Mathematical Association of Victoria
Trial Exam 2010

MATHEMATICAL METHODS (CAS)

STUDENT NAME ____

Written Examination 2

Reading time: 15 minutes
Writing time: 2 hours

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
1	22	22	22
2	4	4	58
			Total 80

Note
- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved CAS calculator (memory DOES NOT need to be cleared) and, if desired, one scientific calculator. For approved computer based CAS, their full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied
- Question and answer book of 22 pages with a detachable sheet of miscellaneous formulas at the back
- Answer sheet for multiple-choice questions.

Instructions
- Detach the formula sheet from the back of this book during reading time.
- Write your **student number** in the space provided above on this page.
- All written responses must be in English.

At the end of the examination
- Place the answer sheet for multiple-choice questions inside the front cover of this book.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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SECTION 1

Instructions for Section 1
Answer all questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is correct for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will not be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.

Question 1

The maximal domain of the function with rule $f(x)=\dfrac{9}{\sqrt{x^2-1}}$ is

A. $(1,\infty)$
B. $[1,\infty)$
C. $(-1, 1)$
D. $(-\infty, -1)\cup(1,\infty)$
E. $(-\infty, -1]\cup[1,\infty)$

Question 2

The inverse of the function $f:(2,\infty)\to R,\ f(x)=1-2\log_e(3x-6)$ is

A. $f^{-1}:(2,\infty)\to R,\ f^{-1}(x)=2\log_e(3x-6)-1$
B. $f^{-1}:R\to R,\ f^{-1}(x)=2\log_e(3x-6)-1$
C. $f^{-1}:R\to R,\ f^{-1}(x)=\dfrac{1}{3}e^{\left(\frac{1-x}{2}\right)}+2$
D. $f^{-1}:(2,\infty)\to R,\ f^{-1}(x)=\dfrac{1}{3}e^{\left(\frac{1-x}{2}\right)}+2$
E. $f^{-1}:R\to R,\ f^{-1}(x)=\dfrac{1}{2}e^{\left(\frac{1-x}{3}\right)}+3$

Question 3

Which one of the following functions satisfies the functional equation $f(4x)=f(x)+2f(2)$?

A. $f:[0,\infty)\to R,\ f(x)=\sqrt{2x}$
B. $f:R\to R,\ f(x)=2x+2$
C. $f:R\to R,\ f(x)=e^x$
D. $f:(0,\infty)\to R,\ f(x)=\log_e(x)$
E. $f:R\backslash\{0\}\to R,\ f(x)=\dfrac{2}{x}$

SECTION 1 (Cont'd)

Question 4

The graphs with equations $y = ax$ and $y = -\frac{1}{x} + 1$, where a is a non-zero real constant, will have two distinct points of intersection if

A. $a > \frac{1}{4}$

B. $a = \frac{1}{4}$

C. $a < \frac{1}{4}$

D. $a \geq \frac{1}{4}$

E. $\{a : -\infty < a < 0\} \cup \left\{a : 0 < a < \frac{1}{4}\right\}$

Question 5

If $\int_1^4 f(x)dx = 5$, then $\int_1^4 (f(x) + 2)dx$ is equal to

A. **4**

B. 7

C. 8

D. 9

E. 11

Question 6

The general solution to the equation
$2\tan(x) + 1 = 3$ is

A. $x = \frac{\pi}{4} + n\pi,\ n \in Z$

B. $x = \frac{\pi}{4} + 2n\pi,\ n \in Z$

C. $x = 2n\pi - \frac{3\pi}{4},\ n \in Z$

D. $x = (4n - 3)\pi,\ n \in Z$

E. $x = \frac{\pi}{4} + n\pi$ or $x = -\frac{\pi}{4} + n\pi,\ n \in Z$

Question 7

The period, amplitude and range of the graph of g with equation $g(x) = -2\cos\left(\frac{x}{3} - 2\right) + 1$ are respectively

A. 6π, 2 and $[-2, 2]$

B. $\frac{2\pi}{3}$, 2 and $[-2, 2]$

C. 6π, -2 and $[-1, 3]$

D. $\frac{2\pi}{3}$, 2 and $[-1, 3]$

E. 6π, 2 and $[-1, 3]$

Question 8

Let $g : R \setminus \{0\} \to R,\ g(x) = \left|\frac{5}{x} + 1\right|$.

$\{x : g(x) \geq 3\} =$

A. $\left\{x : -\frac{5}{4} \leq x \leq \frac{5}{2}\right\}$

B. $\left\{x : -\frac{5}{4} \leq x < 0\right\} \cup \left\{x : 0 < x \leq \frac{5}{2}\right\}$

C. $\left\{x : -\frac{5}{2} \leq x \leq -\frac{5}{4}\right\}$

D. $\left\{x : -\frac{5}{4} < x < 0\right\} \cup \left\{x : 0 < x < \frac{5}{2}\right\}$

E. $\left\{x : -\frac{5}{4} < x < \frac{5}{2}\right\}$

Question 9

The simultaneous linear equations

$2x + (p+5)y = p$

$px + 3y = -1$

where p is a real constant, have **no solution** if

A. $p \in \{-6, 1\}$

B. $p \in R \setminus \{-6, 1\}$

C. $p \in \{-1, 6\}$

D. $p \in R \setminus \{-1, 6\}$

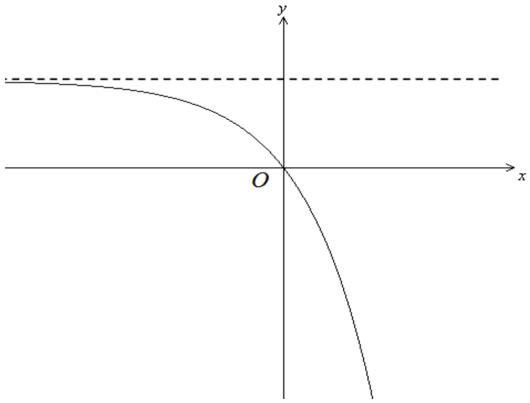
E. $p \in \{-2, 3\}$

SECTION 1 (Cont'd)

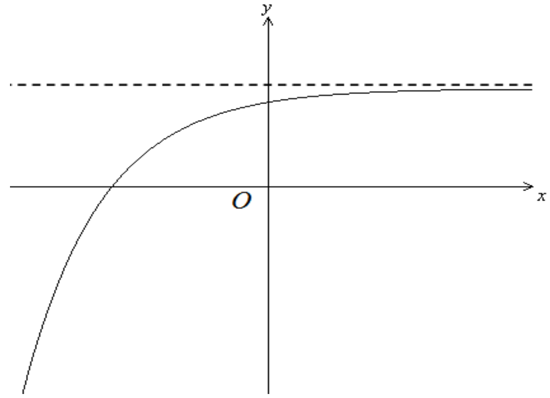
Question 10

The graph of $y = a e^{-(x+h)} + k$, where $a < 0$, $h > 0$ and $k > 0$ could be

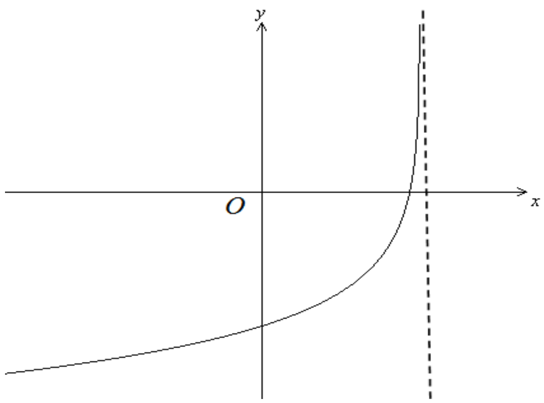
A.



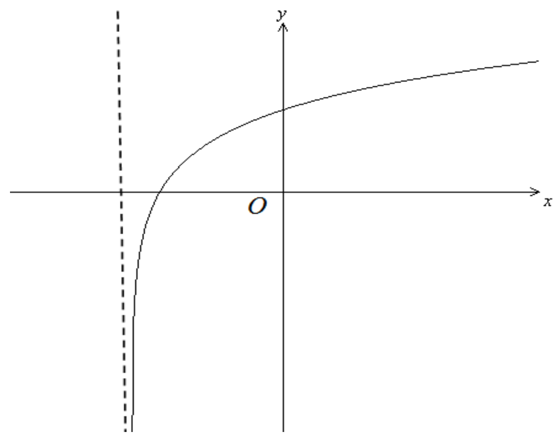
B.



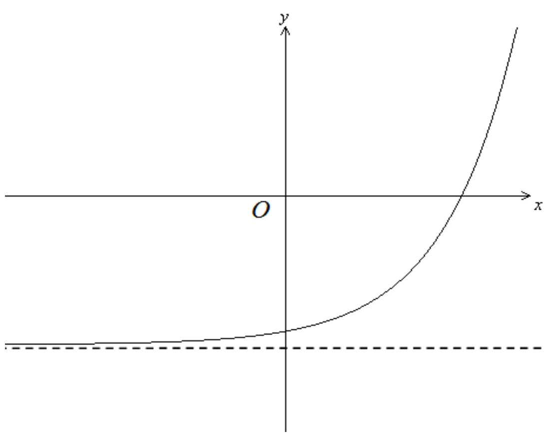
C.



D.



E.



Question 11

The graph of $y = h(x)$ undergoes the following sequence of transformations.

- A dilation of scale factor 2 from the y-axis
- A reflection in the x-axis
- Translations 6 units right and 1 unit up

The image of the graph under these transformations has the equation

A. $y = 1 - h\left(\frac{1}{2}(x-6)\right)$

B. $y = 1 - 2h(x-6)$

C. $y = 2h(6-x) + 1$

D. $y = h(2(6-x)) + 1$

E. $y = h\left(\frac{1}{2}(6-x)\right) + 1$

Question 12

The travel time on a particular train route is a normally distributed random variable with a mean of 45 minutes and a standard deviation of 8 minutes. If Z is the random variable of the standard normal distribution, then the probability that a travel time is more than 55 minutes is equal to

A. $\Pr(Z < 1.25)$

B. $1 - \Pr(Z < 1.25)$

C. $\Pr(Z > -1.25)$

D. $\Pr(Z < 2)$

E. $1 - \Pr(Z < 2)$

Question 13

At a particular railway station, the morning train arrives on time on 70% of weekdays. The probability that the train arrives on time on at least 3 out of 5 weekday mornings is closest to

A. 0.8369
B. 0.1631
C. 0.6913
D. 0.3087
E. 0.3430

Question 14

The table below gives the distribution of the discrete random variable X.

X	0	1	2	3	4
$\Pr(X = x)$	r	0.2	0.35	0.3	$2r$

If r is a real constant, $E(X)$ is equal to

A. 0.05
B. 0.15
C. 1.8
D. 2
E. 2.2

SECTION 1 (Cont'd)

Question 15

The continuous random variable T has the probability density function

$$f(t)=\begin{cases}\dfrac{t+1}{8} & -1\le t\le k\\ 0 & \text{otherwise}\end{cases}$$

The value of k is

A. 0
B. 1
C. 2
D. 3
E. 8

Question 16

For a Markov chain, $S_n = T^n \times S_0$. T and S_0 could be

A. $T=\begin{bmatrix}0.8 & 0.2\\ 0.4 & 0.6\end{bmatrix},\ S_0=\begin{bmatrix}1\\ 0\end{bmatrix}$

B. $T=\begin{bmatrix}1\\ 0\end{bmatrix},\ S_0=\begin{bmatrix}0.8 & 0.2\\ 0.4 & 0.6\end{bmatrix}$

C. $T=\begin{bmatrix}0.3 & 0.9\\ 0.7 & 0.1\end{bmatrix},\ S_0=\begin{bmatrix}15\\ 30\end{bmatrix}$

D. $T=\begin{bmatrix}15\\ 30\end{bmatrix},\ S_0=\begin{bmatrix}0.3 & 0.9\\ 0.7 & 0.1\end{bmatrix}$

E. $T=\begin{bmatrix}0.15 & 0.65\\ 0.75 & 0.35\end{bmatrix},\ S_0=\begin{bmatrix}\frac{1}{3}\\ \frac{2}{3}\end{bmatrix}$

Question 17

Let $f(x)=A(x-B)^{\frac{1}{3}}+C$, where A, B and C are real constants. If the equation of the normal to the curve of f at $x = 3$ is $y = 1$ and $f(0)<0$ then the equation of f could be

A. $f(x)=-(x-3)^{\frac{1}{3}}+1$

B. $f(x)=(x-3)^{\frac{1}{3}}+1$

C. $f(x)=-(x+1)^{\frac{1}{3}}+3$

D. $f(x)=-(x+3)^{\frac{1}{3}}+1$

E. $f(x)=(x-3)^{\frac{1}{3}}-1$

Question 18
The graph of *f* with equation $f(x) = x^4 - x^3 - 3x^2 + 5x - 2$ has

A. a local maximum and a stationary point of inflection

B. a local maximum and two local minimums

C. a local minimum and a stationary point of inflection

D. a local minimum and two local maximums

E. a local minimum and a non-stationary point of inflection

Question 19
If $f:[0,\infty) \to R$ where $f(x) = x^2 + \sqrt{x}$ and $g: R \to R$ where $g(x) = x^2$ then $\frac{d}{dx} f(g(x))$ equals

A. $2x + \frac{1}{2\sqrt{x}}$

B. $4x^3 + 1$

C. $x^4 + \sqrt{x^2}$

D. $\begin{cases} 4x^3 + 1 \text{ for } x > 0 \\ 4x^3 - 1 \text{ for } x < 0 \end{cases}$

E. $\begin{cases} 4x^3 + 1 \text{ for } x \geq 0 \\ 4x^3 - 1 \text{ for } x \leq 0 \end{cases}$

Question 20
The derivative of $\log_e(\sin(2x))$ is not defined when

A. $x = k\pi, k \in Z$ only

B. $\pi k \leq x \leq \frac{(2k+1)\pi}{2}, k \in Z$

C. $x = \frac{k\pi}{2}, k \in Z$ only

D. $\frac{(2k-1)\pi}{2} \leq x \leq \pi k, k \in Z$

E. $x = \frac{k\pi}{2}, k \in R$

SECTION 1 (Cont'd)

Question 21
The approximate area under a curve over a particular interval is found by using left endpoint rectangles. An over estimate for the actual area under the curve over the interval will be found if its equation is

A. $y = e^x$

B. $y = \sqrt{x}$

C. $y = x^{\frac{3}{2}}$

D. $y = -e^x$

E. $y = -e^{-x}$

Question 22

The area enclosed by the curves with equations $f(x) = \sin\left(\frac{x}{2}\right)$ and $g(x) = \cos(x)$ over the interval $[0, 4\pi]$ can be found by evaluating

A. $\int_{\frac{\pi}{3}}^{\frac{4\pi}{3}} \left(f(x) - g(x)\right)dx + \int_{\frac{4\pi}{3}}^{3\pi} \left(g(x) - f(x)\right)dx$

B. $\int_{\frac{\pi}{3}}^{\frac{4\pi}{3}} \left(g(x) - f(x)\right)dx + \int_{\frac{4\pi}{3}}^{3\pi} \left(f(x) - g(x)\right)dx$

C. $\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} \left(f(x) - g(x)\right)dx + \int_{\frac{5\pi}{3}}^{3\pi} \left(g(x) - f(x)\right)dx$

D. $\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} \left(g(x) - f(x)\right)dx + \int_{\frac{5\pi}{3}}^{3\pi} \left(f(x) - g(x)\right)dx$

E. $\int_{1.05}^{5.24} \left(f(x) - g(x)\right)dx + \int_{5.24}^{9.42} \left(g(x) - f(x)\right)dx$

END OF SECTION 1

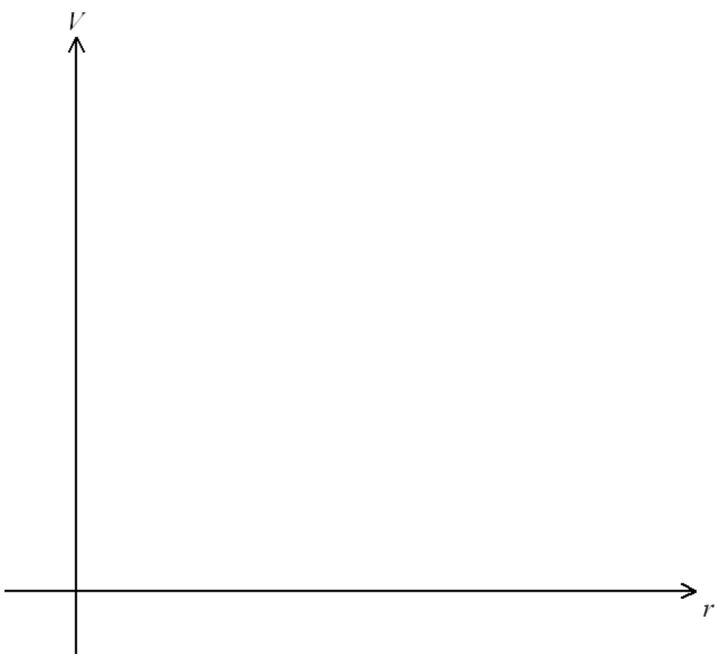
SECTION 2

Instructions for Section 2
Answer all questions in the spaces provided.
A decimal approximation will not be accepted if an exact answer is required to a question.
In questions where more than one mark is available, appropriate working must be shown.
Unless otherwise indicated, the diagrams in this book are not drawn to scale.

Question 1

The volume, V cm^3, of a particular set of right circular cones is given by $V = \frac{1}{3}\pi r^2 h = \pi r^3$, where r is the radius and h the height in cm.

a. Sketch the graph of V on the set of axes below for $0 < r \leq 4$.



2 marks

b. If the radius of one of these cones is 4 cm what would the height of the cone be?

1 mark

SECTION 2 (Cont'd)

c. **i.** Using the approximation $V(r) \approx V(2) + (r-2)V'(2)$, show that the volume of the cone can be approximated by $V(r) \approx 12\pi r - 16\pi$ for r close to 2.

ii. Hence, find the approximate value for the volume when $r = 2.1$ cm. Give your answer in terms of π.

iii. Is the approximation in **part c.ii.** an underestimate or an overestimate of the actual volume? Explain.

2 + 1 + 2 = 5 marks

d. Right endpoint rectangles of width 1 cm are used to estimate the area under the curve of V for $0 < r \leq 4$.

i. What is the approximate area under the curve? Give your answer in terms of π.

ii. By how much is the actual area over estimated? Give your answer in terms of π.

iii. By how much would the average value of V for $0 < r \leq 4$ be overestimated? Give your answer in terms of π.

2 + 2 + 1 = 5 marks

SECTION 2 (Cont'd)

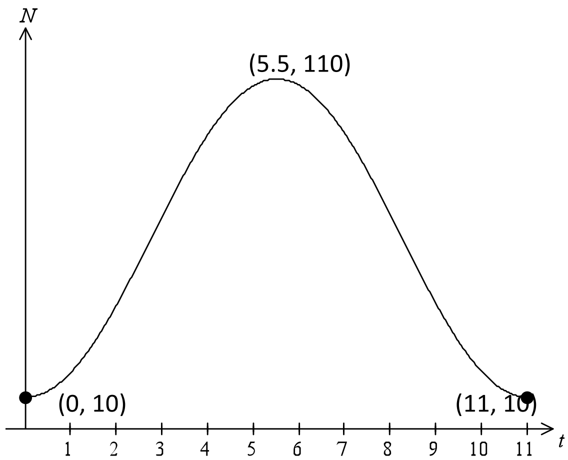
Question 2

Solar flares or "sunspots" are caused by the Sun's magnetic field. The average number of sunspots in any given year follows a periodic cycle, called a solar cycle.

Using historical data, Bryan, a solar astronomer, modelled the number of sunspots during a solar cycle with the function

$N:[0,11] \rightarrow R, \quad N(t) = b - a\cos(nt),$

where N is the number of sunspots t years after the start of a solar cycle and a, b and n are **positive** real constants. The graph of the function is shown.



a. According to this model:

i. How many complete solar cycles have occurred between 1755 and 2008?

ii. What is the range of N?

1 + 1 = 2 marks

b. Show that $n = \dfrac{2\pi}{11}$

1 mark

c. Show that $a = 50$ and $b = 60$.

2 marks

Assume that a new solar cycle began on 1 January 2009.

d. What is the predicted number of sunspots on 1 January 2011, correct to the nearest integer?

1 mark

e. The level of UV radiation increases with the number of sunspots. Bryan proposes to monitor UV radiation levels during the period when $N \geq 80$. For what length of time is $N \geq 80$? Express the answer correct to the nearest month.

2 marks

SECTION 2 (Cont'd)

Bryan's colleague, Mariam, noted that the historical data shows considerable variation in the amplitude of the solar cycle. For the **current cycle**, she proposes the alternative model,

$N_1 : [0,11] \rightarrow R,\ \ N_1(t) = 60 - 50e^{-kt}\cos\left(\frac{2\pi t}{11}\right)$, where k is a positive real constant.

f. If Mariam's model predicts that there will be 89 sunspots on 1 January 2014, show that, correct to two decimal places, $k = 0.10$.

1 mark

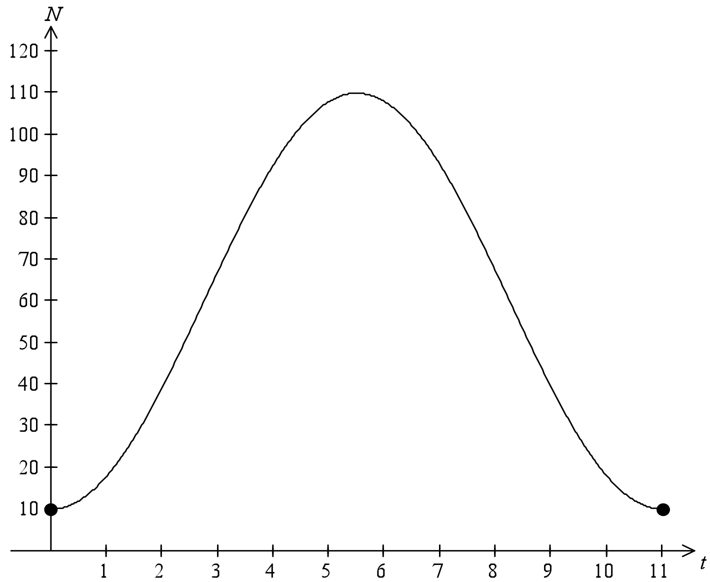
g. Find the points of intersection of the graphs of N and N_1, correct to two decimal places.

2 marks

h. The graph of N is shown below. On the same set of axes, sketch the graph of N_1.

Label the coordinates, correct to one decimal place, of the **points of intersection** with the graphs of N and N_1.

On your graph of N_1, label the turning points and the endpoints with their coordinates, correct to one decimal place.



3 marks
Total 14 marks

SECTION 2 (Cont'd)

WORKING SPACE

Question 3

The records of a large hospital show that the healing time for a particular type of surgical incision is a normally distributed random variable with a mean of 11 days and a standard deviation of 2.5 days. Due to a shortage of hospital places, all patients undergoing this surgery are discharged from this hospital within 13 days of receiving the surgery.

a. Grainger, a hospital supervisor, randomly selects the record of a discharged patient who has undergone this surgery.

i. What is the probability, correct to four decimal places, that the incision had healed by the day the patient was discharged?

ii. Given that, at the time of discharge, the patient's incision had healed, what is the probability that it took more that 10 days to heal? Express the answer correct to three decimal places.

2 + 2 = 4 marks

b. Grainger selects the records of five discharged patients who have undergone this surgery. What is the probability that, for at least one patient, the incision had **not** healed at the time of discharge? Express the answer correct to three decimal places.

2 marks

SECTION 2 (Cont'd)

The hospital has recently purchased new technology for performing this surgical procedure. The manufacturer of the technology claims that the mean healing time will be reduced to 9 days and that for 95% of patients the incision will heal within 12 days.

c. If this claim is true, what is the standard deviation of the healing time when the surgery is performed using this technology? Express your answer in days, correct to two decimal places.

3 marks

After adopting this technology, the length of stay in hospital, T days, for patients undergoing this surgery is a continuous random variable with probability density function given by

$$f(t) = \begin{cases} \frac{3}{64}(t-8)^2(12-t) & 8 \le t \le 12 \\ 0 & \text{otherwise} \end{cases}$$

d. For these patients, what is the **median** length of stay in hospital, correct to two decimal places?

2 marks

During her lunch breaks, Grainger either goes to the canteen or she uses a 'social networking' website, *Tracebook*, to trace her childhood friends. If she uses *Tracebook* one day, there is a 60% chance that she will use *Tracebook* the following day. However, if she goes to the canteen one day, there is a 75% chance that she will go to the canteen the following day.

Grainger went to the canteen on Monday.

e. What is the probability that she will use *Tracebook* during the lunch break either on Tuesday or Wednesday, but not both?

2 marks

f. What is the probability, correct to two decimal places, that she will go to the canteen on Friday?

3 marks
Total 16 marks

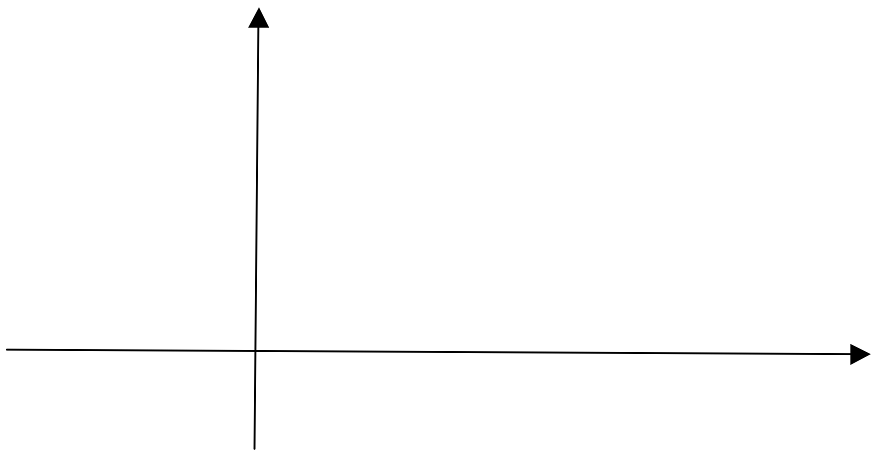
SECTION 2 (Cont'd)

WORKING SPACE

Question 4

Scientists have predicted that a population of insects, $P\ (\times 1000)$, on an island is increasing at a rate $\dfrac{dP}{dt} = 3^{1-t} + 2$, where t is the time in years from January 1st 2009. The initial population was 200 000.

a. **i.** Sketch $\dfrac{dP}{dt}$ against t on the set of axes below.



ii. According to the model, at what rate will the population be increasing in the long term?

2 + 1 = 3 marks

b. **i.** Express P in terms of t.

SECTION 2 (Cont'd)

ii. What was the population, to the nearest thousand, on January 1st 2010?

2 + 1 = 3 marks

The scientists underestimated the effect of climate change. So, they decided to review their model and they produced a hybrid function for $\frac{dP}{dt}$.

$\frac{dP}{dt} = 3^{1-t} + 2$ until $\frac{dP}{dt} = t$. After this date, the scientists predicted that $\frac{dP}{dt}$ would equal the inverse function of $y = 3^{1-t} + 2$.

c. **i.** When does the rule change in the new model? Give your answer in years correct to two decimal places.

ii. Find the new rule for $\frac{dP}{dt}$.

iii. According to the new model, when will the rate of change be zero? Give the date.

iv. What is the predicted maximum number of insects? Give your answer to the nearest whole number.

1 + 2 + 1 + 4 = 8 marks

d. How is climate change going to affect the insects in the long term, according to the scientists?

1 mark

Total 15 marks

END OF QUESTION AND ANSWER BOOKLET

Mathematical Methods (CAS) Formula Sheet

Mensuration

area of a trapezium:	$\frac{1}{2}(a+b)h$	volume of a pyramid:	$\frac{1}{3}Ah$
curved surface area of a cylinder:	$2\pi rh$	volume of a sphere:	$\frac{4}{3}\pi r^3$
volume of a cylinder:	$\pi r^2 h$	area of a triangle:	$\frac{1}{2}bc\sin A$
volume of a cone:	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}\left(x^n\right)=nx^{n-1}$

$\int x^n dx=\frac{1}{n+1}x^{n+1}+c, n\neq -1$

$\frac{d}{dx}\left(e^{ax}\right)=ae^{ax}$

$\int e^{ax}dx=\frac{1}{a}e^{ax}+c$

$\frac{d}{dx}(\log_e(x))=\frac{1}{x}$

$\int \frac{1}{x}dx=\log_e|x|+c$

$\frac{d}{dx}(\sin(ax))=a\ \cos(ax)$

$\int \sin(ax)dx=-\frac{1}{a}\cos(ax)+c$

$\frac{d}{dx}(\cos(ax))=-a\ \sin(ax)$

$\int \cos(ax)dx=\frac{1}{a}\sin(ax)+c$

$\frac{d}{dx}(\tan(ax))=\frac{a}{\cos^2(ax)}=a\ \sec^2(ax)$

product rule: $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$

quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

chain rule: $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$

approximation: $f(x+h)\approx f(x)+hf'(x)$

Probability

$\Pr(A)=1-\Pr(A')$

$\Pr(A\cup B)=\Pr(A)+\Pr(B)-\Pr(A\cap B)$

$\Pr(A|B)=\frac{\Pr(A\cap B)}{\Pr(B)}$

transition matrices: $S_n=T^n\times S_0$

mean: $\mu=\mathrm{E}(X)$

variance: $\mathrm{var}(X)=\sigma^2=\mathrm{E}((X-\mu)^2)=\mathrm{E}(X^2)-\mu^2$

probability distribution		mean	variance
discrete	$\Pr(X=x)=p(x)$	$\mu=\sum x\,p(x)$	$\sigma^2=\sum(x-\mu)^2 p(x)$
continuous	$\Pr(a<X<b)=\int_a^b f(x)dx$	$\mu=\int_{-\infty}^{\infty}x\,f(x)dx$	$\sigma^2=\int_{-\infty}^{\infty}(x-\mu)^2 f(x)dx$

END OF FORMULA SHEET

MULTIPLE CHOICE ANSWER SHEET

Student Name: ....

Circle the letter that corresponds to each correct answer

1	A	B	C	D	E
2	A	B	C	D	E
3	A	B	C	D	E
4	A	B	C	D	E
5	A	B	C	D	E
6	A	B	C	D	E
7	A	B	C	D	E
8	A	B	C	D	E
9	A	B	C	D	E
10	A	B	C	D	E
11	A	B	C	D	E
12	A	B	C	D	E
13	A	B	C	D	E
14	A	B	C	D	E
15	A	B	C	D	E
16	A	B	C	D	E
17	A	B	C	D	E
18	A	B	C	D	E
19	A	B	C	D	E
20	A	B	C	D	E
21	A	B	C	D	E
22	A	B	C	D	E