Year 2011
VCE
Mathematical Methods
CAS
Solutions
Trial Examination 1

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Question 1

$$\Delta = \begin{vmatrix} -2 & k \\ k+1 & -6 \end{vmatrix}$$

$\Delta = 12 - k(k+1) = -k^2 - k + 12$ M1

$\Delta = -(k^2 + k - 12) = -(k+4)(k-3)$

i. for a unique solution $\Delta \neq 0 \Rightarrow k \in R \setminus \{-4, 3\}$ A1

ii. if $k = 3$ $(1)\ -2x + 3y = 5$

$(2)\ 4x - 6y = -10$

These equations represent the same line and are consistent,
so for infinitely many solutions $k = 3$ A1

iii. if $k = -4$ $(1)\ -2x - 4y = -2$

$(2)\ -3x - 6y = -10$

These lines are parallel, they have the same gradient, but different *y*-intercepts,
the equations are inconsistent, so for no solution $k = -4$ A1

Question 2

$f: \quad y = \dfrac{1}{1-2x}$ swapping x and y

$f^{-1} \quad x = \dfrac{1}{1-2y} \quad \Rightarrow 1 - 2y = \dfrac{1}{x} \quad \Rightarrow \quad 2y = 1 - \dfrac{1}{x}$ M1

$y = f^{-1}(x) = \dfrac{1}{2}\left(1 - \dfrac{1}{x}\right)$ A1

domain $f^{-1} =$ range $f = R \setminus \{0\}$

domain $f =$ range $f^{-1} = R \setminus \{\tfrac{1}{2}\}$ A1

Question 3

a. $y = x\sin(2x)$ using the product rule

$\dfrac{dy}{dx} = \sin(2x) + 2x\cos(2x)$ A1

b. Since $\frac{d}{dx}\left(x\sin(2x)\right)=\sin(2x)+2x\cos(2x)$ it follows that

$$2\int x\cos(2x)\,dx = x\sin(2x)-\int \sin(2x)\,dx$$

$$2\int x\cos(2x)\,dx = x\sin(2x)+\frac{1}{2}\cos(2x)$$ M1

$$\int x\cos(2x)\,dx = \frac{1}{2}x\sin(2x)+\frac{1}{4}\cos(2x)+c$$ A1

c.i. $$\int_0^b 2\cos(2x)dx=1$$

$$\left[\sin(2x)\right]_0^b=\sin(2b)-\sin(0)=1$$

$$\sin(2b)=1$$

$$2b=\frac{\pi}{2}$$ M1

$$b=\frac{\pi}{4}$$

ii. $$E(X)=\int_0^{\frac{\pi}{4}} 2x\cos(2x)dx$$

$$E(X)=\left[x\sin(2x)+\frac{1}{2}\cos(2x)\right]_0^{\frac{\pi}{4}}$$ M1

$$E(X)=\left(\left(\frac{\pi}{4}\sin\left(\frac{\pi}{2}\right)+\frac{1}{2}\cos\left(\frac{\pi}{2}\right)\right)-\left(0\times\sin(0)+\frac{1}{2}\cos(0)\right)\right)$$

$$E(X)=\frac{\pi}{4}-\frac{1}{2}=\frac{\pi-2}{4}$$ A1

iii. $$\int_0^m 2\cos(2x)dx=\frac{1}{2}$$ median m A1

$$\left[\sin(2x)\right]_0^m=\sin(2m)-\sin(0)=\frac{1}{2}$$

$$\sin(2m)=\frac{1}{2}$$

$$2m=\frac{\pi}{6}$$

$$m=\frac{\pi}{12}$$ A1

Question 4

Let $y=\log_e\left(\frac{3x^2+4}{4x^2+3}\right)=\log_e\left(3x^2+4\right)-\log_e\left(4x^2+3\right)$

$\frac{dy}{dx}=\frac{6x}{3x^2+4}-\frac{8x}{4x^2+3}$

$\frac{dy}{dx}=\frac{6x\left(4x^2+3\right)-8x\left(3x^2+4\right)}{\left(3x^2+4\right)\left(4x^2+3\right)}$ M1

$\frac{dy}{dx}=\frac{-14x}{12x^4+25x^2+12}$

$b=-14$ A1

$g(x)=12x^4+25x^2+12$ A1

Question 5

$s=\int_0^2 \frac{72}{(3t+2)^2}\,dt$ A1

$s=\left[\frac{-72}{3(3t+2)}\right]_0^2=\left[\frac{-24}{3t+2}\right]_0^2$ M1

$s=\frac{-24}{8}+\frac{24}{2}=-3+12$

$s=9$ metres A1

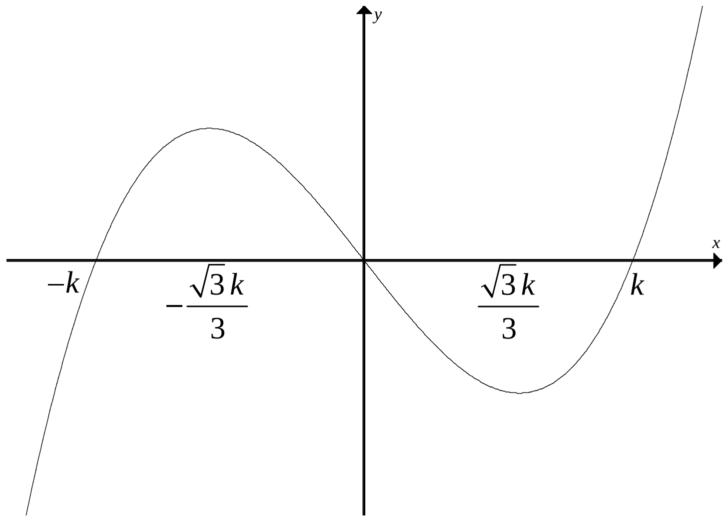
Question 6

a. $f(x)=x^3-k^2x=x\left(x^2-k^2\right)=x(x+k)(x-k)$

the graph crosses the x-axis at $(k,0)$ and $(-k,0)$

$f'(x)=3x^2-k^2=0$ for stationary points

$x^2=\frac{k^2}{2} \quad\Rightarrow\quad x=\pm\frac{k}{\sqrt{3}}=\pm\frac{\sqrt{3}\,k}{3}$ M1



$f'(x) \ge 0 \quad \Rightarrow \left(-\infty, -\frac{\sqrt{3}k}{3}\right] \cup \left[\frac{\sqrt{3}k}{3}, \infty\right)$ A1

b. The area $A = \int_0^k \left(x^3 - k^2 x\right) dx$ is below the *x*-axis so is negative

$A = -\int_0^k \left(x^3 - k^2 x\right) dx = 64$ M1

$-\left[\frac{1}{4}x^4 - \frac{k^2 x^2}{2}\right]_0^k = \left(\frac{k^4}{2} - \frac{k^4}{4}\right) = \frac{k^4}{4} = 64$

$k^4 = 4 \times 64$, since $k > 0$

$k = 4$ A1

Question 7

a. $f(\alpha) = g(\alpha) \quad \Rightarrow \quad \tan(\alpha) = \cos(\alpha)$

$\frac{\sin(\alpha)}{\cos(\alpha)} = \cos(\alpha)$ M1

$\sin(\alpha) = \cos^2(\alpha) = 1 - \sin^2(\alpha)$

$\sin^2(\alpha) + \sin(\alpha) - 1 = 0$ solving using quadratic formula M1

$\sin(\alpha) = \frac{-1 \pm \sqrt{5}}{2}$ but $\sin(\alpha) > 0$ since $\alpha \in \left(0, \frac{\pi}{2}\right)$ M1

$\sin(\alpha) = \frac{1}{2}\left(\sqrt{5} - 1\right)$ shown

b. $f'(x)=\dfrac{1}{\cos^2(x)}$ $\qquad g'(x)=-\sin(x)$

$f'(\alpha)=\dfrac{1}{\cos^2(\alpha)}$ $\qquad g'(\alpha)=-\sin(\alpha)$ A1

Now $f'(\alpha)g'(\alpha)=\dfrac{1}{\cos^2(\alpha)}\times-\sin(\alpha)$

$=-\dfrac{\sin(\alpha)}{\cos(\alpha)}\times\dfrac{1}{\cos(\alpha)}$ M1

$=-\dfrac{f(\alpha)}{g(\alpha)}=-1$ since $f(\alpha)=g(\alpha)$

Since the products of the gradients is -1, the functions intersect at right angles. A1

Question 8

$X \sim Bi(n=6, p="p")$

$\Pr(X=3)=\binom{6}{3}p^3(1-p)^3$

$\Pr(X=4)=\binom{6}{4}p^4(1-p)^2$ M1

$\Pr(X=3)=\Pr(X=4)$

$\dfrac{6\times5\times4}{3\times2}p^3(1-p)^3=\dfrac{6\times5}{2}p^4(1-p)^2$

$20p^3(1-p)^3-15p^4(1-p)^2=0$ A1

$5p^3(1-p)^2\left[4(1-p)-3p\right]=0$

$5p^3(1-p)^2(4-7p)=0$ since $0<p<1$

$p=\dfrac{4}{7}$ A1

Question 9

a. $\int_1^4 \frac{4}{\sqrt{x}}\,dx$

$= 4\int_1^4 x^{-\frac{1}{2}}\,dx = 8\left[\sqrt{x}\right]_1^4 = 8\left(\sqrt{4}-\sqrt{1}\right)$

$= 8$ A1

b. **i.** $f:[-4,-1]\to R \quad f(x)=\frac{-4}{\sqrt{-x}}$

both domain and rule are required A1

ii. Total area $8\times 2+4\times 8$ area of rectangle and symmetry from **a.**

$= 48 \text{ units}^2$ A1

Question 10

$y' = -2\cos\left(\frac{\pi}{3}\left(x'+\frac{3}{\pi}\right)\right)+4$

mapping $y=\cos(x)$ into $\frac{y'-4}{-2}=\cos\left(\frac{\pi}{3}\left(x'+\frac{3}{\pi}\right)\right)$

$y=\frac{y'-4}{-2}$ $\quad x=\frac{\pi}{3}\left(x'+\frac{3}{\pi}\right)$

M1

$y'=-2y+4$ $\quad x'=\frac{3x}{\pi}-\frac{3}{\pi}$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)=\begin{bmatrix} \frac{3}{\pi} & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} -\frac{3}{\pi} \\ 4 \end{bmatrix}$$

$a=\frac{3}{\pi}$, $b=0$, $c=0$, $d=-2$ A1

$h=-\frac{3}{\pi}$ $\quad k=4$ A1

END OF SUGGESTED SOLUTIONS