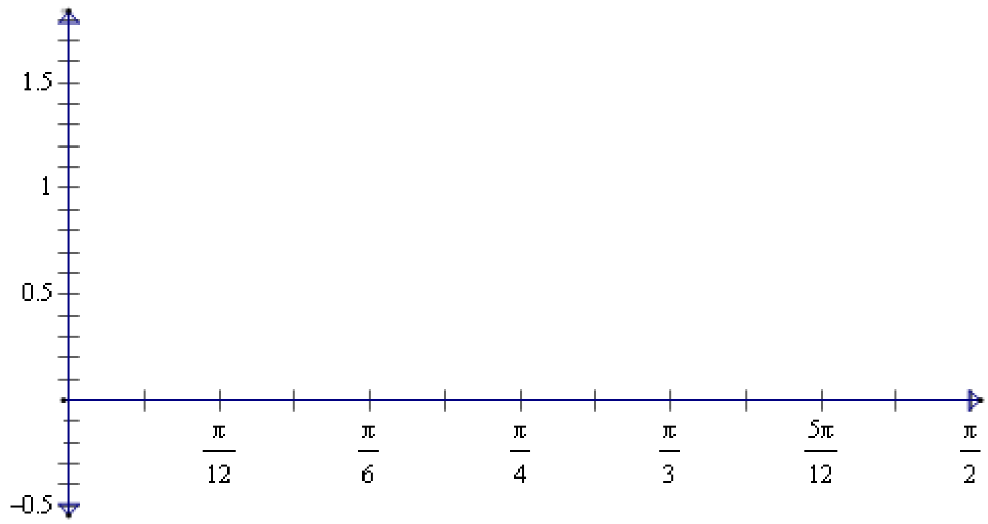
(2 marks)

c) Is the function *f(x)* differentiable at $x=\frac{\pi}{4}$? Explain your answer.

(2 marks)

Task

d) Sketch the graph of $f(x) = |\sin(4x)|\ 0 \le x \le \frac{\pi}{2}$



(3 marks)

e) Set up an expression that will determine the area of $f(x) = |4\sin(x)|\ 0 \le x \le \frac{\pi}{2}$.

(2 marks)

f) Now find the area enclosed by *f(x)* and the x-axis.

(1 mark)

End of Part 1

Task

PART 2

Question 4 (13 marks)

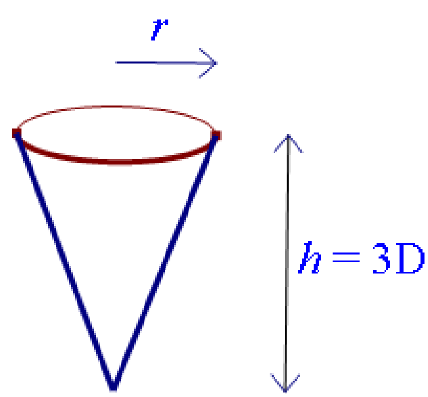
A rocket is propelled by a liquid fuel stored in a conical tank. The volume of fuel used is a constant 100 litres per second.

a) Explain why the value of $\frac{dV}{dt}$ must be negative.

(1 mark)

b) The ratio of the height, h, of the conical tank to its diameter, D, is such that h = 3D. If the volume of any cone is given by $V = \frac{1}{3}\pi r^2 h$, show algebraically that the volume of this tank can be written in terms of r only as:

$$V = 2\pi r^3$$



(2 marks)

c) Find an expression for the rate of change of radius with time.

(3 marks)

Task

d) How fast is the radius of the surface of the fuel changing when the radius is 0.5 m?

(2 marks)

e) If the original volume of fuel in the tank was 1000 L:

i. How long will it take for the tank to be empty?

(1 mark)

ii. For what value of r will $\frac{dr}{dt} = \frac{-25}{6\pi}$ ms^{-2}?

(2 marks)

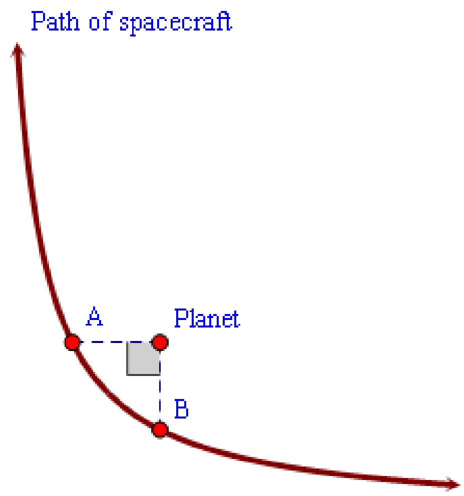
iii. What time does this correspond to?

(2 marks)

Task

Question 5 (12 marks)

A spacecraft is going to use the gravity of a nearby planet to make a change in direction. It has been calculated that the craft will pass through points A and B at a distance of 5 000 km from the planet. The angle formed by A and B with the planet is 90°.



a) Taking 10 000 km to be 1 unit, and the planet to be the origin, write down the coordinates of points A and B.

A = (_____, 0) and B = (0, _____)

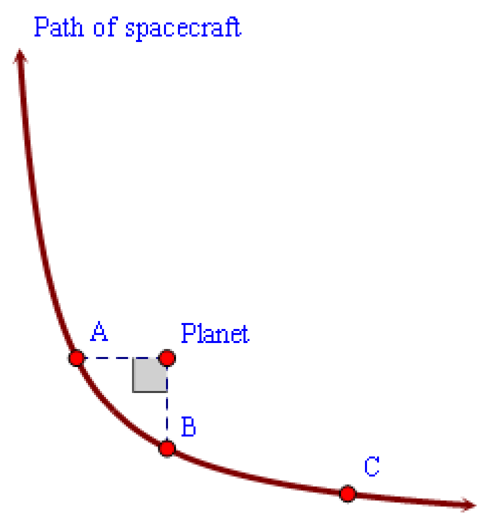
(2 marks)

b) The path of the spacecraft can be modelled by the function $f(x)=\frac{a}{(x+1)}-1$. If a third point C on the path has the co-ordinates $\left(1, \frac{-3}{4}\right)$, show that the value of a is $\frac{1}{2}$.

(2 marks)

Task

c) Taking the path of the craft to be the hyperbola described above, add in axes, asymptotes and an appropriate scale to the diagram shown below.



(3 marks)

d) Using the equation of the hyperbola and the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ set up an equation for finding the distance between the planet and the spacecraft at any time.

(2 marks)

CAS TECHNOLOGY

e) Using your calculator determine the minimum distance between the planet and the craft. Quote your answer to the nearest kilometre.

(3 marks)

Task

Question 6 (15 marks)

A polynomial function of degree 4 is known to pass through the following points:

A(-3, 1), B(-2, 3), C(-1, -1), D(1, -3) and E (2, 1).

In order to determine the values of the constants for this function, the matrix equation below is developed.

$$\begin{bmatrix} 81 & -27 & 9 & -3 & 1 \\ 16 & -8 & 4 & -2 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 16 & 8 & 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{s} \\ \boldsymbol{t} \\ \boldsymbol{u} \\ \boldsymbol{v} \\ \boldsymbol{w} \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \\ -3 \\ 1 \end{bmatrix}$$

a) Write down the value of element 2,4 from the square matrix above.

(1 mark)

CAS TECHNOLOGY

b) Use your calculator to find the determinant of the square matrix.

(1 mark)

Task

CAS TECHNOLOGY

c) Use your calculator to determine the inverse of the square matrix and fill in the blank entries below:

$$\begin{bmatrix} \frac{1}{40} & \frac{-1}{12} & \frac{1}{12} & \frac{-1}{24} & \frac{1}{60} \\ \_\_\_ & \frac{-1}{12} & \frac{1}{6} & \frac{-1}{6} & \frac{1}{12} \\ \frac{-1}{8} & \frac{7}{12} & \frac{-7}{12} & \frac{1}{24} & \frac{1}{12} \\ 0 & \frac{1}{12} & \frac{-2}{3} & \frac{2}{3} & \frac{-1}{12} \\ \frac{1}{10} & \frac{-1}{2} & \_\_\_ & \frac{1}{2} & \frac{-1}{10} \end{bmatrix}$$

(2 marks)

d) When solved, the polynomial is found to have the equation

$$f(x) = \frac{-x^4}{6} + \frac{x^3}{6} + \frac{13x^2}{6} - \frac{7x}{6} - 4$$

Use calculus to determine the derivative of the function.

(2 marks)

e) Using your derivative, find the gradient when x = 1.

(1 mark)

Task

CAS TECHNOLOGY

f) Determine the values of the x-intercepts of the function *f(x)* to 4 decimal places.

x_1 = _________________

x_2 = _________________

x_3 = _________________

x_4 = _________________

(2 marks)

g) Using your results to part f), set up an integrand to evaluate the integral bounded by the lower and upper x-intercepts, the function f(x) and the x-axis.

(3 marks)

CAS TECHNOLOGY

h) Now evaluate the integral in part g).

(1 mark)

i) Find the average value of the f(x) between its lowest and highest x-intercepts.

(2 marks)

End Part 2

Teacher Advice

This is the Analysis Task, as suggested to be undertaken in weeks 12 and 13 in the sample teaching sequence on page 193 of the VCAA Study Design.

This task contributes 20 of the 40 SAC marks in Unit 4.
The coursework scores for this task are:

Outcome 1	7 marks	35%
Outcome 2	8 marks	40%
Outcome 3	5 marks	25%
TOTAL	20 marks	

This weighting can be used in the conversion of their mark out of 50.
For example, a score of 40 results in:

OUTCOME 1	OUTCOME 2	OUTCOME 3
40/80*20*0.35	40/80*20*0.4	40/80*20*0.25
= 3.5	= 4	= 2.5
= 4 (rounded up)	= 4 (rounded down)	= 3 (rounded up)

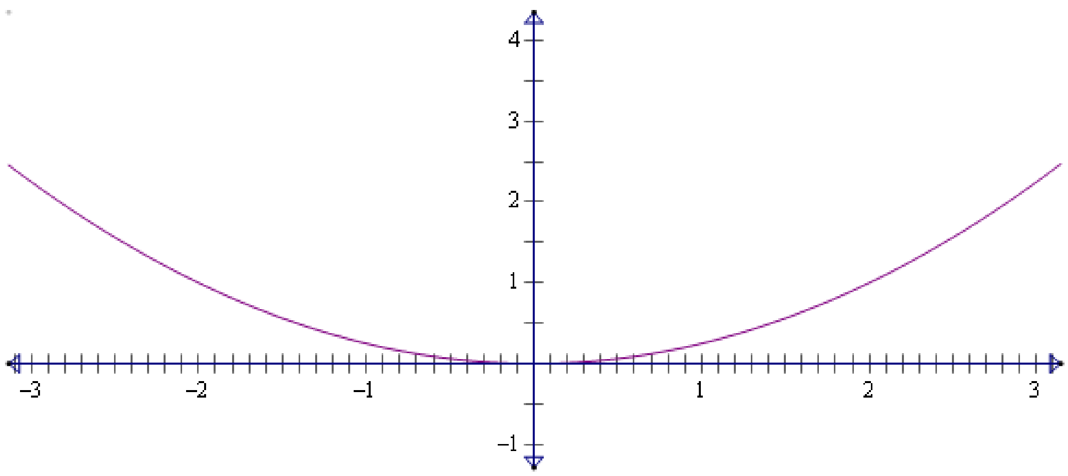
The above can be established in an Excel file.
This QAT has been designed to meet the highest level in the performance descriptors provided by VCAA for each outcome in unit 3 in the Assessment Handbook 2006-14.

Solution Pathway

PART 1

Question 1

a)



1 mark shape, 1 mark position

b)

$\frac{dy}{dx}=\frac{x}{2}$ 1 mark

$\therefore m=\frac{x}{2}$ 1 mark

c)

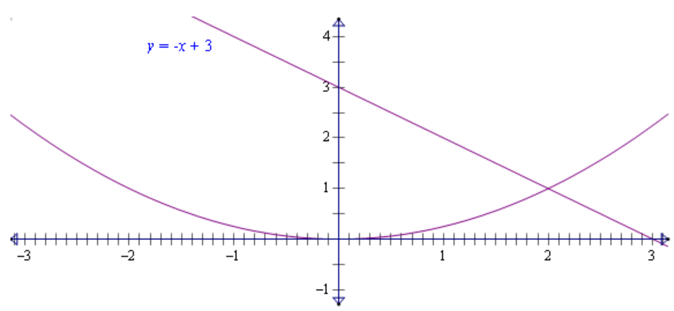
$m_T=\frac{2}{2}=1$

$\therefore m_N=-1$ 1 mark

$y = -1x + c$ at (2, 1)

c = 3

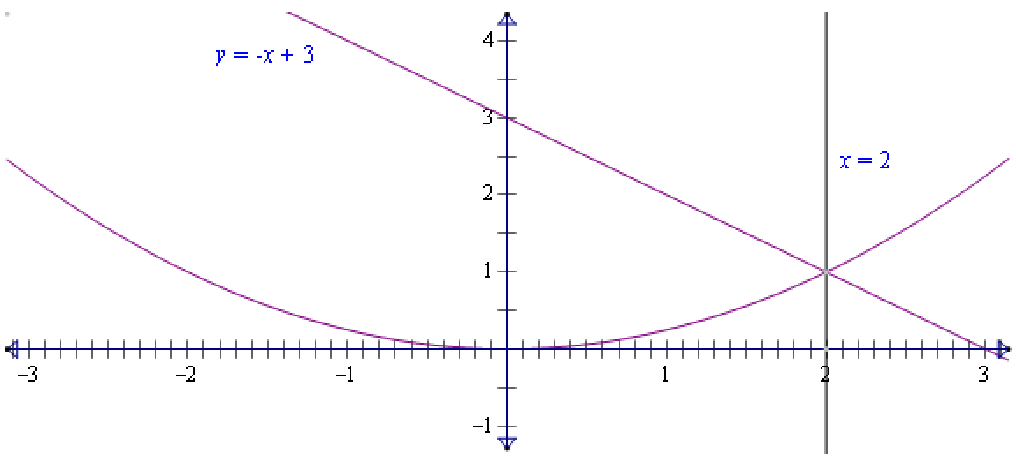
$\therefore y = -x + 3$ 1 mark



Solution Pathway

1 mark for correct placement of line on graph.

d) 1 mark for correct placement.



e)

$\tan^{-1}(m_T) = \tan^{-1}(-1) = 135^o$ 1 mark

The line x = 2 is a vertical line making an angle of 90^o to the x-axis

$\therefore$ Acute angle = $135^o - 90^o = 45^o$ 1 mark

f) As with reflection in the line y = x the x and y values are reflected through 90^o. (1 mark)

(Accept angle of incidence equals angle of reflection.)

g) Students can draw in a line from $\left(1, \frac{1}{4}\right)$ with an angle of $\frac{-3}{4}$ or

$$y = \frac{-3}{4}x + c$$

$$c = y + \frac{3}{4}x$$

$$At \left(1, \frac{1}{4}\right), c = 1$$

1 mark for method, 1 mark for c = 1

h) Both lines have a y-intercept of 1. 1 mark

Solution Pathway

Question 2

a) P_o = 101.325 kPa 1 mark

b)

$$76.143 = 101.325e^{2000\boldsymbol{k}}$$

$$\ln\left(\frac{76.143}{101.325}\right) = 2000\boldsymbol{k}$$

$$\therefore \boldsymbol{k} = \frac{1}{2000}\ln\left(\frac{76.143}{101.325}\right)$$ 1 mark

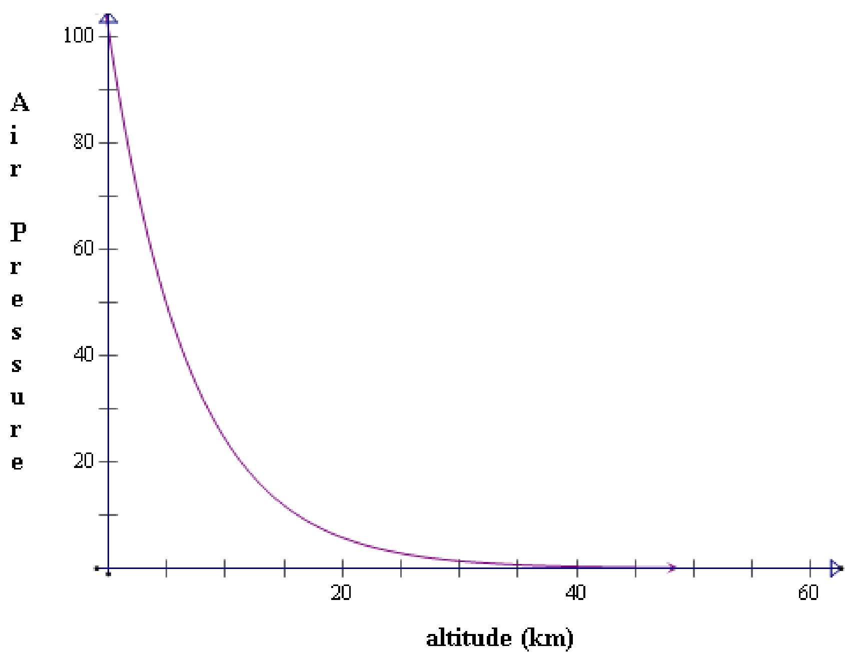
$$\therefore \boldsymbol{k} \approx -0.000143$$ 1 mark

c)

Mountain	Altitude at Summit	Air Pressure at Summit
Mt Kosciusko	2228 m	**76.6798 kPa**
Mt Everest	**8839.38 m**	28.6256 kPa

1 mark for BOTH correct.

d) 1 – shape, 1 y-intercept, 1 – asymptote at y = 0,



Solution Pathway

e) 101325 kg 1 mark

f)

$$\int_{-2}^{2} f(x)dx = \left[\frac{-x^5}{5} + \frac{x^2}{2} + 18x\right]_{-2}^{2}$$ 1 mark

$$= \left[\frac{-32}{5} + \frac{4}{2} + 36\right] - \left[\frac{32}{5} + \frac{4}{2} - 36\right]$$

$$= 59.2$$ 1 mark

g)

$$\int_{-2}^{2} g(x)dx = \int_{-2}^{2} (x+2)(x-1)^3\, dx$$ 1 mark

$$= \int_{-2}^{2} (x^4 - x^3 - 3x^2 + 5x - 2)\, dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4} - x^3 + \frac{5x^2}{2} - 2x\right]_{-2}^{2}$$ 1 mark

$$= -11.6$$ 1 mark

h) Area = 59.2 - -11.6 = 70.4 m^2 1 mark

i) 70.4 x 101325 = 7133280 kg 1 mark

Question 3

a) $tcollect(|a\sin(x)\cos(x) - 2a\sin^3(x)\cos(x)|) = \frac{|a\sin(4x)|}{4}$ 1mark

b) $f'(x) = \begin{cases} 4\cos(4x) & (0, \frac{\pi}{4}] \\ -4\cos(4x) & (\frac{\pi}{4}, \frac{\pi}{2}] \end{cases}$ *etc*

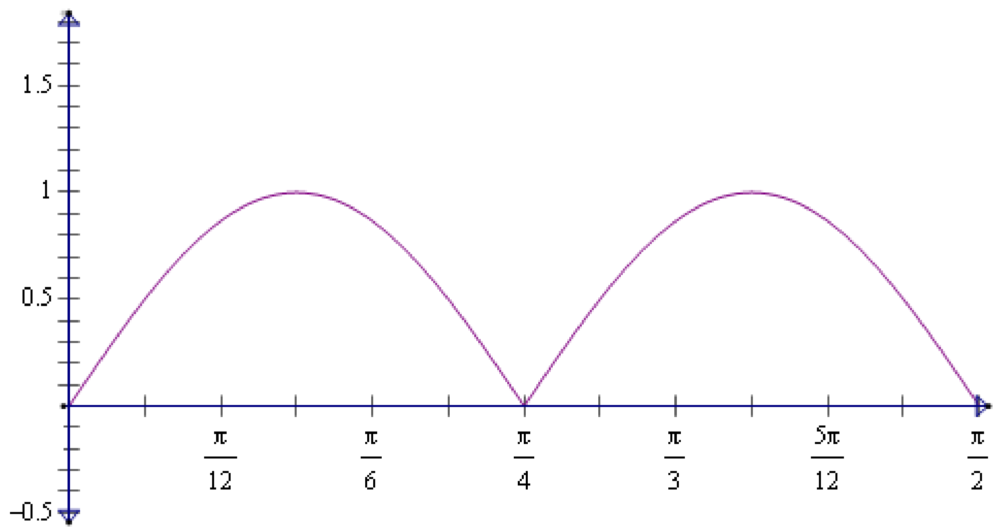
1 mark for each section

c) No. (1 mark)

Solution Pathway

At $x = \frac{\pi}{4}$ the graph is not smooth. (There is a cusp) (1 mark)

d)



Shape – 1, intercepts – 1, Maxima - 1

e) $2\int_0^{\frac{\pi}{4}} \sin(4x)dx$ or $\int_0^{\frac{\pi}{4}} \sin(4x)dx + \left|\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(4x)dx\right|$ 1 mark if partially correct, 2 marks for completely correct

f) $2\left[\frac{-\cos(4x)}{4}\right]_0^{\frac{\pi}{4}} = 1$ 1 mark

Solution Pathway

PART 2

Question 4

a) Fuel is being removed from the tank. 1 mark

b)

$$\begin{aligned} \boldsymbol{h} &= 3\boldsymbol{D} \\ &= 3\times 2\boldsymbol{r} \\ &= 6\boldsymbol{r} \end{aligned}$$ 1 mark

$$\therefore \boldsymbol{V} = \frac{1}{3}\pi \boldsymbol{r}^2 (6\boldsymbol{r})$$ 1 mark

$$= 2\pi \boldsymbol{r}^3$$

c)

$$\frac{d\boldsymbol{V}}{d\boldsymbol{r}} = 6\pi \boldsymbol{r}^2 \quad \text{and} \quad \frac{d\boldsymbol{V}}{d\boldsymbol{t}} = -100$$ 1 mark

$$\frac{d\boldsymbol{r}}{d\boldsymbol{t}} = \frac{d\boldsymbol{r}}{d\boldsymbol{V}} \times \frac{d\boldsymbol{V}}{d\boldsymbol{t}}$$

$$= \frac{1}{6\pi \boldsymbol{r}^2} \times -100$$ 1 mark

$$= \frac{-50}{3\pi \boldsymbol{r}^2}$$ 1 mark

d)

$$\frac{d\boldsymbol{r}}{d\boldsymbol{t}} = \frac{-50}{3\pi\left(\frac{1}{2}\right)^2}$$ 1 mark

$$= \frac{-200}{3\pi}$$

$$\approx -21.22\,\text{ms}^{-1}$$ 1mark

Solution Pathway

e) i. $\frac{1000}{100} = 10 \text{ seconds}$ 1 mark

ii. $\frac{-25}{6\pi} = \frac{-50}{3\pi \boldsymbol{r}^2}$ 1 mark

$\boldsymbol{r}^2 = 4$

$\boldsymbol{r} = \pm 2$

Take positive value only since r is a length, r = 2 m. 1 mark.

iii. $\text{Volume used} = 1000 - \frac{1}{3}\pi \times 8$

$= 991.622 \, \boldsymbol{L}$ 1 mark

$\text{Time taken} = \frac{991.622}{100}$

$= 9.92 \text{ seconds}$ 1 mark

Question 5

a) A (**-0.5**, 0) and B (0, **-0.5**) 1 mark each

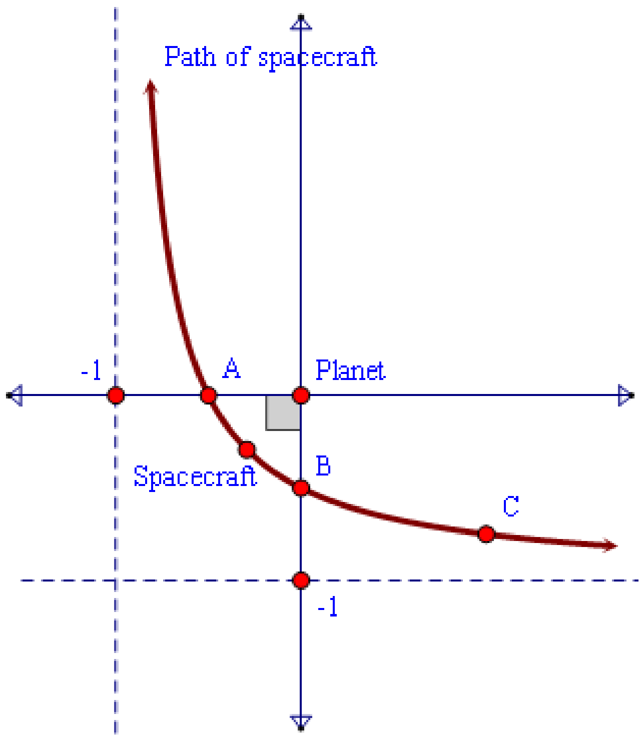
b) $\frac{-3}{4} = \frac{\boldsymbol{a}}{1+1} - 1$ 1 mark

$\frac{1}{4} = \frac{\boldsymbol{a}}{2}$ 1 mark

$\boldsymbol{a} = \frac{1}{2}$

Solution Pathway

c) Axes correct – 1, Asymptotes correct – 1, Scale correct - 1



d)

$$d=\sqrt{x^2+\left(\frac{1}{2(x+1)}-1\right)^2}$$ 1 mark for each component

e)

$$solve\left(\frac{d}{dx}\left(\sqrt{x^2+\left(\frac{1}{2(x+1)}-1\right)^2}\right)=0,x\right)$$

$x=\dfrac{\sqrt{2}-2}{2}$ 1 mark

$\therefore d=\sqrt{2}-1=0.4142\ units$ 1 mark

$d=4142\,km$ 1 mark

Solution Pathway

Question 6

a) $a_{24} = -2$ 1 mark

b) det(A) = 2280 1 mark

c) Entry values are 0 and 1. 2 marks

d)

$$f'(x) = \frac{-4x^3}{6} + \frac{3x^2}{6} + \frac{26x}{6} - \frac{7}{6}$$
$$= \frac{-2x^3}{3} + \frac{x^2}{2} + \frac{13x}{3} - \frac{7}{6}$$

1 mark for half of the terms correct. 2 marks for all correct.

e)

$$f'(1) = \frac{-2}{3} + \frac{1}{2} + \frac{13}{3} - \frac{7}{6}$$
$$= \frac{-17}{3}$$

1 mark

f) 1 mark for 2 of 4, 2 marks for all 4.

x_1 = -3.1080

x_2 = -1.2253

x_3 = 1.7674

x_4 = 3.5659

g) $$\int_{-3.1080}^{3.5659} \frac{-x^4}{6} + \frac{x^3}{6} + \frac{13x^2}{6} - \frac{7x}{6} - 4\,dx$$

Terminals correct – 1 mark, Expression correct – 1 mark, Notation correct – 1 mark

h)

$$Average\ value = \frac{1}{x_{max} - x_{min}} \times \int_{x_{min}}^{x_{max}} f(x)dx$$
$$= \frac{1}{3.5659 - -3.1080} \times \int_{-3.1080}^{3.5659} f(x)dx$$
$$= -0.0127$$

1 mark