Trial Examination 2011

VCE Mathematical Methods (CAS) Units 3 & 4

Written Examination 2

Suggested Solutions

SECTION 1

1	**A**	B	C	D	E
2	**A**	B	C	D	E
3	A	**B**	C	D	E
4	A	B	C	**D**	E
5	A	**B**	C	D	E
6	A	**B**	C	D	E
7	A	B	C	**D**	E
8	A	B	C	D	**E**
9	**A**	B	C	D	E
10	A	B	**C**	D	E
11	A	B	C	**D**	E

12	A	B	**C**	D	E
13	A	**B**	C	D	E
14	**A**	B	C	D	E
15	A	B	C	**D**	E
16	**A**	B	C	D	E
17	A	B	C	D	**E**
18	A	B	**C**	D	E
19	A	B	C	D	**E**
20	A	B	**C**	D	E
21	A	B	C	**D**	E
22	A	B	C	D	**E**

SECTION 1

Question 1 **A**

Using CAS, define $f(x) = 4 - 2x - x^2$ and $g(x) = |x - 7|$.

$f(g(5))$ is found directly on CAS or alternatively:

$f(g(x)) = f(|x - 7|) = 4 - 2|x - 7| - (|x - 7|)^2$

There is no need to simplify this. Let $x = 5$ to directly give:

$$\begin{aligned} f(g(5)) &= 4 - 2|5 - 7| - (|5 - 7|)^2 \\ &= 4 - 4 - 4 \\ &= -4 \end{aligned}$$

Question 2 **A**

$$\lim_{h \to 0} \frac{\log_e(e + h) - 1}{h} = \lim_{h \to 0} \frac{\log_e(e + h) - \log_e(e)}{h}$$

This is the first principle's definition of $f\,'(e)$, where $f(x) = \log_e(x)$.

Question 3 **B**

$$\begin{aligned} g(x) &= f(1 - 3x) + 2 \\ &= f\left(-3\left(x - \frac{1}{3}\right)\right) + 2 \end{aligned}$$

This means f undergoes a reflection in the y-axis: $f(x) \to f(-x)$; $(3, -11) \to (-3, -11)$,

followed by a dilation away from the y-axis of scale factor $\frac{1}{3}$: $f(-x) \to f(-3x)$; $(-3, -11) \to (-1, -11)$,

followed by a translation of $\frac{1}{3}$ to the right:

$f(-3x) \to f\left(-3\left(x - \frac{1}{3}\right)\right)$, i.e. $f(1 - 3x)$; $(-1, -11) \to \left(\frac{-2}{3}, -11\right)$.

Finally, it is translated 2 units vertically up: $f(1 - 3x) \to f(1 - 3x) + 2$; $\left(\frac{-2}{3}, -11\right) \to \left(\frac{-2}{3}, -9\right)$.

Question 4 **D**

The graph of $f(x) = 1 - 5\cos(2x - \pi)$ clearly shows an amplitude of 5, a range of $[-4, 6]$ and the period is π, so **A**, **B** and **C** are correct.

Consider alternative **D**. If $g(x) = 5\cos(2x - \pi)$, then a vertical translation of 1 unit up means $g(x) = 5\cos(2x - \pi) + 1$. If we now reflect in the x-axis, we get $f(x) = -(5\cos(2x - \pi) + 1) = -5\cos(2x - \pi) - 1$ which is incorrect.

Alternative **E** shows $h(x) = -5\cos(2x)$ shifted $\frac{\pi}{2}$ units right giving $h(x) = -5\cos\left(2\left(x - \frac{\pi}{2}\right)\right) = -5\cos(2x - \pi)$. Translating this 1 unit up results in $f(x) = 1 - 5\cos(2x - \pi)$.

Question 5 **B**

From the matrix equation $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2x+1 \\ 4y-2 \end{bmatrix}$

i.e. $x' = 2x + 1$ and $y' = 4y - 2$.

Therefore $x = \frac{x' - 1}{2}$ and $y = \frac{y' + 2}{4}$.

So $y = x^3$ becomes $\frac{y' + 2}{4} = \left(\frac{x' - 1}{2}\right)^3$

$y = 4\left(\frac{x-1}{2}\right)^3 - 2 \Rightarrow y = \frac{1}{2}(x-1)^3 - 2$

Question 6 **B**

During the first second, the position of the body changes (in the negative direction) by an amount equal to the area of the triangle, i.e. $\frac{1}{2} \times 1 \times 4 = 2$.

Notice this means the body moves to the left by 2 m during this time.

From $1 < t < 2$, the body moves in the opposite direction (right) by $\frac{1}{2} \times 1 \times 3 = 1.5$. So the net result in the first 2 seconds is a move of 0.5 units to the left.

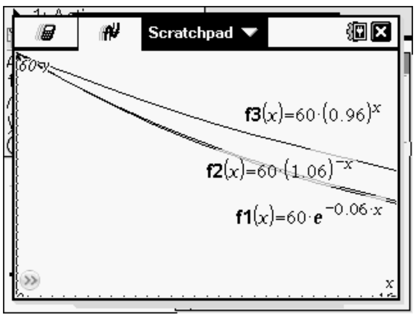
So $x(2) < x(0)$. Only alternatives **A** and **B** satisfy this requirement.

From $2 < t < 4$, the body moves further right by $\frac{1}{2}(2 + 1) \times 3 = 4.5$.

After this, the body moves back to the left once again. Therefore $x(6) < x(4)$. Combining these results gives $x(2) < x(0) < x(6) < x(4)$.

Question 7 **D**

There are many varied approaches which could be used, ranging from statistical regression to trial and error. One quick approach is to graph the 3 exponential functions in an appropriate window using 0 to 16 for $x(n)$ and 0 to 60 for $y(N)$.



$N = 60(0.96)^n$ follows the data more closely over the domain than the others. Clearly the data is not negative so alternative **E** is rejected. Alternative **C** is quadratic. A quick look at its graph also shows it matches the data only for very small values of n.

Question 8 E

Given $p(x) = 5x^{2k+1} - 10x^{2k} + 3x^{2k-1} + 5$, the remainder when divided by $x + 1$ is $p(-1)$.

Now $p(-1) = 5(-1)^{2k+1} - 10(-1)^{2k} + 3(-1)^{2k-1} + 5$

As *k* is a positive integer, $2k \pm 1$ is odd and $2k$ is even.

Thus $p(-1) = 5(-1) - 10(1) + 3(-1) + 5$

$= -5 - 10 - 3 + 5 = -13$

Question 9 A

An inverse exists if the function is one-to-one.

It is tempting to think of cubics as many-to-one functions but both *f* and *g* are one-to-one.

A sketch of their graphs on CAS quickly shows this is the case. Alternatively, the derivatives of *f* and *g* are shown algebraically:

$$f'(x) = x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$$

$$g'(x) = 3x^2 + 2x + 1 = (x + 1)^2 + 2x^2$$

Clearly each of these derivatives is always positive. Thus *f* and *g* are increasing functions and hence one-to-one.

The graph of $h(x) = x^{-\frac{2}{3}}$ is symmetrical about the *y*-axis and hence the function is not one-to-one.

Question 10 C

A dilation from the *x*-axis by a factor of $\frac{1}{2}$ means $\cos(3x) \to \frac{1}{2}\cos(3x)$ and $\sin(3x) \to \frac{1}{2}\sin(3x)$.

A dilation from the *y*-axis by a factor of 3 means $x \to \frac{1}{3}x$ so we have new equations $y_1 = \frac{1}{2}\cos(x)$ and $y_2 = \frac{1}{2}\sin(x)$.

These graphs meet when $\frac{1}{2}\sin(x) = \frac{1}{2}\cos(x) \Rightarrow \tan(x) = 1$.

$x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$ are the first two positive solutions with $y = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ and $\cos\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$.

Thus the graphs intersect at $\left(\frac{\pi}{4}, \frac{\sqrt{2}}{4}\right)$ and $\left(\frac{5\pi}{4}, -\frac{\sqrt{2}}{4}\right)$ after the dilations have occurred.

Question 11 D

$$\int_p^3 \frac{1}{3}x^2\,dx = 1$$

$$\left[\frac{1}{9}x^3\right]_p^3 = 1 \qquad \text{Or using CAS: solve}\left(\int_p^3 \frac{1}{3}x^2\,dx = 1, p\right)$$

$$3^3 - p^3 = 9$$

$$p^3 = 18$$

$$p = \sqrt[3]{18}$$

Question 12 **C**

$$X: \text{ number of goals scored in 30 attempts}$$

$$X \sim \text{Bi}(n = 30, p = 0.7)$$

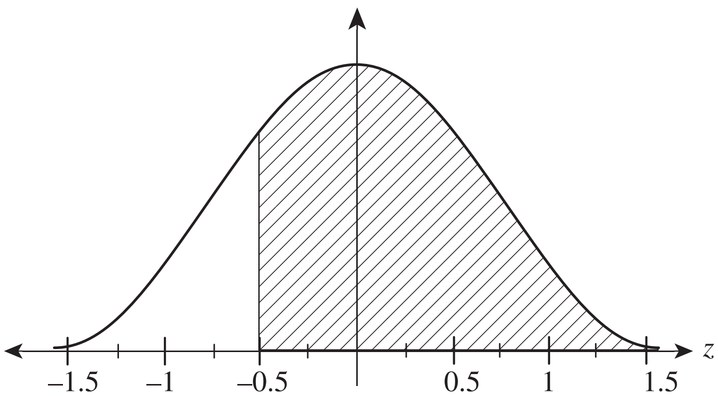
$$\mu = 30 \times 0.7 = 21$$

$$\sigma = \sqrt{30 \times 0.7 \times 0.3} = 2.51$$

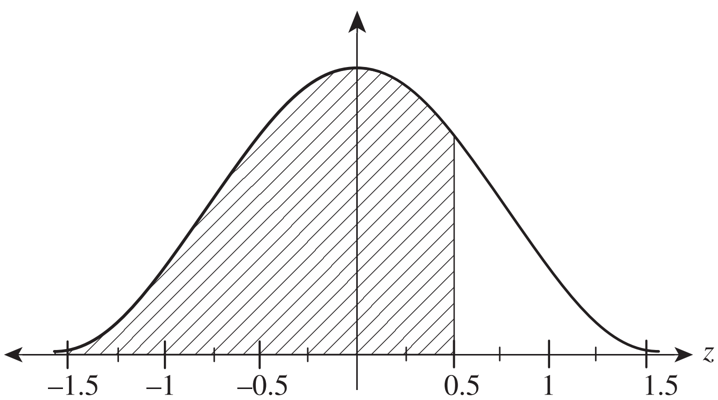
$$\begin{aligned}\Pr(\mu - \sigma < X < \mu + \sigma) &= \Pr(21 - 2.57 < X < 21 + 2.51)\\ &= \Pr(18.49 < X < 23.51)\\ &= \Pr(19 \le X \le 23)\\ &= 0.6812\end{aligned}$$

Question 13 **B**

$$\begin{aligned}\Pr(47.5 < X < 57.5) &= \Pr\left(\frac{47.5 - 50}{5} < Z < \frac{57.5 - 50}{5}\right)\\ &= \Pr(-0.5 < Z < 1.5)\end{aligned}$$



is equivalent to:



Question 14 A

$$\begin{aligned}\Pr(\text{at least 1 boy and 1 girl}) &= 1 - \Pr(BBB) - \Pr(GGG)\\ &= 1 - \left(\frac{4}{12}\cdot\frac{3}{11}\cdot\frac{2}{10}\right) - \left(\frac{8}{12}\cdot\frac{7}{11}\cdot\frac{6}{10}\right)\\ &= 1 - \frac{1}{55} - \frac{14}{55}\\ &= 1 - \frac{3}{11}\\ &= \frac{8}{11}\end{aligned}$$

Question 15 D

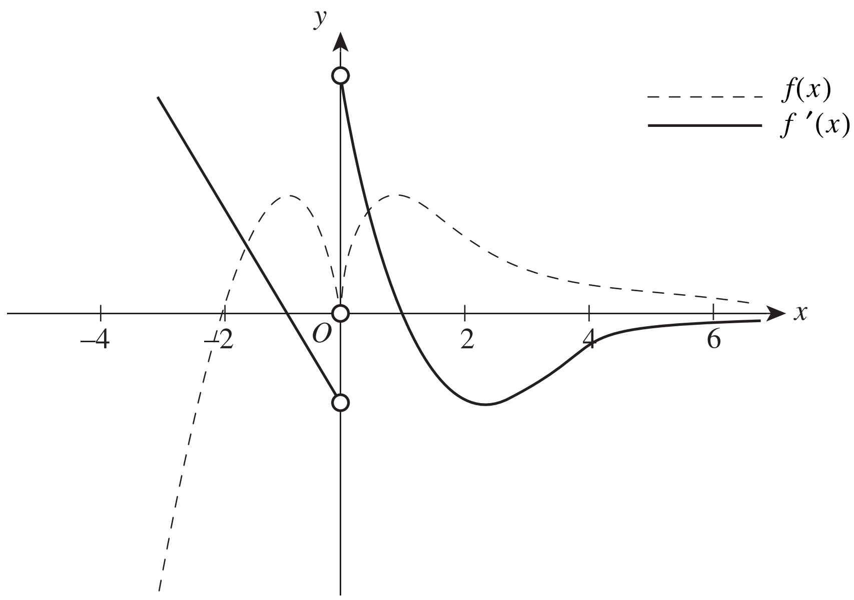
$$\begin{aligned}\Sigma\Pr(X = x) &= 1\\ p + p + p + 2p + 3p + 2p &= 1\\ 10p &= 1\\ p &= 0.1\end{aligned}$$

x	1	2	4	6	8	10
$\Pr(X \le x)$	0.1	0.2	0.3	0.5	0.8	1

As $\Pr(X \le 6) = 0.5$, the median is halfway between 6 and the next value of X; here 8.

$$\therefore \text{median} = \frac{6+8}{2} = 7$$

Question 16 A



For:

$x < -1$, $f'(x) > 0$ i.e. graph of $f'(x)$ is above the x-axis.

$x = -1$, $f(x)$ has a turning point, $\therefore f'(x) = 0$.

$-1 < x < 0$, $f'(x) < 0$ i.e. graph of $f'(x)$ is below the x-axis.

$x = 0$, $f'(x)$ does not exist.

$0 < x < 1$, $f'(x) > 0$ i.e. graph of $f'(x)$ is above the x-axis.

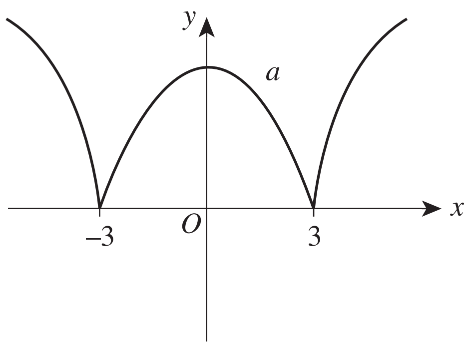
$x = 1$, $f(x)$ has a turning point $\therefore f'(x) = 0$.

$x > 1$, $f'(x) < 0$ i.e. graph of $f'(x)$ is below x-axis.

As $x \to \infty$, $f'(x) \to 0$.

Question 17 E

$f(x) = |9 - x^2|$ has graph:



Clearly at $x = \pm 3$ there are cusps, $\therefore$ $f'(-3)$ and $f'(3)$ do not exist.

Alternatively using CAS:

$f'(x) = 2x\text{sign}(x^2 - 9)$

$\lim\limits_{x \to 3^-} f'(x) = -6$

$\lim\limits_{x \to 3^+} f'(x) = 6$ $\quad\therefore$ the limit does not exist and $f'(3)$ does not exist.

Note: sign(0) is undefined.

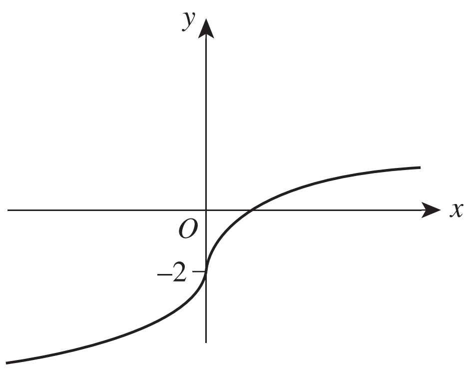
Question 18 C

Using a CAS calculator:

$\text{normalLine}\left(x^{\frac{3}{5}} - 2, x, 0\right)$ or $\text{normal}\left(x^{\frac{3}{5}} - 2, x, 0\right)$ gives –2, which we interpret to mean $y = -2$.

OR

A graph of $y = f(x)$ is:

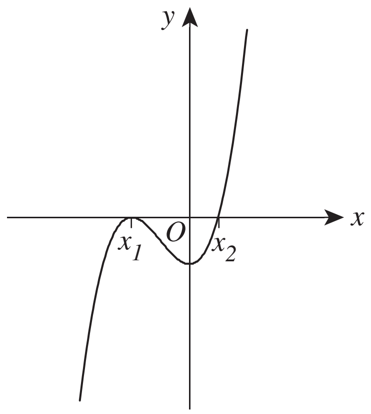


$\frac{d}{dx} f(x)$ at $x = 0$ is undefined.

$\therefore$ gradient of normal at $x = 0$ is 0.

$\therefore$ equation of normal at $(0, -2)$ is $y = -2$.

Question 19 E



Let x_1 be the first x-intercept, x_2 the second x-intercept, so therefore $f'(x_1) = 0$ and $f'(x_2) = 0$.

$\therefore$ On the graph of $y = f(x)$ there will stationary points at x_1 and x_2.

To the left of x_1, the gradient is negative. To the right of x_1, the gradient is negative.
$\therefore$ At x_1, there is a point of inflexion.

To the left of x_2, the gradient is negative. To the right of x_2, the gradient is positive.
$\therefore$ At x_2, there is a local minimum.

E is the only graph which has these two features.

Question 20 C

Given $\displaystyle\int_0^a f(x)\,dx = -2$

$\therefore \displaystyle\int_0^{\frac{a}{2}} f(2x)\,dx = -1$ as $f(2x)$ is a dilation of $\frac{1}{2}$ parallel to the x-axis.

$$\text{Also, } \int_0^{\frac{a}{2}} 2f(2x)\,dx = 2\int_0^{\frac{a}{2}} f(2x)\,dx$$
$$= 2 \times -1$$
$$= -2$$

Question 21 D

As A and B are independent, so therefore are A' and B'.

$\therefore \Pr(A'|B') = \Pr(A') = 1 - p$

OR

$\Pr(A \cap B) = pq$ this gives the following probability table:

$\cap$	A	A'	
B	pq	$q - pq$	q
B'	$p - pq$	$1 + pq - p - q$	$1 - q$
	p	$1 - p$	1

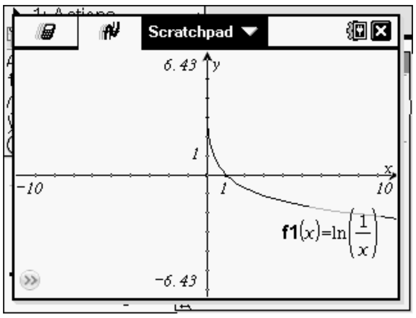
$$\Pr(A|B') = \frac{\Pr(A' \cap B')}{\Pr(B')} = \frac{1 + pq - p - q}{1 - q}$$

$$= \frac{(1-p)(1-q)}{(1-q)}$$

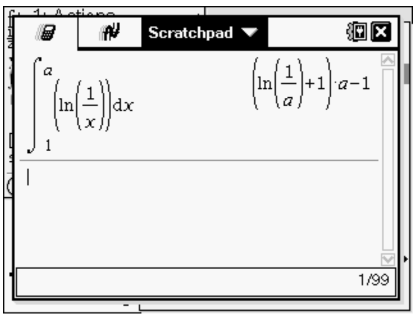
$$= 1 - p$$

Question 22 E

A graph of $y = \log_e\left(\frac{1}{x}\right)$ shows that as $a > 1$, the area required is below the x-axis.



Using a CAS calculator:



Therefore area required is $-\left(a\left(\log_e\left(\frac{1}{a}\right) + 1\right) - 1\right) = 1 - a\left(\log_e\left(\frac{1}{a}\right) + 1\right)$.

$$= 1 - a + a\log_e(a)$$

SECTION 2

Question 1

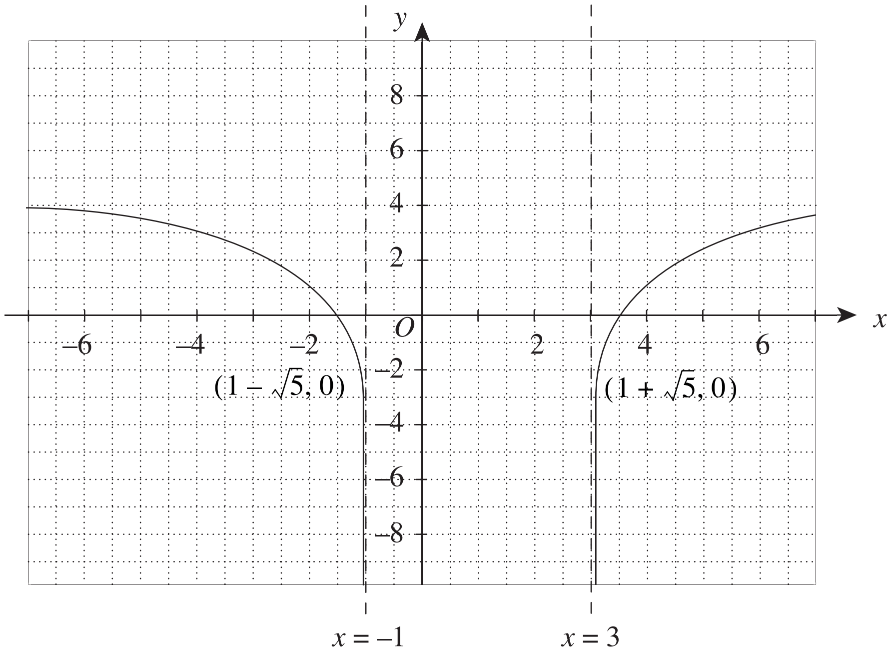
a. For f to be defined, $((x-1)^2 - m) > 0 \Rightarrow (x-1)^2 > m$. M1

Thus $x - 1 < -\sqrt{m}$ or $x - 1 > \sqrt{m}$

$\Rightarrow x < 1 - \sqrt{m}$ or $x > 1 + \sqrt{m}$

Hence the domain of f is $(-\infty, 1-\sqrt{m}) \cup (1+\sqrt{m}, \infty)$. A1

b.



x-intercepts at $(1+\sqrt{5}, 0)$, $(1-\sqrt{5}, 0)$ A1

Asymptotes with equations $x = -1$ and $x = 3$ A1

Graph shape and location A1

c. Given that $x < -2$, $1 - \sqrt{m} = -2 \Rightarrow m = 9$. M1 A1

d. Given that $g : (-\infty, -2) \to R$, $g(x) = \log_e((x-1)^2 - m)$, the inverse is given by:

$$x = \log_e((y-1)^2 - m)$$ M1

$$e^x = ((y-1)^2 - m)$$

$$y - 1 = \pm\sqrt{e^x + m}$$

$$y = 1 \pm \sqrt{e^x + m}$$

As $\mathrm{dom}(g) = \mathrm{ran}(g^{-1}) = (-\infty, -2)$, $g^{-1}(x) = 1 - \sqrt{e^x + m}$. A1

Domain $= (\log_e(9-m), \infty)$, Range $= (-\infty, -2)$. A1

e. **i.** $f(x) = \log_e((x-1)^2 - m) \Rightarrow f'(x) = \dfrac{2(x-1)}{((x-1)^2 - m)}$ A1

A turning point occurs if $f'(x) = 0 \Rightarrow x = 1$.

In addition, $(x-1)^2 - m > 0$ for the function to be continuous and have a turning point.

Hence $m < 0$ and $x = 1$. A1

ii. Starting with $f(x) = \log_e((x-1)^2 + 1)$, we get:

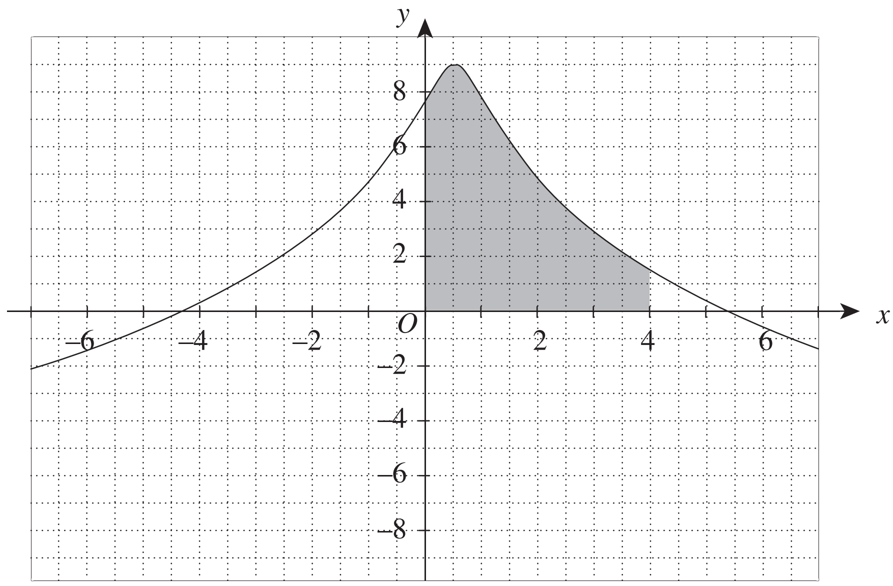
$y = -\log_e((x-1)^2 + 1)$ (reflection in the x-axis)

which becomes: $y = -\log_e((2x-1)^2 + 1)$ (dilation from the y-axis, factor $\frac{1}{2}$) M1

which becomes: $y = -2\log_e((2x-1)^2 + 1)$ (dilation from the x-axis, factor 2) A1

iii. If we translate $y = -2\log_e((2x-1)^2 + 1)$ A units up, we get:

$y = -2\log_e((2x-1)^2 + 1) + A$



Solve on CAS: $\displaystyle\int_0^4 (-2\log_e((2x-1)^2 + 1) + A)\,dx = 23.5$ (or equivalent) M1

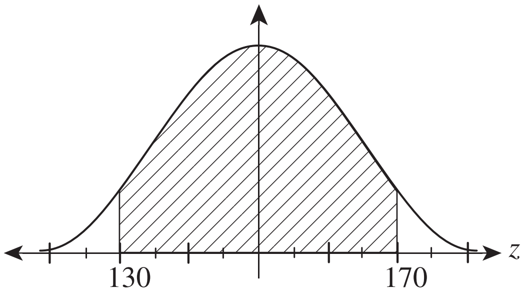
Gives $A = 10.0015$, so $A = 10$, correct to the nearest integer A1

Question 2

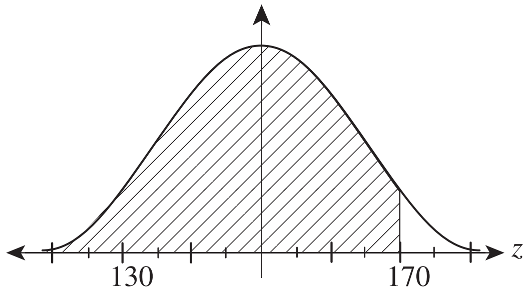
a. $\Pr(130 < X < 170) = 0.5763 = 57.63\%$ A1

b. $\Pr(130 < X < 170) \geq 0.8$

i.e.



into



$\therefore \Pr(X < 170) \geq 0.9$ A1

$\Pr(Z < z) \geq 0.9$

$Z_1 = \text{invnorm}(0.9)$

$\frac{170 - 150}{\sigma} \geq 1.28$

$\therefore \sigma \leq \frac{20}{1.28} = 15.625$ M1

To achieve this, $\sigma = 15$, not 16, to the nearest gram. A1

c. $(0.6)^5 = 0.07776$ A1

d. $\binom{5}{3}(0.6)^3(0.4)^2 = 0.3456$ A1

e. $\Pr(\text{AB or BA}) = 0.6 \times 0.2 + 0.4 \times 0.6$ M1

$= 0.36$ A1

f. $p = 0.8$

$$T = \begin{bmatrix} A\backslash A & A\backslash B \\ B\backslash A & B\backslash B \end{bmatrix} = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix}$$

$$\therefore \Pr(X_4) = T^3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.744 \\ 0.256 \end{bmatrix}$$ M1

$\therefore 0.256$ A1

g. For a steady state, choose a large value of *n*, e.g. *n* = 50.

$$T^{50} = \begin{bmatrix} 0.75 & 0.75 \\ 0.25 & 0.25 \end{bmatrix}$$

$\therefore 0.75$ or 75% A1

h. $\therefore \begin{bmatrix} p & (1.4-p) \\ (1-p) & (p-0.4) \end{bmatrix}^3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$ or *n*spire, solve$\left(\begin{bmatrix} p & (1.4-p) \\ (1-p) & (p-0.4) \end{bmatrix}^3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}, p \right)$ M1

$$\begin{bmatrix} 4p^3 - 7.6p^2 + 5.16p - 0.56 \\ -4(p-1)(p^2 - 0.9p + 0.39) \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$$ A1

Using CAS: solving $4p^3 - 7.6p^2 + 5.16p - 0.56 = 0.8$

$$p = 0.8591$$ A1

Question 3

a. The maximum height is given by the maximum value of:

$$f(x) = b\left[\frac{1}{2} + \frac{1}{2}\sin\left(\frac{\pi(4x-d)}{2d}\right)\right]$$

$$f(x)_{\max} = b\left[\frac{1}{2} + \frac{1}{2}\right] = b$$ A1

b. **i.** Using **part a.** and the graph, $b = 0.1$. The width of the hump is 0.5, so $d = 0.5$. A2

ii. Using $b = 0.1$ and $d = 0.5$, we have

$$f(x) = 0.1\left[\frac{1}{2} + \frac{1}{2}\sin\left(\frac{\pi(4x-0.5)}{2 \times 0.5}\right)\right] = \frac{1}{20}\left[1 + \sin\left(4\pi x - \frac{\pi}{2}\right)\right]$$ M1

$$\sin\left(\theta - \frac{\pi}{2}\right) = \sin\left(-\left(\frac{\pi}{2} - \theta\right)\right) = -\sin\left(\frac{\pi}{2} - \theta\right) = -\cos(\theta)$$

As $f(x) = \frac{1}{20}\left[1 + \sin\left(4\pi x - \frac{\pi}{2}\right)\right] = \frac{1}{20}(1 - \cos(4\pi x))$ A1

iii. As $y = \frac{1}{20}(1 - \cos(4\pi x))$ describes this hump, CAS gives $\frac{dy}{dx} = \frac{\pi}{5}\sin(4\pi x)$ M1

This means the magnitude of the maximum gradient is $\frac{\pi}{5}$. A1

Thus the maximum angle of the hump is $\tan^{-1}\left(\frac{\pi}{5}\right) = 32.14°$, which means the speed hump is **not** of the described design. A1

iv. $f(x) = b\left[\frac{1}{2} + \frac{1}{2}\sin\left(\frac{\pi(4x-d)}{2d}\right)\right]$ can be re-expressed on CAS as $f(x) = \frac{1}{2}b[1 - \cos(4\pi x)]$ as the value of *d* remains the same, i.e. $d = 0.5$.

This gives $f'(x) = 2\pi b\sin(4\pi x)$. M1

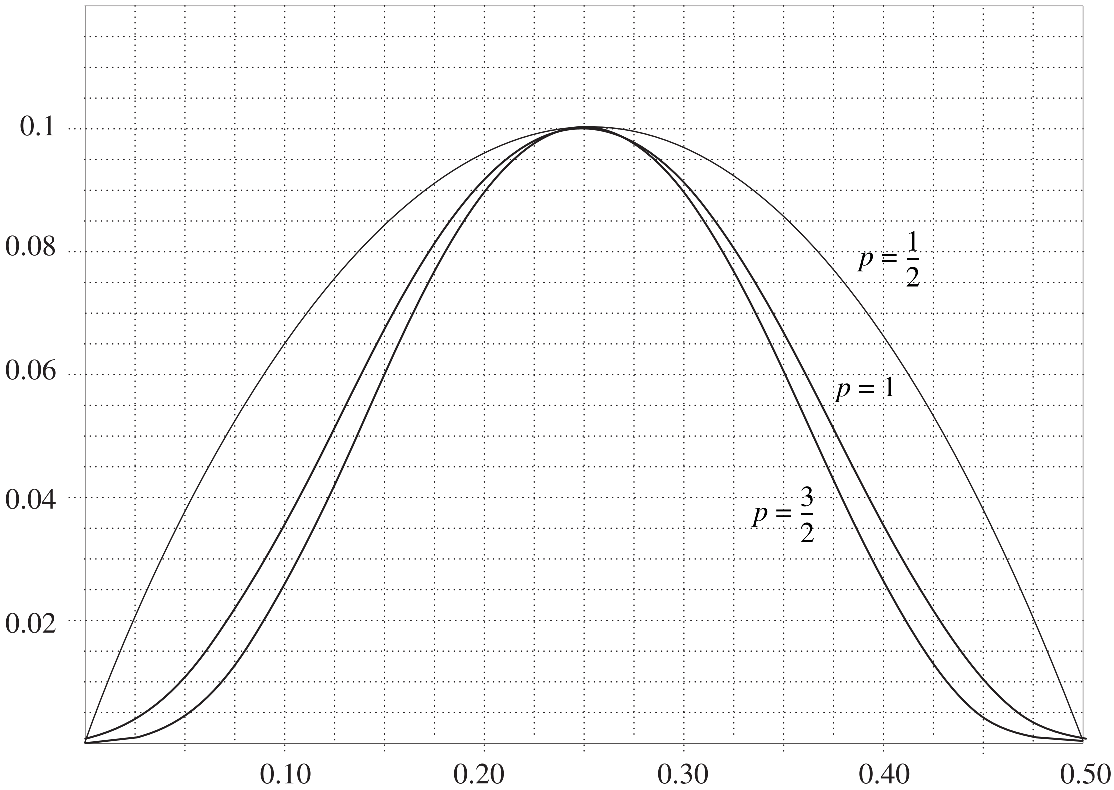
Thus the maximum value for the gradient is $2\pi b$.

So $2\pi b = \tan(15°) = 2 - \sqrt{3} \Rightarrow b = \frac{2-\sqrt{3}}{2\pi}$ M1

Our original value for *b* was $\frac{1}{10}$, hence $\frac{1}{10}k = \frac{2-\sqrt{3}}{2\pi} \Rightarrow k = \frac{5(2-\sqrt{3})}{\pi}$.

So a dilation of factor $\frac{5(2-\sqrt{3})}{\pi}$ from the *x*-axis is required. A1

c. **i.**



One mark per graph A2

ii. Too steep at the start and end of the hump. A1

d. **i.** Now the area between each curve is given by:

$$A = \int_0^{0.5} 0.1[0.5 + 0.5\sin(\pi(4x - 0.5))] - 0.1[0.5 + 0.5\sin(\pi(4x - 0.5))]^{1.5} dx$$ A1

The volume difference will be given by:

$$V = 3\int_0^{0.5} 0.1[0.5 + 0.5\sin(\pi(4x - 0.5))] - 0.1[0.5 + 0.5\sin(\pi(4x - 0.5))]^{1.5} dx$$ A1

ii. Calculating by CAS, we get 0.011338 cubic metres. As there are 5 humps, we have 0.057 cubic metres. A1

Question 4

a. $f\,'(x) = -(x-3)^2(x-2)(5x-12) = 0$ for stationary points. M1

$\therefore x = 3$ or 2 or $\frac{12}{5}$

At $x = 2$, a local minimum A1

$x = \frac{12}{5}$, a local maximum A1

$x = 3$, a stationary point of inflexion A1

b. If $a = b$. A1

i.e. $g(x) = (a - x)^5$, its nature will be a point of inflexion. A1

c. $g(x) = (a - x)^2(b - x)^3$

$g'(x) = -(x - b)^2(x - a)(5x - 3a - 2b) = 0$ for stationary point M1

$x = a$ or $x = b$ or $x = \frac{3a + 2b}{5}$ A1

d. Using CAS:

Define $h(x)$.

Determine $h'(x) = (a - x)^{m-1}(b - x)^{n-1}((m + n)x - (mb + na))$ M1 A1

Solve $h'(x) = 0$

Answer $x = a$ or b or $\frac{an + bm}{m + n}$

e. For any point to be equidistant from $x = a$ and $x = b$, $\therefore x = \frac{a + b}{2}$.

$\therefore$ from **Question 4 part d.**, $\frac{an + bm}{m + n} = \frac{a + b}{2}$.

Solving for either m or n gives $m = n$. A1