QATs
Quality Assessment Tasks

# UNIT 3 ■ OUTCOMES 1 & 3

# VCE Mathematical Methods (CAS)

## SCHOOL-ASSESSED COURSEWORK

## Introduction

**Outcome 1**
Define and explain key concepts as specified in the content from the areas of study, and apply a range of related mathematical routines and procedures.

**Outcome 3**
Select and appropriately use technology to develop mathematical ideas, produce results and carry out analysis in situations requiring problem-solving, modelling or investigative techniques or approaches.

**Task:**
*Test – multiple-choice, short-answer and extended response items.*

The task has been designed to allow achievement up to and including the highest level in the Performance Descriptors and meets a broad range of **key knowledge** and **key skills** related to each outcome.

This is one of two tests. The two tests contribute 20 of the total marks (60) allocated for SAC in Unit 3.

This task will be marked out of 40 and then will be converted to a proportion of the contribution of this task to SAC in this unit.

The marks for each question are indicated in brackets.

The test is of 90 minutes duration. The formula sheet for end-of-year examinations may be used in the tests, and you may bring up to four A4 (two sides of two pages) of summary notes into the test.

Access to an approved CAS calculator will be allowed.

CAS TECHNOLOGY Indicates where use of the technology is specifically required in order to answer the question.

Answer in spaces provided.

NAME:

# Task

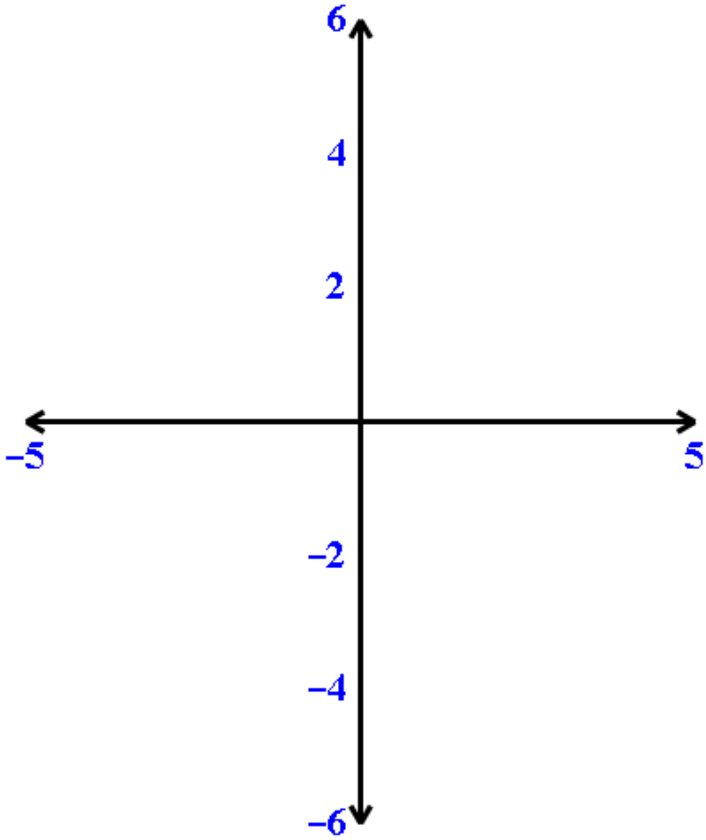
MARKS: /20 /10 /10

**Part A: Short Response**

**Question 1**

a) Use calculus to determine the derivative of $y = \frac{2}{x^2} + \frac{x^2}{2} - 2$

b) Use the derivative to find the coordinates of the turning points of the function.




(1 + 2 = 3 marks)

**Question 2**

a) A tangent to the function $y = 4\sqrt{x-2}$ has a gradient of 2. Find the coordinates of the point of contact between the function and this tangent.

(2 marks)

## Task

b) Now find the equation of the tangent at this point.

(2 marks)

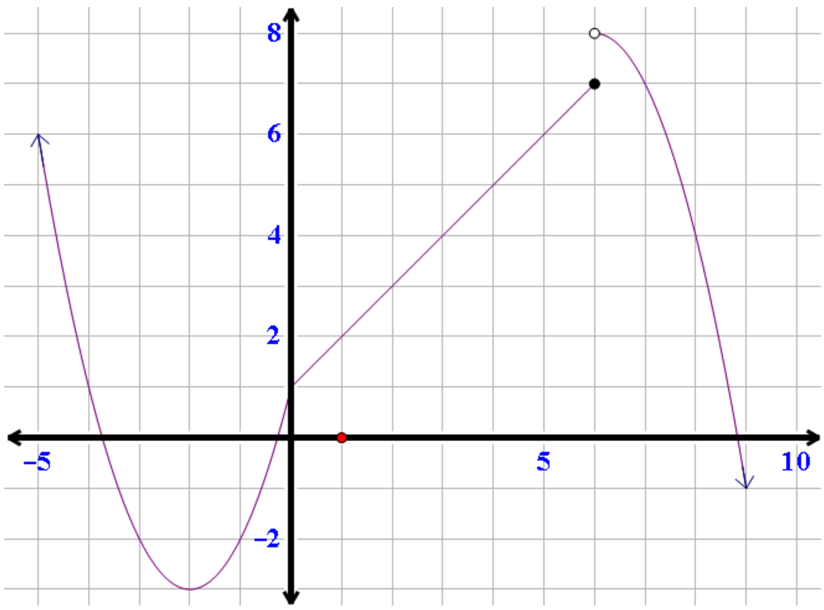
**Question 3**

$f(x) = \sin(2x) + 1$ Use Calculus to find an approximation for $f(\pi + 0.1)$

(3 marks)

**Question 4**

For the graph of the function shown:



**Task**

a) For what values of x is the function discontinuous?

b) For what values of the x is the function differentiable?

(1 + 3= 4 marks)

**Question 5**

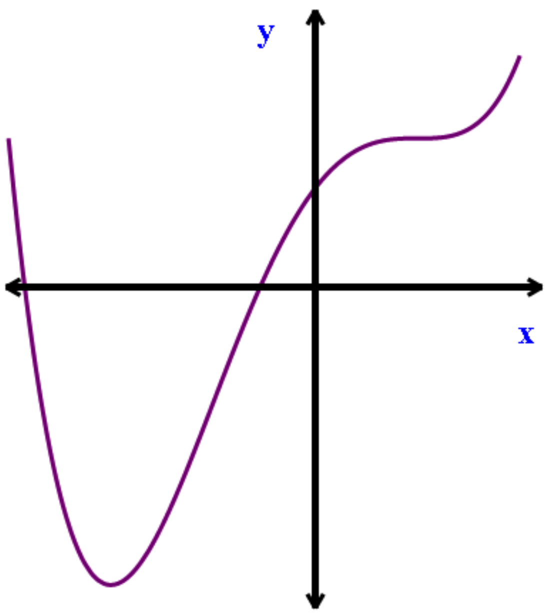
What is the average rate of change for the function $f(x) = e^{x} - (x+2)^{2}$ between x = 0 and x = 3? Express your answer in exact form.

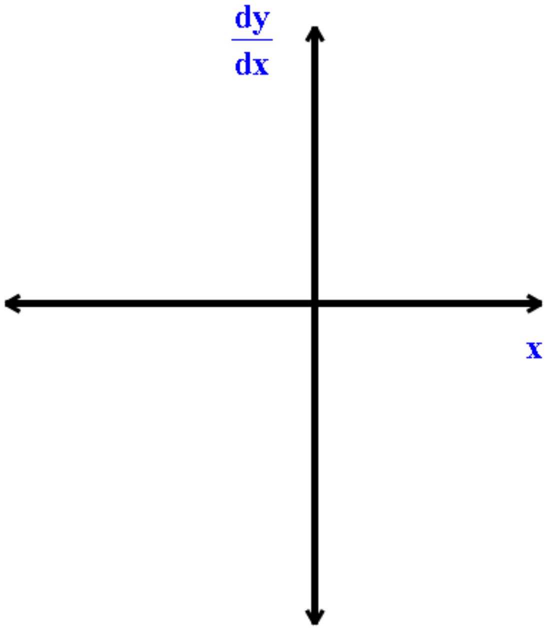
(3 marks)

# Task

**Question 6**

On the left of the diagram below is drawn the graph of a function. On the second set of axes, on the right, draw the derivative graph for this function.



## Task

**Part B: Multiple Choice and Extended Response**

*One mark for each correct multiple choice. Circle your correct response.*

CAS TECHNOLOGY

**Question 7**

The derivative of $\log_{10} x$ is:

A. $\dfrac{1}{x}$

B. $\dfrac{1}{x.\log_{10} x}$

C. $\dfrac{1}{x.\ln(10)}$

D. $\dfrac{1}{\ln x}$

E. $\dfrac{1}{10x}$

**Question 8**

The derivative of $e^x.\tan(x)$ is:

A. $e^x.\tan(x)+\dfrac{e^x}{\sec^2(x)}$

B. $e^x.\tan(x)+e^x.\sec^2(x)$

C. $e^x+\dfrac{e^x}{\sec^2(x)}$

D. $e^x.\left(\tan(x)+\cos^2(x)\right)$

E. $e^x+\sec^2(x)$

## Task

**Question 9**

The equation of the normal to the curve $y = x^3 - 6x^2 + 9x - 2$ at the point (3, -2) is:

A. $y = -2$

B. $y = 3x^2 - 12x + 9$

C. $y = -2x - 2$

D. $x = 3$

E. undefined

**Question 10**

The function $f(x) = \dfrac{x^3 - 7x + 6}{x - 2}$ is:

A. Continuous and differentiable for all $x \in R$

B. Continuous for $x \in R$ and differentiable for all $x \in R \backslash \{2\}$

C. Continuous and differentiable for all $x \in R \backslash \{-2\}$

D. Continuous for $x \in R \backslash \{2\}$ and differentiable for all $x \in R$

E. Continuous and differentiable for all $x \in R \backslash \{2\}$

**Question 11**

The $\lim\limits_{x \to 3} \left( \dfrac{1}{x^2 - 2x} - 3 \right)$ equals

A. Undefined

B. -3

C. 2

D. $\dfrac{-8}{3}$

E. 3

## Task

**Question 12**

The derivative of $\dfrac{\sin^2 x}{x^2}$ is:

A. $\dfrac{2\sin(x)}{x^2}\left(\cos(x)-\dfrac{\sin(x)}{x}\right)$

B. $\dfrac{2\sin(x)\cos(x)}{x^2}-\dfrac{2\sin(x)}{x^3}$

C. $\dfrac{2\sin(x)\cos(x)-2\sin(x)}{x^2}$

D. $\dfrac{2}{x^2}\left(\cos(x)-\dfrac{\sin(x)}{x}\right)$

E. $\dfrac{2\sin(x)}{x^3}\left(\cos(x)-\sin(x)\right)$

**Question 13**

Given that $f(x)=x^3-8$ and $g(x)=|x|$ then the derivative of $h(x) = f(g(x))$ is:

A. $h'(x)=3x^2$

B. $h'(x)=\begin{cases}-3x^2 & x\geq 2\\ 3x^2 & x<2\end{cases}$

C. $h'(x)=\begin{cases}-3x^2 & x\leq 2\\ 3x^2 & x>2\end{cases}$

D. $h'(x)=\begin{cases}-3x^2 & x>2\\ 3x^2 & x<2\end{cases}$

E. $h'(x)=\begin{cases}-3x^2 & x<2\\ 3x^2 & x>2\end{cases}$

## Task

**Question 14**

A function is locally linear wherever it is smooth. Which of the following functions is locally linear but not differentiable at $x = 3$?

A. $|x-3|$

B. $\sqrt{x-3}$

C. $\sqrt[5]{x-3}$

D. $(x-3)^{-1}$

E. $\dfrac{x^3-3x^2}{x-3}$

**Question 15**

The function $y = x^3 - 15x^2 + 75x - 124$ is strictly increasing over the domain:

A. $x < 5 \cup x > 5$

B. $x \geq 5$ only

C. $x > 5$ only

D. $x < 5$ only

E. R

**Question 16**

The instantaneous rate of change of the function $f(x) = \ln(\sin(x))$ when $x = \dfrac{\pi}{6}$ is

A. $\dfrac{\cos(x)}{\sin(x)}$

B. $\dfrac{1}{\sin(x)}$

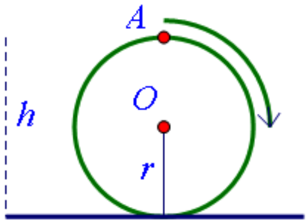
C. $-\ln(2)$

D. $\sqrt{3}$

E. undefined

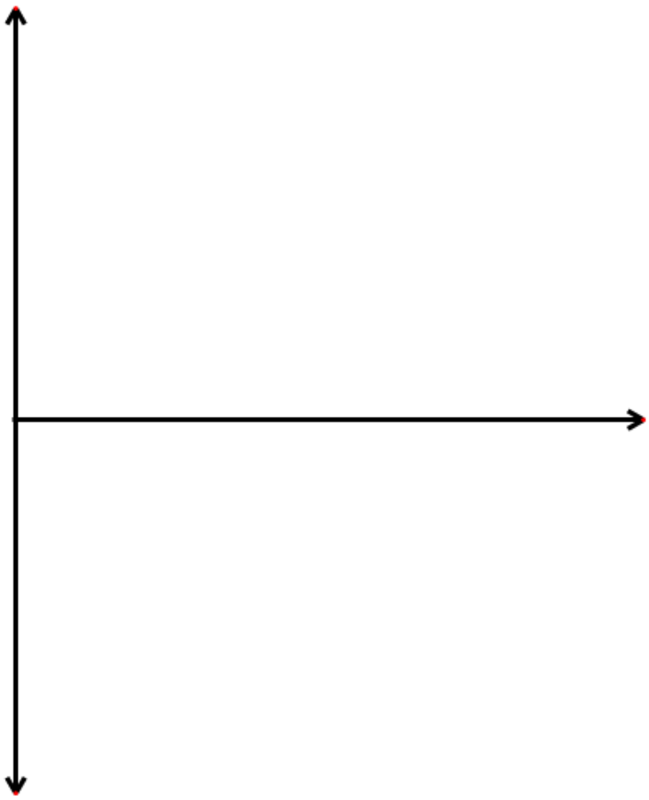
## Task

**Question 17 (Extended reponse)**




A wheel has a radius of 5m and spins vertically at a rate of two revolutions per second.

a) Sketch the graph of point A for one revolution. Assume that A starts at the top of the wheel.



(2 marks)

b) Show that the height of point A is given by $h = 5\cos(4\pi t) + 5$

(2 marks)

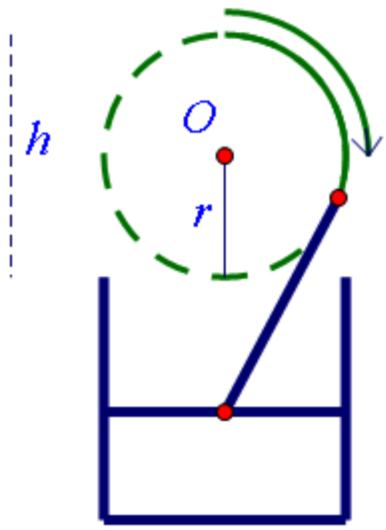
## Task

c) What is the instantaneous rate of change of height of point A?

(2 marks)

d) Attached to the wheel is a piston and cylinder. The volume of the cylinder is given by $V = 25\pi h$.

Find an equation for the rate of change of volume with respect to height.




(1 mark)

e) Now find an equation for the rate of change of volume with respect to time.

(2 marks)

f) Finally, find the instantaneous rate of change of volume when t = 0.25 seconds.

(1 mark)

# Teacher Advice

This is the Test 2 – Differentiation, as suggested to be undertaken in week 17 in the **sample teaching sequence** on page 142 of the VCAA Study Design.

The test has been designed so that *the Short response section can be done as a technology free task if teachers so wish*.

This test covers assessment in:
- Functions and graphs
- Transformations
- Polynomial functions
- Exponential and logarithmic functions
- Circular Functions
- Differentiation of all of the above functions

This test contributes 10 of the 60 SAC marks in Unit 3.
The coursework scores for this test are:
Outcome 1 7.5 marks 75%
Outcome 3 2.5 marks 25%
TOTAL 10 marks
This weighting can be used in the conversion of their mark out of 40.
For example, a score of 28 results in:

OUTCOME 1

$$\frac{28}{40}\times 10\times 0.75 = 5.25$$
$$= 5 \text{ (rounded)}$$

OUTCOME 3

$$\frac{28}{40}\times 10\times 0.25 = 1.75$$
$$= 2 \text{ (rounded)}$$

The above can be established in an Excel file.

This QAT has been designed to meet the highest level in the performance descriptors provided by VCAA for each outcome in unit 3 in the VCAA Mathematics Study Design February 2010.

## Solution Pathway

**Question 1**

a) Rearrange the function first, then differentiate:

$$y = 2x^{-2} + \frac{x^2}{2} - 2$$

$$\frac{dy}{dx} = -4x^{-3} + x$$

b) Let the derivative equal zero and solve for x:

$$0 = -4x^{-3} + x$$

$$4x^{-3} = x$$

$$\frac{4}{x^3} = x$$

$$4 = x^4$$

$$\therefore x = \pm\sqrt{2}$$

$$\therefore y = \frac{2}{2} + \frac{2}{2} - 2 = 0$$

Hence the coordinates are $(-\sqrt{2}, 0)$ ***and*** $(\sqrt{2}, 0)$

**Question 2**

a) Find the derivative and let it equal 2.

$$y = 4(x-2)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = 2(x-2)^{-\frac{1}{2}}$$

$$\textbf{\textit{So}}\ 2 = 2(x-2)^{-\frac{1}{2}}$$

$$1 = (x-2)^{-\frac{1}{2}}$$

$$1 = \frac{1}{(x-2)^{\frac{1}{2}}}$$

$$\therefore (x-2)^{\frac{1}{2}} = 1$$

$$\therefore \quad x - 2 = 1$$

$$x = 3 \qquad \therefore y = 4$$

b) Equation of the tangent is: $y = 2x+c$ at (3, 4).

**Solution Pathway**

Therefore c = 4 – 6 = -2

Equation of the tangent is $y = 2x - 2$

**Question 3**

The linear approximation formula is:

$$f(x+h) \approx f(x) + h.f'(x)$$
$$h = 0.1$$
$$f'(x) = 2.\cos(2x)$$
$$f(\pi + 0.1) \approx (\sin(2\pi) + 1) + 0.1 \times 2\cos(2\pi)$$
$$\approx 1 + 0.2$$
$$\approx 1.2$$

**Question 4**

a) The graph is discontinuous at x = 6 only.

b) The graph is differentiable wherever the function is smooth and continuous, so for $x \in R \setminus \{0, 6\}$

(1 mark for each of the values where x is not differentiable, 1 mark for notation)

**Question 5**

Average rate of change uses gradient formula, so need the y-values that correspond to the x values given.

$f(0) = 1 - 4 = -3$ 1 mark

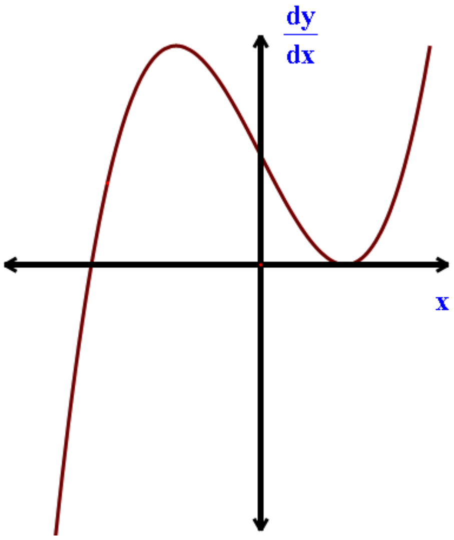
$f(3) = e^3 - 25$ 1 mark

$$\text{Average rate of change} = \frac{e^3 - 25 - (-3)}{3}$$
$$= \frac{e^3 - 22}{3}$$

## Solution Pathway

**Question 6**

1 mark for left x-intercept, 1 mark for right intercept being a turning point, 1 mark for shape.




**Question 7**

Answer **C**. The derivative of ln(x) is $\frac{1}{x}$. The base of this expression though is 10 so before differentiating it is necessary to convert to base *e*. ln(x) = $\frac{\log_e x}{\log_e 10} = \frac{\ln x}{\ln 10}$

**Question 8**

Answer **B**. Use the Product Rule.

$$e^x.\tan(x) \quad Let\ u = e^x \quad and\ v = \tan(x)$$

$$\frac{du}{dx} = e^x \qquad \frac{dv}{dx} = \sec^2(x)$$

$$\frac{dy}{dx} = e^x.\sec^2(x) + \tan(x).e^x$$

***Rearranging gives***

$$\frac{dy}{dx} = e^x.\tan(x) + e^x.\sec^2(x)$$

## Solution Pathway

**Question 9**

Answer **D**. Derivative is $3x^2 - 12x + 9$. The gradient of the tangent is zero. The gradient of the normal is undefined. This means that the normal is a vertical line with equation x = 3.

**Question 10**

Answer **E**. The function is discontinuous at x=2 and therefore not differentiable at this point.

**Question 11**

Answer **D**. The function is continuous at x = 3.Substitute x = 3 and the results in $\frac{-8}{3}$

**Question 12**

Answer **A**. Use quotient rule and then rearrange the result.

$$\text{Let } u = \sin^2 x \quad \text{and } v = x^2$$

$$\frac{du}{dx} = 2\sin(x).\cos(x) \qquad \frac{dv}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{x^2.2\sin(x).\cos(x) - 2x.\sin^2 x}{(x^2)^2}$$

$$\frac{dy}{dx} = \frac{x^2.2\sin(x).\cos(x) - 2x.\sin^2 x}{x^4}$$

$$= \frac{x.2\sin(x).\cos(x) - 2\sin^2 x}{x^3}$$

$$= \frac{2\sin(x)}{x^2}\left(\cos(x) - \frac{\sin(x)}{x}\right)$$

**Question 13**

Answer **E**.

$$h(x) = \left|x^3 - 8\right|$$

$$= \begin{cases} -(x^3 - 8) & x \le 2 \\ x^3 - 8 & x > 2 \end{cases} \quad \text{(domain is split at x = 2 because of the modulus function)}$$

$$\therefore h'(x) = \begin{cases} -3x^2 & x < 2 \\ 3x^2 & x > 2 \end{cases} \quad \text{since function is discontinuous at x=2}$$

## Solution Pathway

**Question 14**

Answer **C**.

| | | |
|---|---|---|
| A. | $\lvert x-3 \rvert$ | The modulus function is not smooth at x = 3. |
| B. | $\sqrt{x-3}$ | The function has a terminating point at x=3 and so is not continuous at that point. |
| C. | $\sqrt[5]{x-3}$ | Thr function is continuous as x=3 but has an undefined derivative since the tangent is vertical. |
| D. | $(x-3)^{-1}$ | The function has an asymptote at x = 3 and so is not continuous. |
| E. | $\dfrac{x^3-3x^2}{x-3}$ | The function has a discontinuity at x=3 so the point does not exist. |

**Question 15**

Answer **E**. To be strictly increasing f(a) < f( b) for all a  < b. The function has a stationary point of inflection at x = 5 so the condition holds for all R.

**Question 16**

Answer **D**.

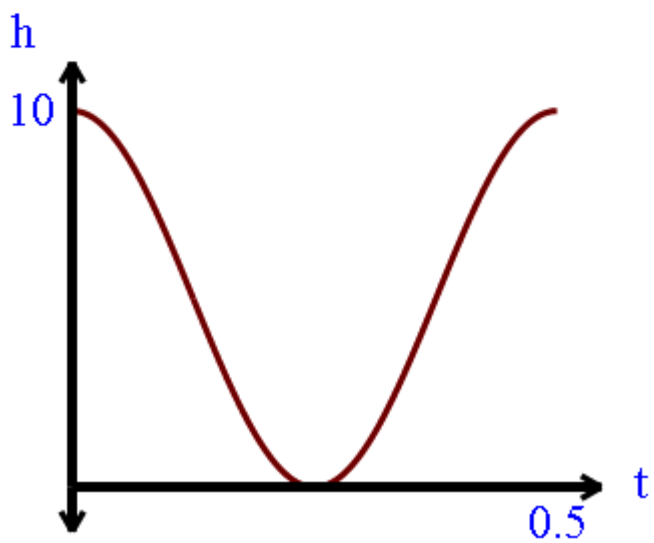
$$\frac{d(\ln(\sin(x)))}{dx}=\frac{\cos(x)}{\sin(x)}$$

$$\text{at } x=\frac{\pi}{6}, \text{rate of change}=\sqrt{3}$$

## Solution Pathway

**Question 17**

a) 1 mark shape, 1 mark values




b) The graph is a cosine with an amplitude of 5, a k value of 5, and a period of 0.5. Therefore $n = \frac{2\pi}{0.5} = 4\pi$ . (1 mark for amplitude and vertical translation, and one mark for n).

c) $\frac{dh}{dt} = -20\pi \sin(4\pi t)$ 1 mark for sine, 1 mark for $-20\pi$ .

d) $\frac{dV}{dh} = 25\pi$

e) Use related rates:

$$\begin{aligned}\frac{dV}{dt} &= \frac{dV}{dh} \times \frac{dh}{dt} \\ &= 25\pi \times (-20\pi \sin(4\pi t)) \\ &= -500\pi^2 \sin(4\pi t)\end{aligned}$$

f) Let t = 0.25 $\frac{dV}{dt} = -500\pi^2 \sin(\pi) = 0$