The Mathematical Association of Victoria
Maths Methods CAS 2011
Trial Written Examination 2 – SOLUTIONS

1. C	**2. A**	**3. B**	**4. C**	**5. E**	**6. D**	**7. A**	**8. B**
9. A	**10. E**	**11. C**	**12. B**	**13. B**	**14. C**	**15. D**	**16. B**
17. A	**18. C**	**19. E**	**20. D**	**21. A**	**22. C**		

Question 1 **Answer C**

The domain of *fg* is the intersection of the domain of *f* and the domain of *g*.

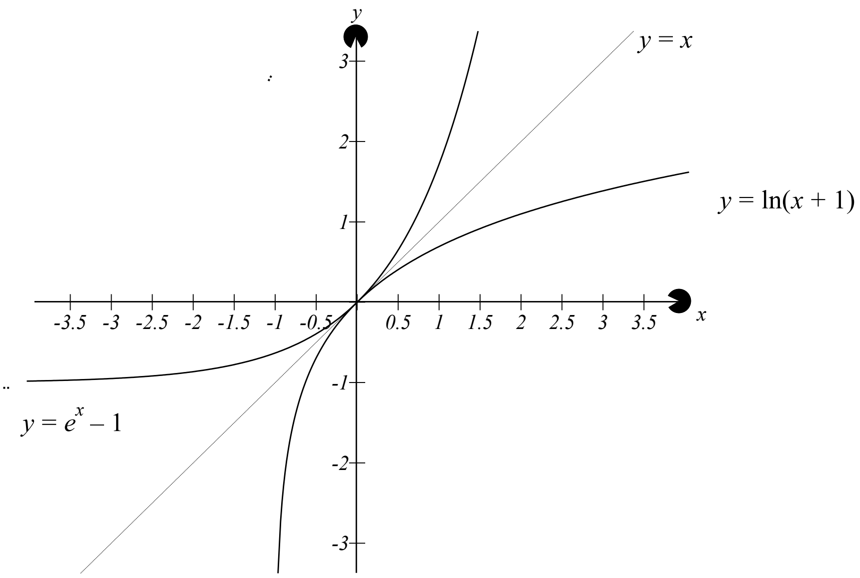
The domain of *f* is $\left(-\infty, \frac{1}{2}\right]$ and the domain of *g* is $R \setminus \left\{\frac{1}{2}\right\}$.

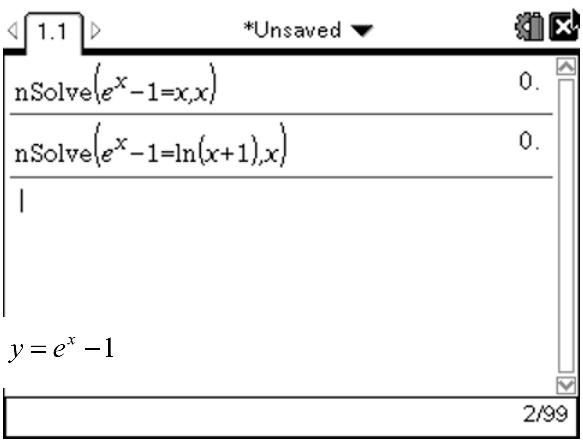
Hence the domain of *fg* is $\left(-\infty, \frac{1}{2}\right)$.

Question 2 **Answer A**

f and its inverse, f^{-1} will have two points of intersection when *f* and $y = x$ have two points of intersection. This will occur when $k < -1$.

The graphs with equations $y = e^x - 1$, $y = x$ and $y = \log_e(x+1)$, where $k = -1$ are shown below. There is only one point of intersection which occurs when $e^x - 1 = x$. Hence $x = 0$ as $e^0 - 1 = 0$.



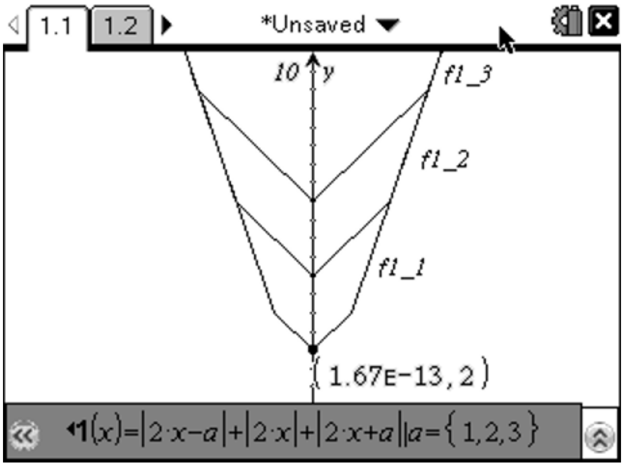


Question 3 **Answer B**

$f(x) = |2x + a| + |2x| + |2x - a|$.

The minimum value occurs when $x = 0$. $f(x) = |a| + |-a| = 2a$.

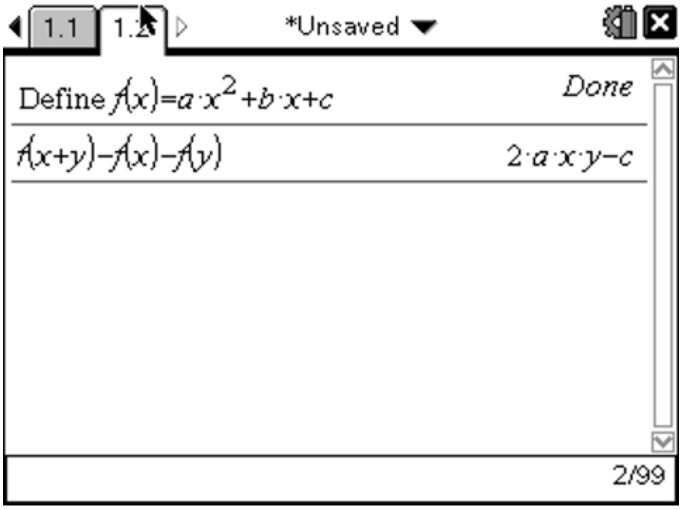
The minimum value is $2a$.



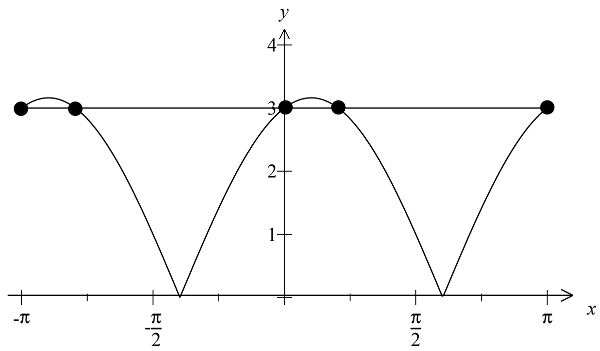
Question 4 **Answer C**

$f(x) = ax^2 + bx + c,\ f(y) = ay^2 + by + c$

$$\begin{aligned} f(x+y) &= a(x+y)^2 + b(x+y) + c \\ &= ax^2 + 2axy + ay^2 + bx + by + c \\ &= ax^2 + bx + c + ay^2 + by + 2axy \\ &= f(x) + f(y) - c + 2axy \end{aligned}$$



Question 5 **Answer E**



5 points of intersection

Question 6 **Answer D**

$$2\sin(2x) = 1$$

$$\sin(2x) = \frac{1}{2}$$

$$2x = 2n\pi + \frac{\pi}{6},\ (2n+1)\pi - \frac{\pi}{6}$$

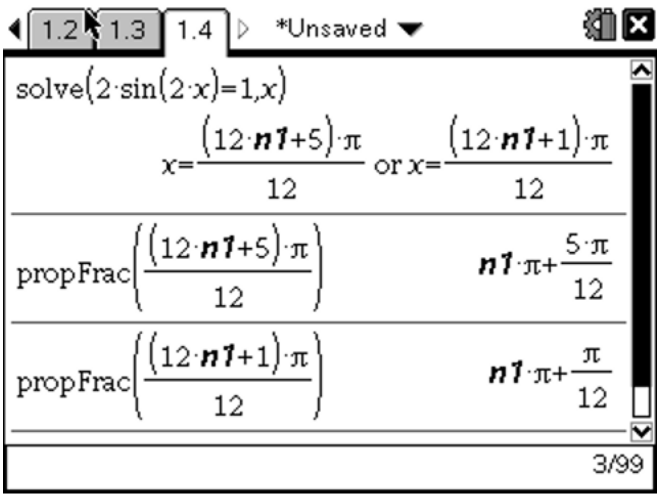
$$x = n\pi + \frac{\pi}{12},\ n\pi + \frac{\pi}{2} - \frac{\pi}{12}$$

$$= n\pi + \frac{\pi}{12},\ n\pi + \frac{5\pi}{12},\ n \in Z$$

OR

$$x = \frac{(12n+5)\pi}{12},\ \frac{(12n+1)\pi}{12}$$

$$= n\pi + \frac{5\pi}{12},\ n\pi + \frac{\pi}{12},\ n \in Z$$



Question 7 **Answer A**

$$f(x) = 2\tan\left(3x + \frac{\pi}{2}\right) + 1,\ -\frac{\pi}{3} \le x \le \frac{\pi}{3}$$

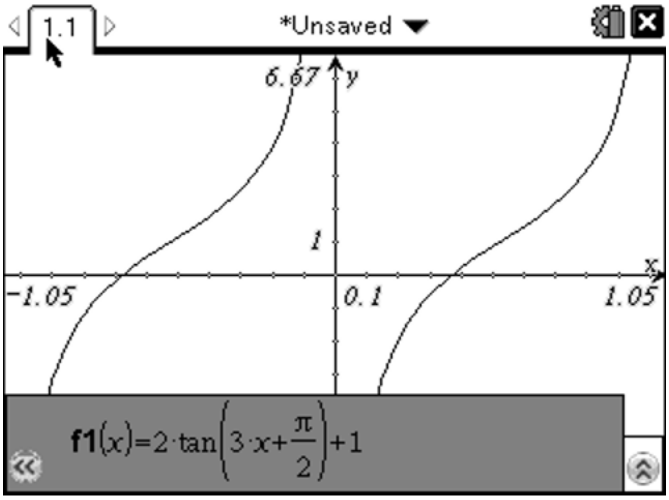
$$= 2\tan\left(3(x + \frac{\pi}{6})\right) + 1,\ -\frac{\pi}{3} \le x \le \frac{\pi}{3}$$

The period is $\frac{\pi}{3}$.

The graph of g with equation $g(x) = 2\tan(3x) + 1$ has asymptotes at $x = -\frac{\pi}{6}$, $x = \frac{\pi}{6}$ and $x = \frac{\pi}{2}$.

Translate $\frac{\pi}{6}$ units to the left.

Thus asymptotes have equations $x = -\frac{\pi}{3}$, $x = 0$ and $x = \frac{\pi}{3}$.



Question 8 **Answer B**

The equation of f can be written in the form $f(x) = A(x - B)^3 + C$.
There is a stationary point of inflection at (2, 4).

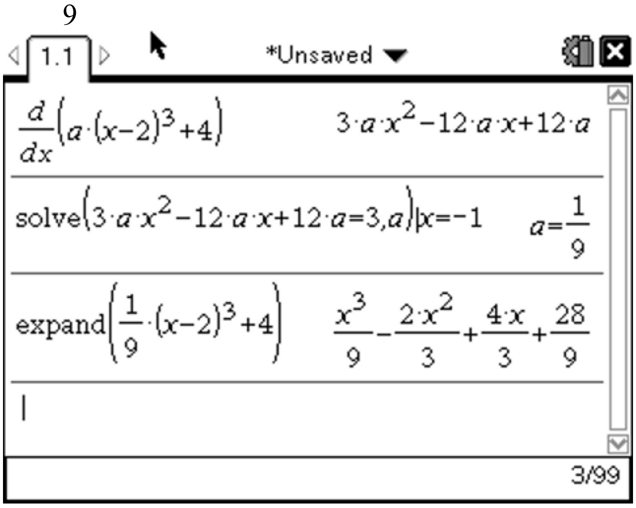
Hence $f(x) = A(x-2)^3 + 4$.

$f'(-1) = 3$

$f'(x) = 3A(x-2)^2$

$27A = 3$

$A = \frac{1}{9}$

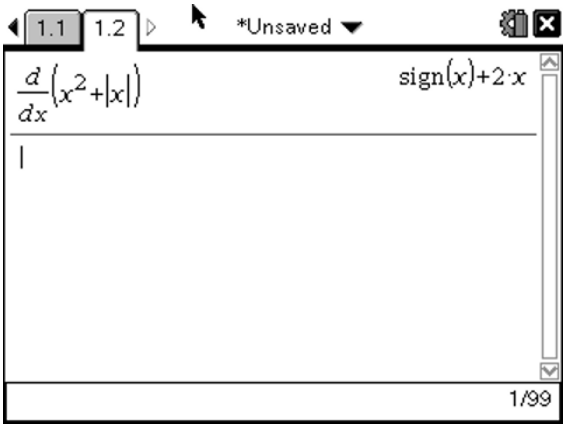


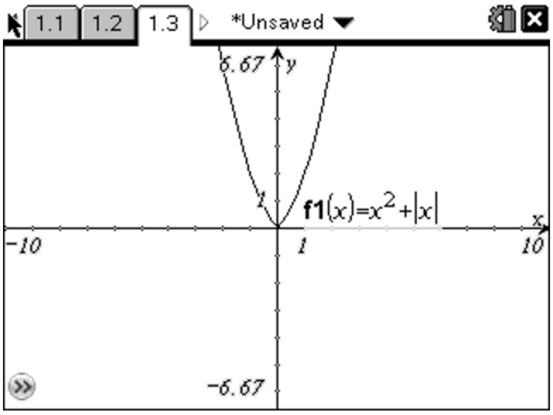
Question 9 **Answer A**

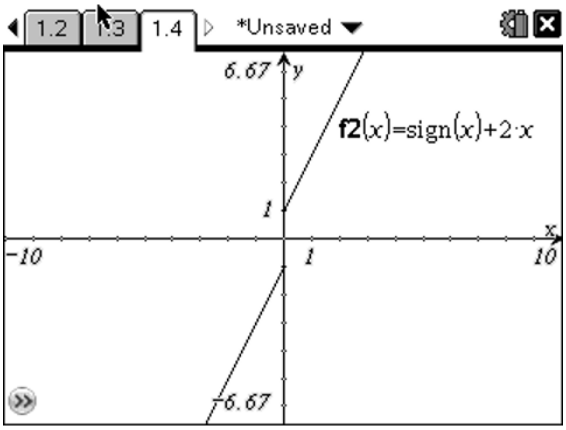
$h(x) = x^2 + x$ and $g(x) = |x|$

$h(g(x)) = (|x|)^2 + |x| = x^2 + |x|$

$$h'(g(x))g'(x) = \begin{cases} 2x+1, & x > 0 \\ 2x-1 & x < 0 \end{cases}$$



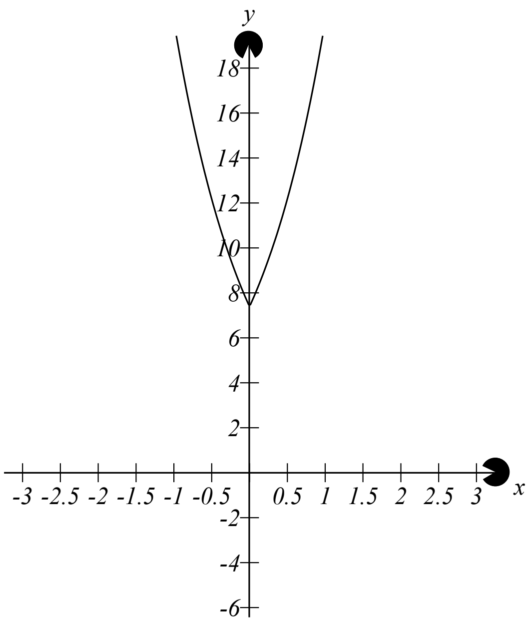




Question 10 **Answer E**

$f(x) = e^{|x|+2}$

By close inspection of the graph near $x = 0$, f cannot be differentiated at $x = 0$.



Question 11 **Answer C**

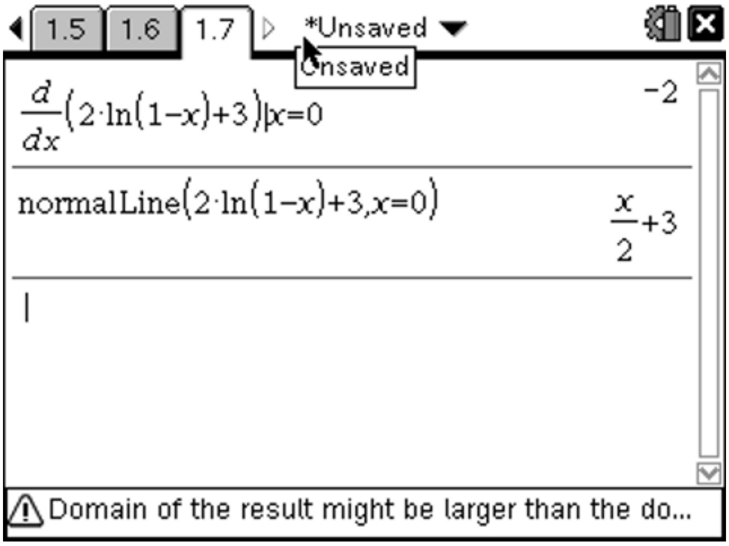
$y = f(x) = 2\log_e(1-x) + 3$

$f'(x) = \dfrac{-2}{1-x}$

Gradient of the tangent

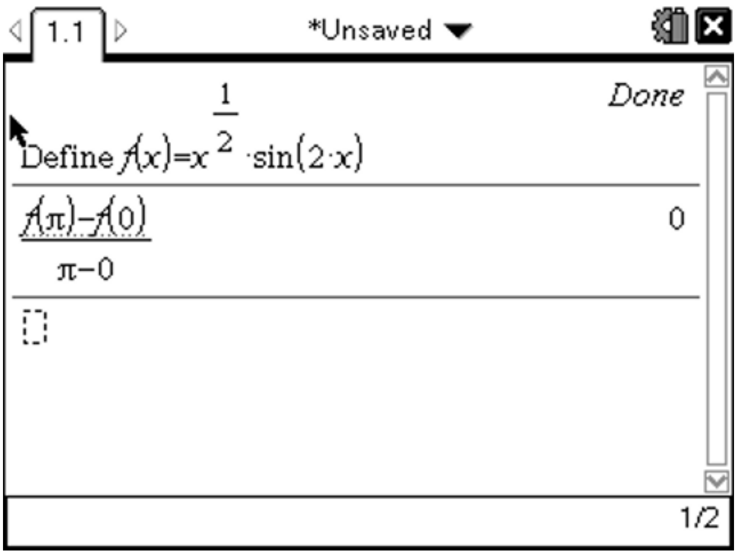
$f'(0) = -2$

Gradient of the normal is $\frac{1}{2}$.



Question 12 **Answer B**

Average rate of change = $\dfrac{f(\pi) - f(0)}{\pi - 0} = 0$



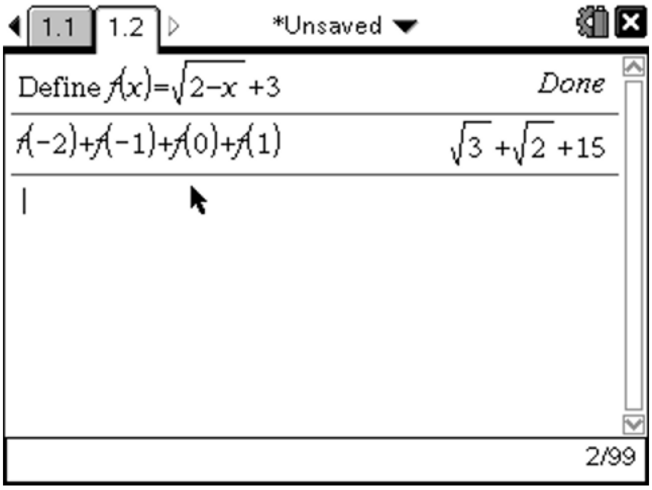
Question 13 **Answer B**

Area $\approx f(-2)+f(1)+f(0)+f(1)$

$= 5+\sqrt{3}+3+\sqrt{2}+3+4$

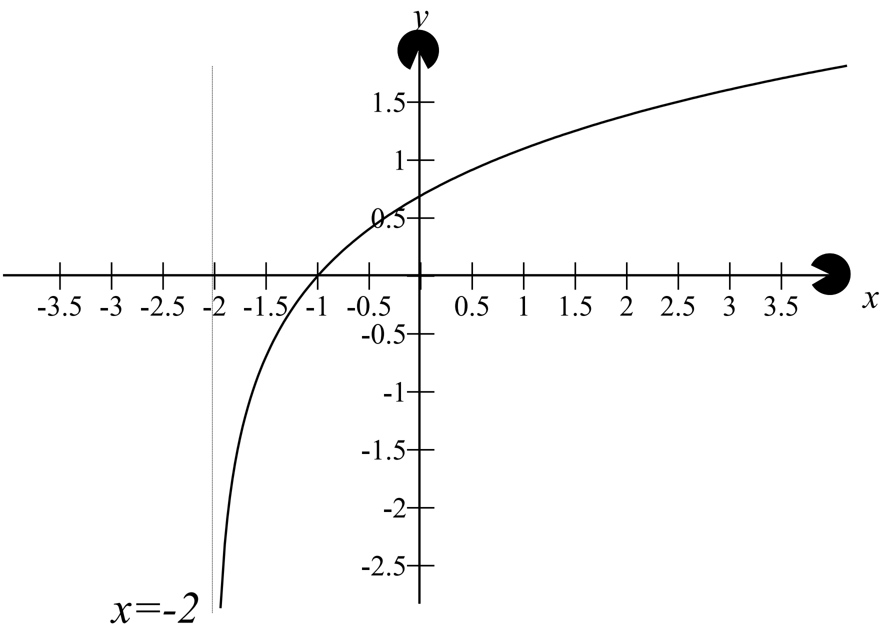
$= 15+\sqrt{3}+\sqrt{2}$

The rectangles are above the curve.
Hence an overestimate of the actual area.



Question 14 **Answer C**

The graph of $y=\log_e(x+2)$ is shown below.



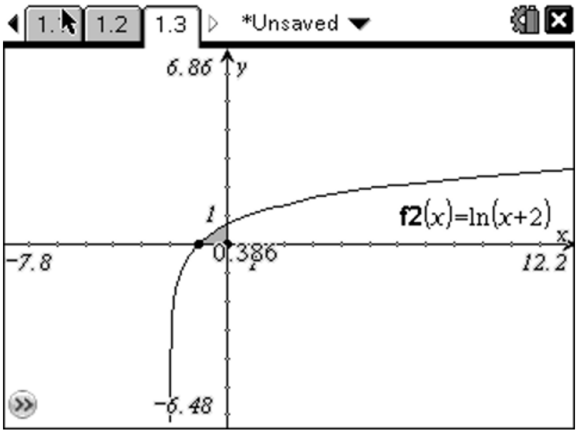
$x=-2$

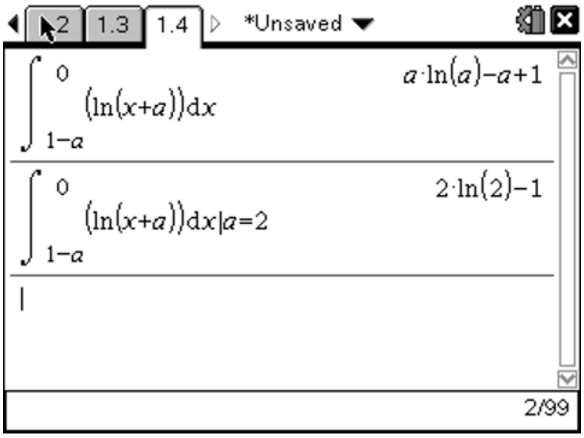
Solve $\log_e(x+a)=0$ for x.

$x+a=e^0$

$x=1-a$

Area $= \int_{1-a}^{0} f(x)dx$





Question 15 **Answer D**

$$\frac{d\left(x\sin(x)\right)}{dx}=\sin(x)+x\cos(x)$$

$$\Rightarrow x\sin(x)=\int\left(\sin(x)+x\cos(x)\right)dx$$

$$x\sin(x)=\int(\sin(x))\,dx+\int(x\cos(x))\,dx$$

$$\Rightarrow \int(x\cos(x))dx=x\sin(x)-\int(\sin(x))\,dx$$

Question 16 **Answer B**

Find coordinates of *B***.**

$\sin(x)+2x=2x$

$\sin(x)=0$

$x=0,\ \pi,\ 2\pi$

B has x value of π

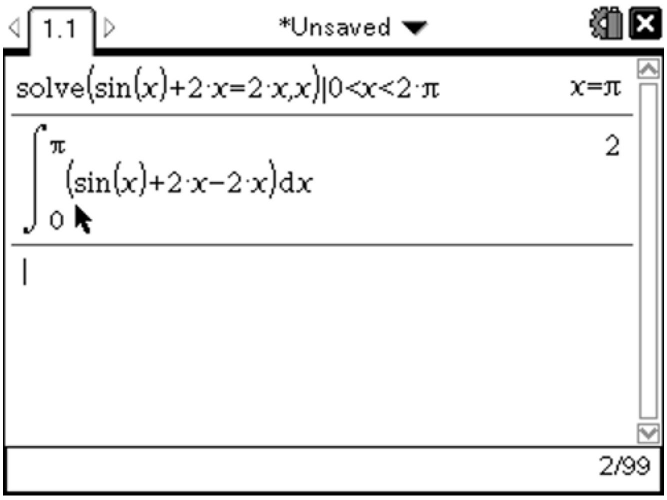
Area between curves:

$$A=\int_0^{\pi}(\sin(x)+2x-2x)\ dx$$

$$=[-\cos(x)]_0^{\pi}$$

$$=-\cos(\pi)+\cos(0)$$

$$=2$$



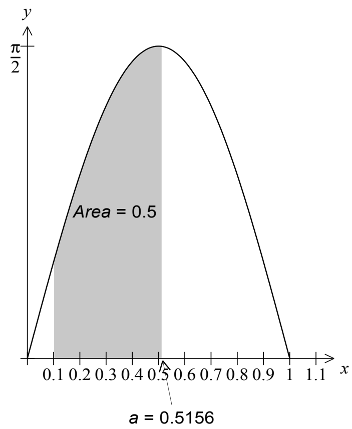
Question 17 **Answer A**

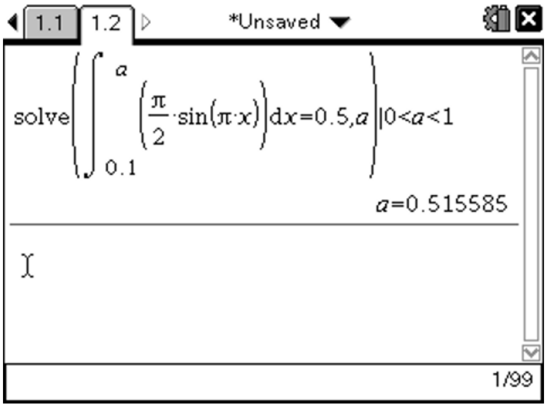
$\int_{0.1}^{a} \frac{\pi}{2}\sin(\pi x)dx = 0.5$

$a = -0.5156;\ 0.5156$

By domain restriction,

$a = 0.5156$





Question 18 **Answer C**

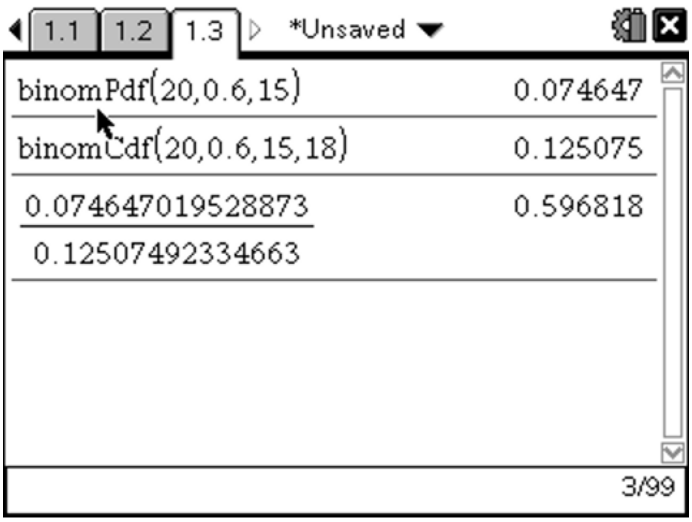
Binomial $n = 20,\ p = 0.6$

$\Pr(X = 15 | 15 \leq X \leq 18) = \frac{\Pr(X = 15)}{\Pr(15 \leq X \leq 18)}$

$= \frac{\Pr(X = 15)}{\Pr(X = 15) + \Pr(X = 16) + \Pr(X = 17) + \Pr(X = 18)}$

$= \frac{0.074647}{0.074647 + 0.034991 + 0.01235 + 0.003087}$

$= 0.5968$



Question 19 **Answer E**

$\Pr(A)\times\Pr(B)=0.4\times0.3=0.12$

$\Pr(A\cap B)=0.2$

A and *B* are not independent

Question 20 **Answer D**

The initial state matrix is $S_0=\begin{bmatrix}0.5\\0.5\end{bmatrix}$

The transition matrix is $T=\begin{array}{c} \\ R_1\\ L_1\end{array}\!\!\begin{array}{cc} R_0 & L_0\\ \left[\begin{matrix}0.8 & 0.6\\ 0.2 & 0.4\end{matrix}\right.& \left.\right]\end{array}$

$$T=\begin{matrix} & R_0 & L_0\\ R_1 & 0.8 & 0.6\\ L_1 & 0.2 & 0.4\end{matrix}$$

The next state matrix is given by the following.

$$S_1=\begin{matrix} & R_0 & L_0\\ R_1 & 0.8 & 0.6\\ L_1 & 0.2 & 0.4\end{matrix}\begin{bmatrix}0.5\\0.5\end{bmatrix}$$

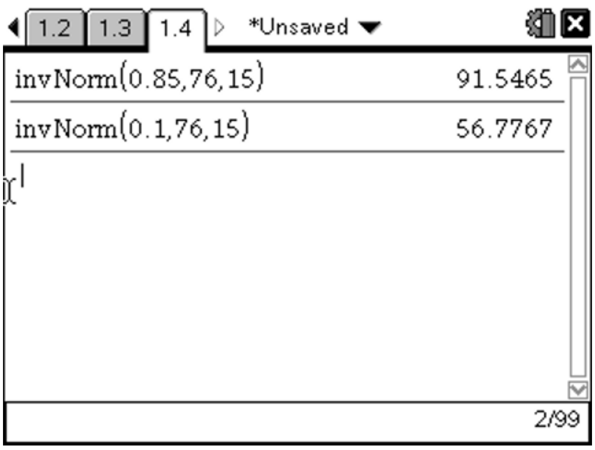
Question 21 **Answer A**

$E(X)=(-1+2+3-4+5-6)\times\frac{1}{6}=-\frac{1}{6}$

The player losses $\$\frac{1}{6}$.

Question 22 **Answer C**

Score A 92; score a pass 57



Solutions to the Extended Answer Section

Question 1

a. i. $h(x)=\begin{cases} -2x+2, & 0\le x\le 1 \\ 0, & 1<x\le 2 \\ 2x-4, & 2<x\le 3 \end{cases}$

$-2x+2$ **1A**

0 **1A**

$2<x\le 3$ **1A**

ii. $\int_0^3 h(x)dx = \text{Area of the two triangles} = 2\,\text{units}^2$ **1A**

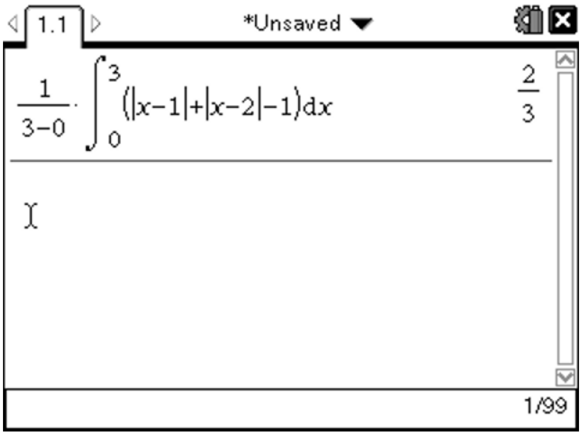
$3\times \text{Average Value} = 2$

Average Value = $\frac{2}{3}$ **1A**

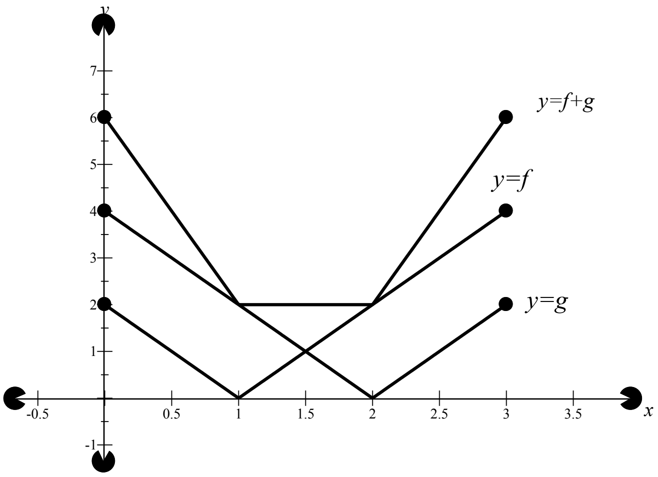
OR **OR**

$\frac{1}{3-0}\int_0^3 h(x)dx$ **1A**

$=\frac{2}{3}$ **1A**



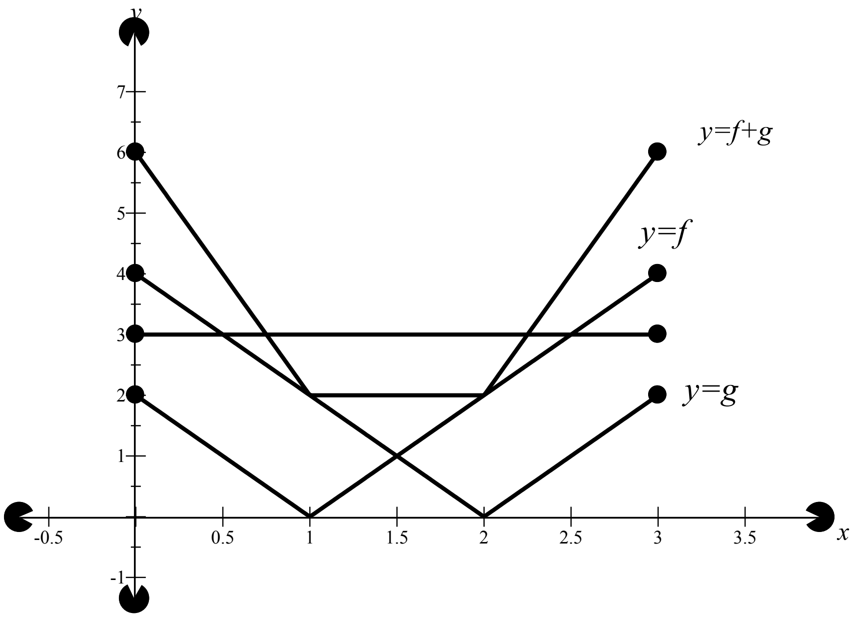
b. i.



Correct shape for $y = f + g$ **1A**

Correct endpoints and closed circles **1A**

ii.



Correct line $y = \frac{10}{3}$ **1A**

iii. There are many different approaches to this question. Some are outlined below.

The required area = the area of the triangles above the line $y = \frac{10}{3}$ because $\frac{10}{3}$ is the average value.

The equation of the first line segment is $y = -4x + 6$.

$\frac{10}{3} = -4x + 6$

$x = \frac{2}{3}$ **1A**

Area of both triangles

$= \text{base} \times \text{height}$

$= \frac{2}{3} \times \left(6 - \frac{10}{3}\right)$ **1A**

$= \frac{16}{9}$ units^2 **1A**

OR **OR**

$f(x) + g(x) = |2(x-1)| + |2(x-2)|$

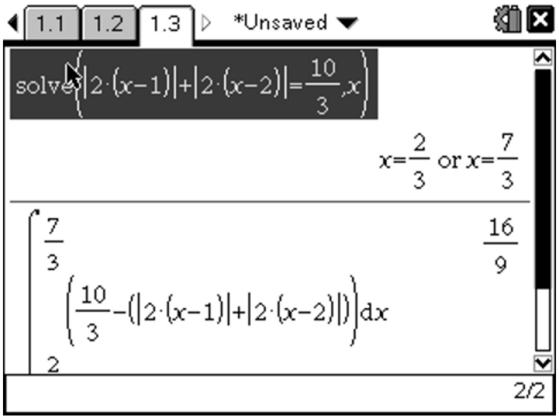
Solve $f(x) + g(x) = \frac{10}{3}$

$x = \frac{2}{3}$ or $x = \frac{7}{3}$ **1A**

(or could use the first method to get $x = \frac{2}{3}$ and then the other x value is $2 + \frac{1}{3}$)

$$= \int_{\frac{2}{3}}^{\frac{7}{3}} \left(\frac{10}{3} - (f(x) + g(x)) \right) dx$$ **1A**

$$= \frac{16}{9} \text{ units}^2$$ **1A**



OR **OR**

Area of the trapezium $= \dfrac{h(a+b)}{2}$

The equation of the first line segment is $y = -4x + 6$.

$$\frac{10}{3} = -4x + 6$$

$x = \dfrac{2}{3}$ or $x = 2 + \dfrac{1}{3} = \dfrac{7}{3}$ **1A**

Area $= \dfrac{\left(\frac{10}{3} - 2\right)\left(2 + \frac{7}{3} - \frac{2}{3}\right)}{2}$ **1A**

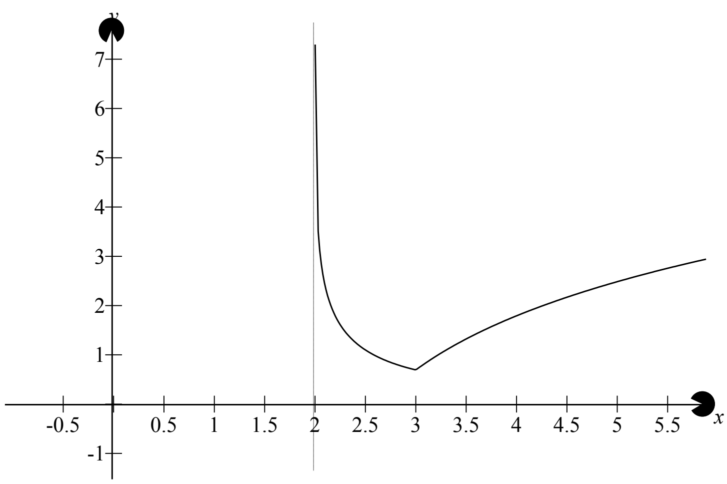
$= \dfrac{16}{9}$ units^2 **1A**

c. i. There is an asymptote at $x = 2$.

$a = 2$ **1A**

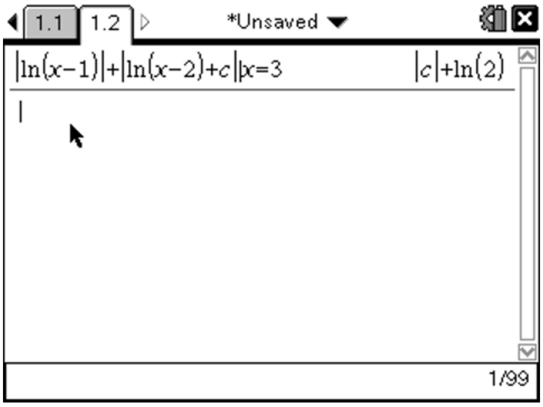
$\text{dom}(p) = \text{dom}(y = |\log_e(x-1)|) \cap \text{dom}(y = |\log_e(x-2)|)$

The graph of $p(x) = |\log_e(x-1)| + |\log_e(x-2)|$



ii. $(3, \log_e(2)+c)$ **1A**

$p(3) = |\log_e(3-1)| + |\log_e(3-2)| + c = \log_e(2) + c$



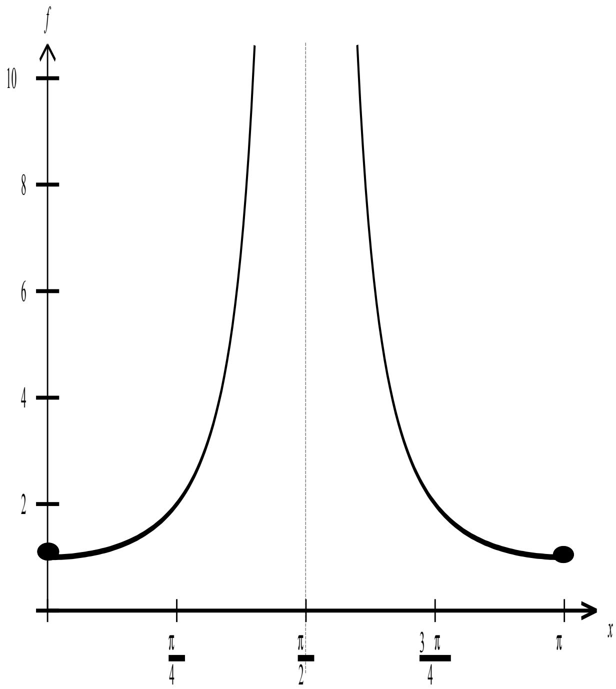
Question 2

a. $\dfrac{dy}{dx} = \dfrac{\cos(x)\cos(x) + \sin(x)\sin(x)}{\cos^2(x)}$ quotient rule

Since $\cos^2(x) + \sin^2(x) = 1$

$\dfrac{dy}{dx} = \dfrac{1}{\cos^2(x)}$ **1M**

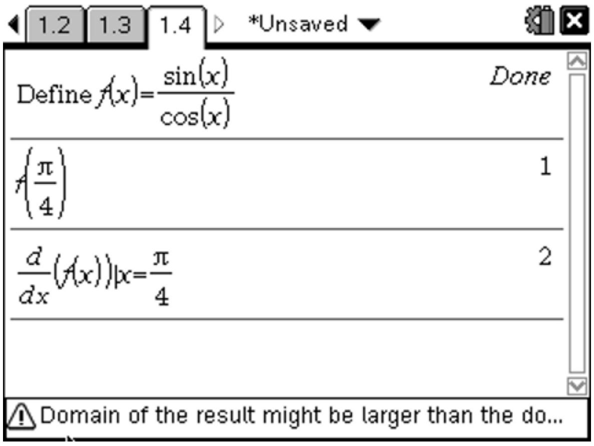
b.



Correct Shape and endpoints **1A**
Asymptote **1A**

c. i. $y = \tan\left(\dfrac{\pi}{4}\right) = 1$ **1A**

ii. $\frac{dy}{dx} = \frac{1}{\cos^2\left(\frac{\pi}{4}\right)} = \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2} = 2$ **1A**



d. domain: $\theta \in \left[0, \frac{\pi}{2}\right)$ **1A**

e. Let the path of the rocket be h km.

$\tan\theta = \frac{h}{3} \Rightarrow h = 3\tan(\theta)$ **1M**

f. From **part a.**, $\frac{dh}{d\theta} = \frac{3}{\cos^2(\theta)}$ **1A**

The rate at which the height of the rocket is changing with respect to the angle, θ. **1A**

g. $\frac{d\theta}{dt} = 20 \times \frac{\pi}{180}$ radians/second **1M**

$\frac{dh}{dt} = \frac{dh}{d\theta} \times \frac{d\theta}{dt}$ **1M**

$= \frac{3}{\cos^2(\theta)} \times \frac{\pi}{9}$

$= \frac{2\pi}{3}$ kilometres/second **1A**

h. i. $\tan\alpha = \frac{3}{4} \Rightarrow \alpha = 0.6435$ radians **1A**

ii. $h = 4\tan(\alpha) \Rightarrow \frac{dh}{d\alpha} = \frac{4}{\cos^2(\alpha)}$

$$\frac{d\alpha}{dt} = \frac{d\alpha}{dh} \times \frac{dh}{dt}$$ **1A**

$$= \frac{\cos^2(\alpha)}{4} \times \frac{2\pi}{3}$$ **1M**

$$= \frac{\pi \times \cos^2(0.6435)}{6}$$

$= 0.34$ radians/second (2 dp) **1A**

Question 3

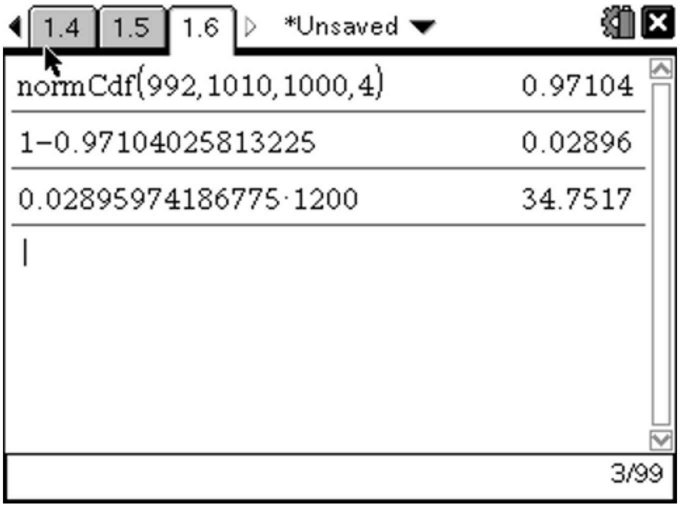
a. X is the weight of chocolate statues

$X \sim N(1000, 16)$

$\Pr(992 < X < 1010) = 0.97104 = 0.9710$ (4 dp) **1A**

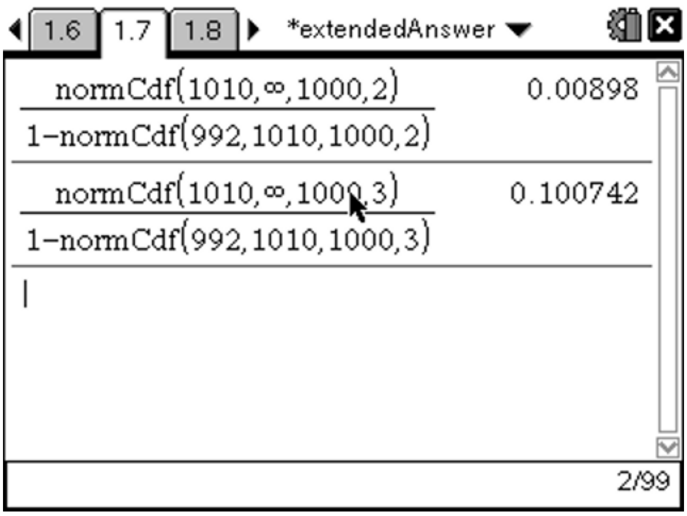
b. $\Pr(\text{rejected}) = 1 - 0.97104 = 0.0290$

Number rejected $= 1200 \times 0.0290 = 34.7 = 35$ statues **1A**



c. Trial and error (technology) using N(1000, {2,3,4}) with $\frac{\Pr(X > 1010)}{1 - \Pr(992 < X < 1010)}$. **1M**

Standard deviation of 3 **1A**



d. Y is the number rejected out of 5.

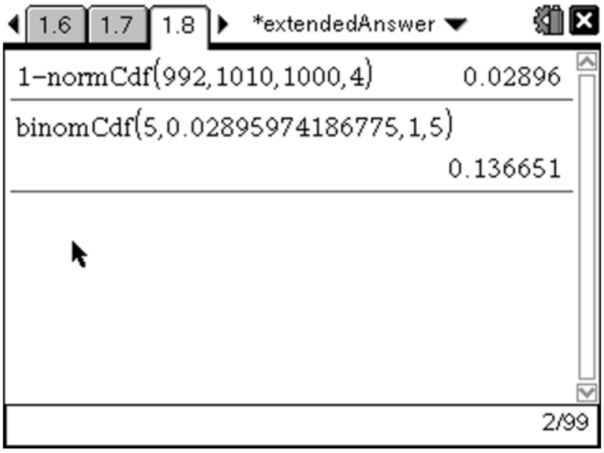
$Y \sim \text{Bi}(5, 0.0290)$

$Y = \{0, 1, 2, 3, 4, 5\}$ **1M**

$\Pr(Y \geq 1) = 1 - \Pr(Y = 0)$

$= 1 - 0.9710^5$

$= 0.13665 = 0.137 \text{ (3 dp)}$ **1A**



e. i. $\Pr(b \mid b) = 0.85;\ \Pr(b \mid b') = 0.3;\ \Pr(b' \mid b) = 0.15;\ \Pr(b' \mid b') = 0.7$

transition matrix $T = \begin{bmatrix} 0.85 & 0.3 \\ 0.15 & 0.7 \end{bmatrix}$

$\Pr(bbb \mid b) = 0.85^3 = 0.614125 = 0.6141 \text{ (4 dp)}$ **1A**

ii. $\Pr(b'bb \mid b) + \Pr(bb'b \mid b) + \Pr(bbb' \mid b)$ **1M**

$0.15 \times 0.3 \times 0.85 + 0.85 \times 0.15 \times 0.3 + 0.85^2 \times 0.15 = 0.18485 = 0.1849 \text{ (4 dp)}$ **1A**

iii. $\begin{bmatrix} 0.85 & 0.3 \\ 0.15 & 0.7 \end{bmatrix} \begin{bmatrix} x \\ 1-x \end{bmatrix} = \begin{bmatrix} x \\ 1-x \end{bmatrix}$

$x = \frac{2}{3}$ **1A**

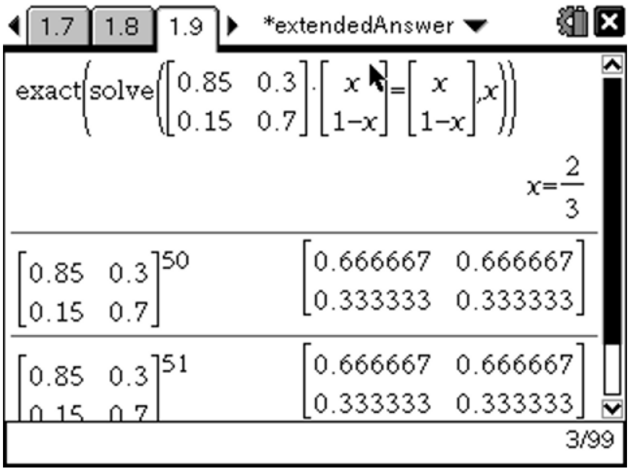
OR **OR**

$t = \begin{bmatrix} 0.85 & 0.3 \\ 0.15 & 0.7 \end{bmatrix} \Rightarrow t^{\infty} = \begin{bmatrix} 0.6667 & 0.6667 \\ 0.3333 & 0.3333 \end{bmatrix}$ (technology)

$x = \frac{2}{3}$ **1A**

OR **OR**

$x = \frac{0.3}{0.15 + 0.3} = \frac{2}{3}$ **1A**

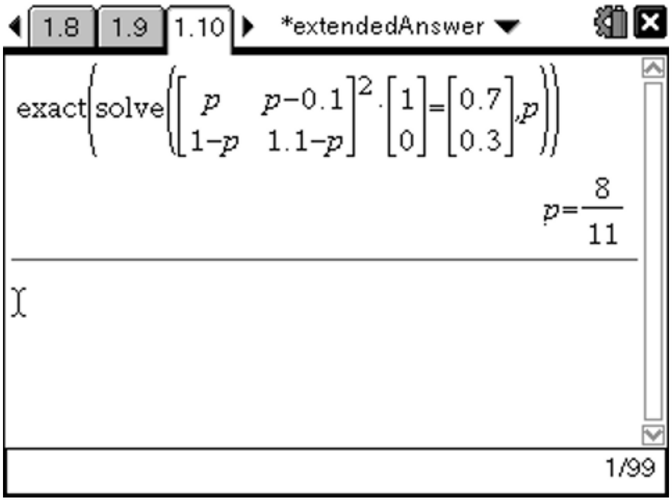


f. Let $t = \begin{bmatrix} p & p-0.1 \\ 1-p & 1-(p-0.1) \end{bmatrix} = \begin{bmatrix} p & p-0.1 \\ 1-p & 1.1-p \end{bmatrix}$ $\quad s = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ **1A**

$\Pr(\text{buy 3rd month}): t^2 s = \begin{bmatrix} 1.1p-0.1 \\ \ldots\ldots \end{bmatrix}$

$1.1p - 0.1 = 0.7$

$p = \dfrac{8}{11}$ **1M**



g. Let $T = \begin{bmatrix} \frac{8}{11} & \frac{8}{11} - 0.1 \\ \frac{3}{11} & 1.1 - \frac{8}{11} \end{bmatrix} = \begin{bmatrix} \frac{8}{11} & \frac{69}{110} \\ \frac{3}{11} & \frac{41}{110} \end{bmatrix}$ **1M**

$\Pr(\text{no buys}) = \Pr(b'b' \mid b) = \dfrac{3}{11} \times \dfrac{41}{110} = \dfrac{123}{1210}$

$\Pr(1 \text{ buy}) = \Pr(bb' \mid b) + \Pr(b'b \mid b) = \dfrac{8}{11} \times \dfrac{3}{11} + \dfrac{3}{11} \times \dfrac{69}{110} = \dfrac{447}{1210}$

$\Pr(2 \text{ buys}) = \Pr(bb \mid b) = \dfrac{8}{11} \times \dfrac{8}{11} = \dfrac{64}{121}$

x	0	1	2
$\Pr(X = x)$	$\dfrac{123}{1210}$	$\dfrac{447}{1210}$	$\dfrac{64}{121}$

1M

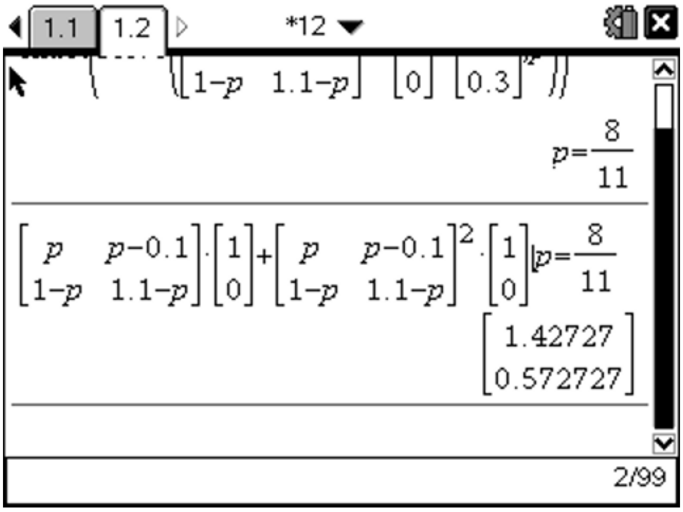
$E(X) = 0 \times \dfrac{1123}{12100} + 1 \times \dfrac{447}{1210} + 2 \times \dfrac{64}{121} = 1.427\ldots = 1.4$ month (1 dp) **1A**

OR **OR**

$$\begin{bmatrix} \frac{8}{11} & \frac{69}{110} \\ \frac{3}{11} & \frac{41}{110} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{8}{11} & \frac{69}{110} \\ \frac{3}{11} & \frac{41}{110} \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{157}{110} \\ \frac{63}{110} \end{bmatrix} \approx \begin{bmatrix} 1.427 \\ 0.573 \end{bmatrix}$$

2M

1.4 months **1A**



Question 4

a. i. x-intercept, $f(x) = 0$

$$0 = a(x-b)^5 + c$$

$$(x-b)^5 = -\frac{c}{a}$$

$$x = b + \sqrt[5]{-\frac{c}{a}}$$

$$\left(b + \sqrt[5]{-\frac{c}{a}}, 0\right)$$ **1A**

or $\left(b - \sqrt[5]{\frac{c}{a}}, 0\right)$

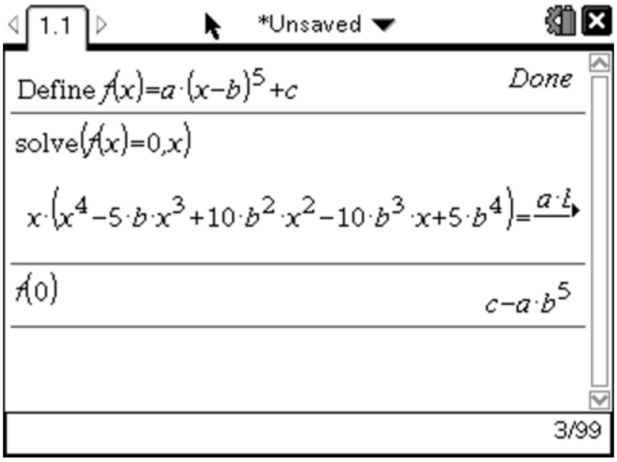
Note the TI-nspire CAS did not solve this equation

ii. y-intercept, $x = 0$

$$f(x) = a(x-b)^5 + c$$

$$f(0) = a(-b)^5 + c = -ab^5 + c$$

$$(0, -ab^5 + c)$$ **1A**



iii. The value of the x-intercept is the same as the y-intercept when the graph passes through (0, 0) only.

The graph of f is a quintic polynomial function which has its stationary point of inflection in the first quadrant. Hence when the y-intercept is positive, the x-intercept will be negative and vice-versa.

The stationary point of inflection is at (b, c). **1A**

Solve the y-intercept equal to 0 for c

$$-ab^5 + c = 0$$

$$c = ab^5$$

The stationary point of inflection is at (b, ab^5) **1A**

iv. The x-intercept is $b + \sqrt[5]{-\frac{c}{a}}$

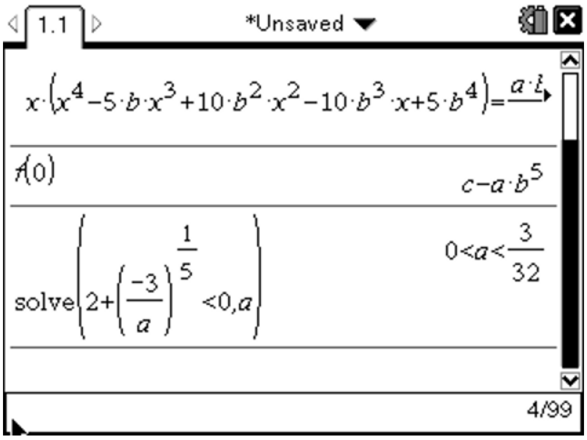
$$b + \sqrt[5]{-\frac{c}{a}} < 0$$

Solve $2 + \sqrt[5]{-\frac{3}{a}} < 0$ for a **1A**

$$\sqrt[5]{-\frac{3}{a}} < -2$$

$$-\frac{3}{a} < -32,\ a > 0$$

$$0 < a < \frac{3}{32}$$ **1A**

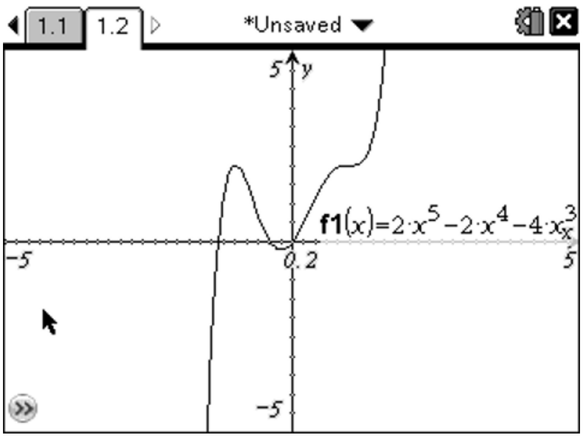


b. i. There are three stationary points. **1A**

Since g can be written in the form $g(x)=A(x-B)^3(x-C)^2+D$ and $A>0$,

there is a local maximum at (C, D), then a local minimum and then a stationary point of inflection at (B, D)

This can also be seen by graphing g.

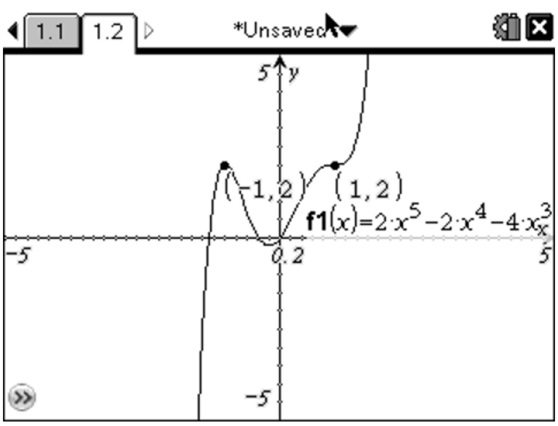


ii. $A=2$

Method 1

From the graph of g, Any 2 correct **1A** All correct **2A**

$B=1,\ C=-1$ and $D=2$



Note that this method does not always give exact values.

OR Method 2

$A = 2$

Solve the derivative of g equal to zero.

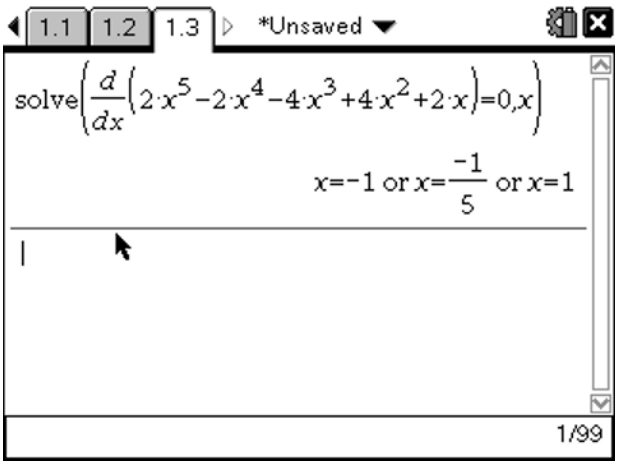
$B = 1,\ C = -1$

$g(x) = 2(x-1)^3(x+1)^2 + D$

The graph of g passes through (0, 0)

$-2 + D = 0$

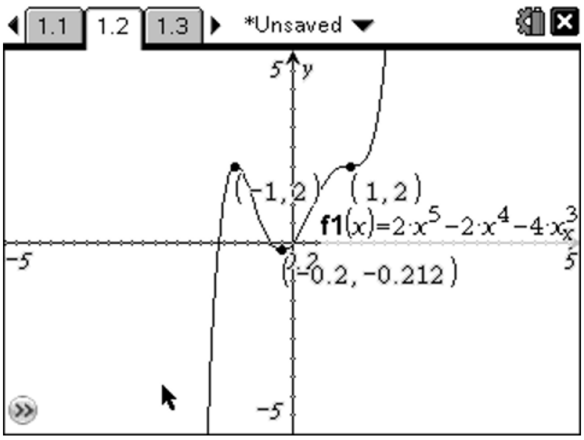
$D = 2$ Any 2 correct **1A** All correct **2A**

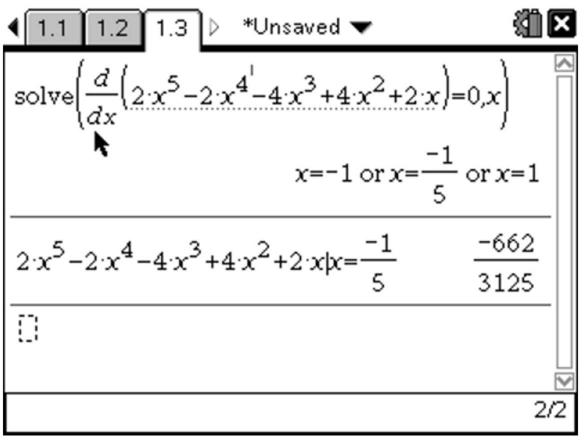


iii. The exact value for the y-coordinate is not given on the graph screen.

$g(-0.2) = -\dfrac{662}{3125}$

$\left(-\dfrac{1}{5}, -\dfrac{662}{3125}\right)$ **1A**





iv.

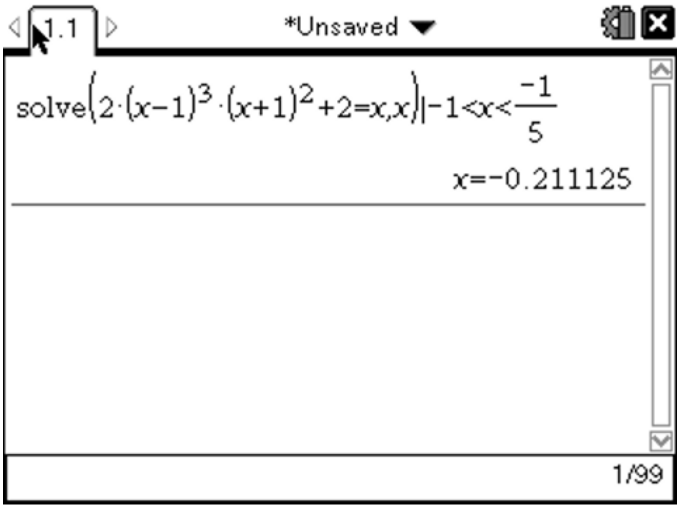
$(-\infty, -1] \cup [-\frac{1}{5}, \infty)$ **2A**

v. There is no need to find the equation of the inverse.

Solve $g_1^{-1} = x$ **1M**

$x = -0.211$

$(-0.211, -0.211)$ **1A**



END OF SOLUTIONS