Year 2010

VCE

Mathematical Methods CAS

Solutions

Trial Examination 2

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SECTION 1

ANSWERS

1		A		B		C		**D**		E
2		A		B		C		D		**E**
3		A		B		C		D		**E**
4		A		**B**		C		D		E
5		A		B		**C**		D		E
6		A		B		**C**		D		E
7		**A**		B		C		D		E
8		A		B		C		D		**E**
9		A		B		**C**		D		E
10		A		B		C		**D**		E
11		**A**		B		C		D		E
12		**A**		B		C		D		E
13		A		B		**C**		D		E
14		A		**B**		C		D		E
15		A		B		C		D		**E**
16		**A**		B		C		D		E
17		A		B		C		**D**		E
18		A		B		C		**D**		E
19		**A**		B		C		D		E
20		A		**B**		C		D		E
21		A		**B**		C		D		E
22		A		B		**C**		D		E

SECTION 1

Question 1 **Answer D**

$f(x)=1+3g(x)$ for $0\le x\le 4$

$$\int_0^4 \left(f(x)-g(x)\right)dx$$

$$=\int_0^4 \left(1+3g(x)-g(x)\right)dx$$

$$=\int_0^4 \left(1+2g(x)\right)dx$$

$$=[x]_0^4+2\int_0^4 \left(g(x)\right)dx$$

$$=4+2\int_0^4 \left(g(x)\right)dx$$

Question 2 **Answer E**

All of **A. B. C.** and **D.** are true, the graph is not defined at $x=-a$,
and the graph is not continuous at $x=-a$
E. is false.

Question 3 **Answer E**

$$f(x+h)\approx f(x)+hf'(x)$$

with $f(x)=\cos(x)$ $f'(x)=-\sin(x)$ $x=\frac{\pi}{6}^c$ $\left(30^0\right)$ $h=-\frac{\pi}{180}$ $\left(-1^0\right)$

$$\cos\left(29^0\right)=f\left(\frac{\pi}{6}\right)+\left(-\frac{\pi}{180}\right)f'\left(\frac{\pi}{6}\right)$$

$$\cos\left(29^0\right)=\cos\left(\frac{\pi}{6}\right)+\left(-\frac{\pi}{180}\right)\left(-\sin\left(\frac{\pi}{6}\right)\right)$$

$$\cos\left(29^0\right)=\frac{\sqrt{3}}{2}+\frac{\pi}{360}$$

Question 4 **Answer B**

$f(x)=\dfrac{\sqrt{x^2+16}}{g(x)}$ using the quotient rule $g(3)=3$ and $g'(3)=-1$

$f'(x)=\dfrac{\frac{1}{2}\frac{2x}{\sqrt{x^2+16}}g(x)-g'(x)\sqrt{x^2+16}}{\left[g(x)\right]^2}$, if $x=3$ $\sqrt{x^2+16}=5$

$f'(3)=\dfrac{\frac{3}{5}\text{x}3+1\text{x}5}{9}=\dfrac{1}{9}\left(\dfrac{9}{5}+5\right)=\dfrac{34}{45}$

Question 5 **Answer C**

$\dfrac{dy}{dx}=\dfrac{1}{2x-9} \quad \Rightarrow \quad y=\displaystyle\int\dfrac{1}{2x-9}dx$

$y=\dfrac{1}{2}\log_e|2x-9|+C$, now when $x=\dfrac{5}{2}$ $y=0$

$0=\dfrac{1}{2}\log_e|5-9|+C=\dfrac{1}{2}\log_e|-4|+C=\dfrac{1}{2}\log_e(4)+C \Rightarrow C=-\dfrac{1}{2}\log_e(4)$

$y=\dfrac{1}{2}\log_e|2x-9|-\dfrac{1}{2}\log_e(4)=\dfrac{1}{2}\log_e\left(\dfrac{|2x-9|}{4}\right)$

when $x=0$ $y=\dfrac{1}{2}\log_e\left(\dfrac{|-9|}{4}\right)=\dfrac{1}{2}\log_e\left(\dfrac{9}{4}\right)=\log_e\left(\sqrt{\dfrac{9}{4}}\right)=\log_e\left(\dfrac{3}{2}\right)$

Question 6 **Answer C**

Let $f(x)=x^5-bx+c$ $\qquad f(-1)=1$

$f(-1)=(-1)^5+b+c=-1+b+c=1 \quad \Rightarrow \quad b+c=2$

$f'(x)=5x^4-b$ $\qquad f'(-1)=0$ since it is a stationary point.

$f'(-1)=5(-1)^4-b=5-b=0 \quad \Rightarrow \quad b=5$ and $c=-3$

Question 7 **Answer A**

$V = L^3 \Rightarrow \frac{dV}{dL} = 3L^2$ given that $\frac{dV}{dt} = p$

$\frac{dL}{dt} = \frac{dL}{dV} \cdot \frac{dV}{dt} = \frac{p}{3L^2}$

Question 8 **Answer E**

None of **A. B. C.** and **D.** are true, however since

$y = -3\log_e\left(\frac{x}{2}\right) = -3\left(\log_e(x) - \log_e(2)\right) = -3\log_e(x) + 3\log_e(2) = -3\log_e(x) + \log_e(8)$

From the graph of $y = \log_e(x)$, a reflection in the *x*-axis, gives $y = -\log_e(x)$ then a dilation by a scale factor of 3 parallel to the *y*-axis, gives $y = -3\log_e(x)$, followed by a translation of $\log_e(8)$ units up and parallel to the *y*-axis, gives $y = -3\log_e(x) + \log_e(8)$.

Question 9 **Answer C**

$f: \quad y = \frac{1}{x+a}$ dom $f = R \setminus \{-a\} = \text{ran } f^{-1}$

$f^{-1} \quad x = \frac{1}{y+a}$ transposing

$y + a = \frac{1}{x} \quad y = \frac{1}{x} - a = \frac{1-ax}{x}$ but ran $f = \text{dom } f^{-1} = R \setminus \{0\}$, so that

$f^{-1}: R \setminus \{0\} \to R$, $f^{-1}(x) = \frac{1-ax}{x}$

Question 10 **Answer D**

All of **A. B. C.** and **E.** are true, however if $a < 0$ and *n* is even,
the point (h,k) is a local maximum.

Question 11 **Answer A**

at $x = \alpha$ $f'(\alpha) = 0$
if $x < \alpha$ $f'(x) > 0$
if $x > \alpha$ $f'(x) < 0$
local maximum at $x = \alpha$

at $x = \beta$ $f'(\beta) = 0$
if $x < \beta$ $f'(x) < 0$
if $x > \beta$ $f'(x) > 0$
local minimum at $x = \beta$

Question 12 **Answer A**

For $y = 2\cos(2x)$, the average rate of change

$$\frac{y\left(\frac{\pi}{8}\right) - y(0)}{\frac{\pi}{8} - 0} = \frac{2\cos\left(\frac{\pi}{4}\right) - 2\cos(0)}{\frac{\pi}{8}} = \frac{8}{\pi}\left(2\left(\frac{\sqrt{2}}{2}\right) - 2\right) = \frac{8\left(\sqrt{2} - 2\right)}{\pi}$$

Question 13 **Answer C**

$v(t) = \dfrac{72}{(3t+2)^2}$ initial speed $v(0) = \dfrac{72}{4} = 18\,\text{m/s}$

$$s = \int_0^2 \frac{72}{(3t+2)^2}\,dt$$

$$s = \left[\frac{72}{-3(3t+2)}\right]_0^2 = -24\left(\frac{1}{8} - \frac{1}{2}\right) = 9$$

Question 14 **Answer B**

$g(x) = x^2 \quad g'(x) = 2x \quad f(x) = \cos(2x) \quad f'(x) = -2\sin(2x)$

$$\frac{d}{dx}\left[g(f(x))\right] = \frac{d}{dx}\left[g(\cos(2x))\right]$$

$$= \frac{d}{dx}\left[\cos^2(2x)\right]$$

$$= -4\sin(2x)\cos(2x) = 2(-2\sin(2x))\cos(2x) = 2f'(x)f(x)$$

Alternatively $\dfrac{d}{dx}\left[g(f(x))\right] = f'(x)g'(f(x)) = -2\sin(2x)2(\cos(2x)) = 2f(x)f'(x)$

Question 15 **Answer E**

$y = \dfrac{1}{x}$ into $y = \dfrac{4}{8-2x} - 2 \Rightarrow y + 2 = \dfrac{-2}{x-4} \Rightarrow \dfrac{y+2}{-2} = \dfrac{1}{x-4}$

$y' = \dfrac{y+2}{-2}$ and $x' = x - 4$ become

$x = x' + 4$ and $y = -2y' - 2$ in matrix form

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

Question 16 **Answer A**

Let $y_1 = x^2 + mx + m$ and $y_2 = 2mx + 2m$ $\quad y_1 = y_2$

$x^2 + mx + m = 2mx + 2m$

$x^2 - mx - m = 0$

$\Delta = b^2 - 4ac = (-m)^2 - 4(1)(-m) = m^2 + 4m = m(m+4)$

For the graphs to intersect at more than one point, we require

$\Delta > 0 \quad \Rightarrow \quad m > 0$ or $m < -4$ or $m \in (-\infty, -4) \cup (0, \infty)$

Question 17 **Answer D**

$X \stackrel{d}{=} \text{Bi}(n = ?, p = ?)$

$\Pr(\text{more than one}) = \Pr(X > 1)$

$\Pr(X > 1) = 1 - [\Pr(X = 0) + \Pr(X = 1)]$

$\Pr(X > 1) = 1 - \left(0.75^7 + 7(0.25)(0.75)^6\right)$

Now $\Pr(X = 0) = q^n$ and $\Pr(X = 1) = npq^{n-1}$

$n = 7$, $q = 0.75$ and $p = 0.25$

Question 18 **Answer D**

$\Pr(A \cap B) = \frac{1}{5} \neq \Pr(A)\Pr(B) = \frac{8}{15} \text{ x } \frac{1}{3}$ not independent

$\Pr(A \cap B) \neq 0$ not mutually exclusive

$\Pr(A' \cap B') = \frac{1}{3}$

$\Pr(A' \cup B') = \frac{7}{15} + \frac{2}{3} - \frac{1}{3} = \frac{4}{5}$

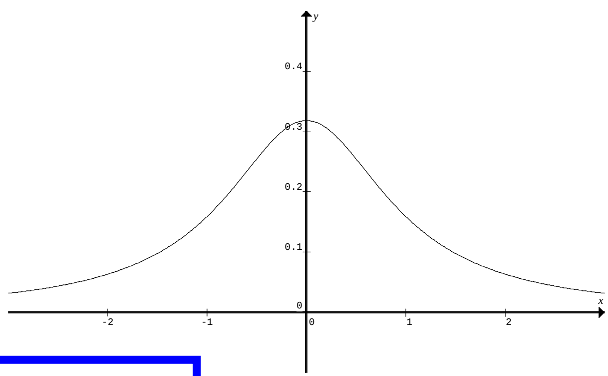
$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$\Pr(A \cup B) = \frac{8}{15} + \frac{1}{3} - \frac{1}{5} = \frac{2}{3}$

	A	A'	
B	$\frac{1}{5}$	$\frac{2}{15}$	$\frac{1}{3}$
B'	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$
	$\frac{8}{15}$	$\frac{7}{15}$	

Question 19 **Answer A**

$E(X) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx$, this is undefined and does not exist, the mode is $x = 0$



Question 20 **Answer B**

Let $A_1 = \int_{-1}^{0} f(x)\,dx < 0 \quad A_2 = \int_{0}^{3} f(x)\,dx \quad A_3 = \int_{3}^{5} f(x)\,dx < 0$

The required area is $A = -A_1 + A_2 - A_3$

$$A = -\int_{-1}^{0} f(x)\,dx + \int_{0}^{3} f(x)\,dx - \int_{3}^{5} f(x)\,dx$$

$$A = \int_{0}^{-1} f(x)\,dx + \int_{0}^{3} f(x)\,dx + \int_{5}^{3} f(x)\,dx$$

Question 21 **Answer B**

$X \stackrel{d}{=} N(\mu, \sigma^2)$, the normal curve is $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ for $x \in R$

So that $f(x) = \dfrac{1}{\sqrt{10\pi}} e^{-\frac{(x-5)^2}{10}}$ for $x \in R \quad \Rightarrow \quad \sqrt{10\pi} = \sigma\sqrt{2\pi}$

$\mu = 5$ and $\sigma = \sqrt{5}$

Question 22 **Answer C**

X	0	1	2
$\Pr(X = x)$	$4c$	$3c$	$2c$

$$\sum \Pr(X = x) = 4c + 3c + 2c = 9c = 1 \quad \Rightarrow \quad c = \frac{1}{9}$$

X	0	1	2
$\Pr(X = x)$	$\frac{4}{9}$	$\frac{1}{3}$	$\frac{2}{9}$

Since $\Pr(X = 0) > \Pr(X = 1)$ and $\Pr(X = 0) > \Pr(X = 2)$ the mode is zero

Since $\Pr(X = 0) < \frac{1}{2}$ but $\Pr(X = 0) + \Pr(X = 1) > \frac{1}{2}$ the median is 1

END OF SECTION 1 SUGGESTED ANSWERS

SECTION 2

Question 1

a.i $f(x) = ax^3 - 6ax + 12$ for $a > 0$

$f'(x) = 3ax^2 - 6a$

for stationary points $f'(x) = 0$

$f'(x) = 3ax^2 - 6a = 0$

$3a(x^2 - 2) = 0$ A1

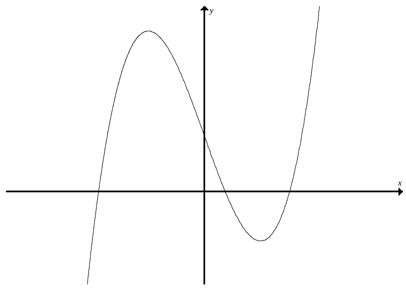
$x^2 = 2$

$x = \pm\sqrt{2}$

ii. $f(\sqrt{2}) = a(\sqrt{2})^3 - 6a\sqrt{2} + 12 = 12 - 4\sqrt{2}$

$f(-\sqrt{2}) = a(-\sqrt{2})^3 + 6a\sqrt{2} + 12 = 12 + 4\sqrt{2}$

since $a > 0$



$(-\sqrt{2}\ , 12 + 4a\sqrt{2})$ is a local maximum A1

$(\sqrt{2}\ , 12 - 4a\sqrt{2})$ is a local minimum. A1

if there are three *x*-intercepts

$12 - 4a\sqrt{2} < 0 \quad \Rightarrow \quad a\sqrt{2} > 3 \quad \Rightarrow a > \dfrac{3}{\sqrt{2}}$

$a > \dfrac{3\sqrt{2}}{2}$ A1

iii. $(\sqrt{2}\ , 12 - 4a\sqrt{2})$ is a local minimum, if there is only one *x*-intercept

$12 - 4a\sqrt{2} > 0 \quad \Rightarrow \quad a\sqrt{2} < 3 \quad$ since $a > 0$

$0 < a < \dfrac{3\sqrt{2}}{2}$ shown M1

iv. $f(-1) = -a + 6a + 12 = 12 + 5a > 0$ since $0 < a < \dfrac{3\sqrt{2}}{2}$ M1

since $f(-1) > 0$ and $f(0) = 12$ and $f(1) = 12 - 5a > 0$ and

$f'(0) = -6a < 0$ and $f'(x) < 0$ for $-1 \le x \le 1$ since $[-1,1] \subset [-\sqrt{2}, \sqrt{2}]$,

there is no zero in $[-1, 1]$ A1

b.i. $g(x) = 12\tan(x) - \dfrac{5}{\cos(x)} + \cos(x)$

$g'(x) = \dfrac{12}{\cos^2(x)} - \dfrac{5\sin(x)}{\cos^2(x)} - \sin(x)$

$g'(x) = \dfrac{12 - 5\sin(x) - \sin(x)\cos^2(x)}{\cos^2(x)}$ M1

$g'(x) = \dfrac{12 - 5\sin(x) - \sin(x)\left(1 - \sin^2(x)\right)}{\cos^2(x)}$

$g'(x) = \dfrac{\sin^3(x) - 6\sin(x) + 12}{\cos^2(x)}$ A1

$g'(x) = \dfrac{f\left(\sin(x)\right)}{\cos^2(x)}$ with $a = 1$

ii. since $a = 1$ is inside $0 < a < \dfrac{3\sqrt{2}}{2}$ and $f(x) \neq 0$ for $x \in [-1,1]$

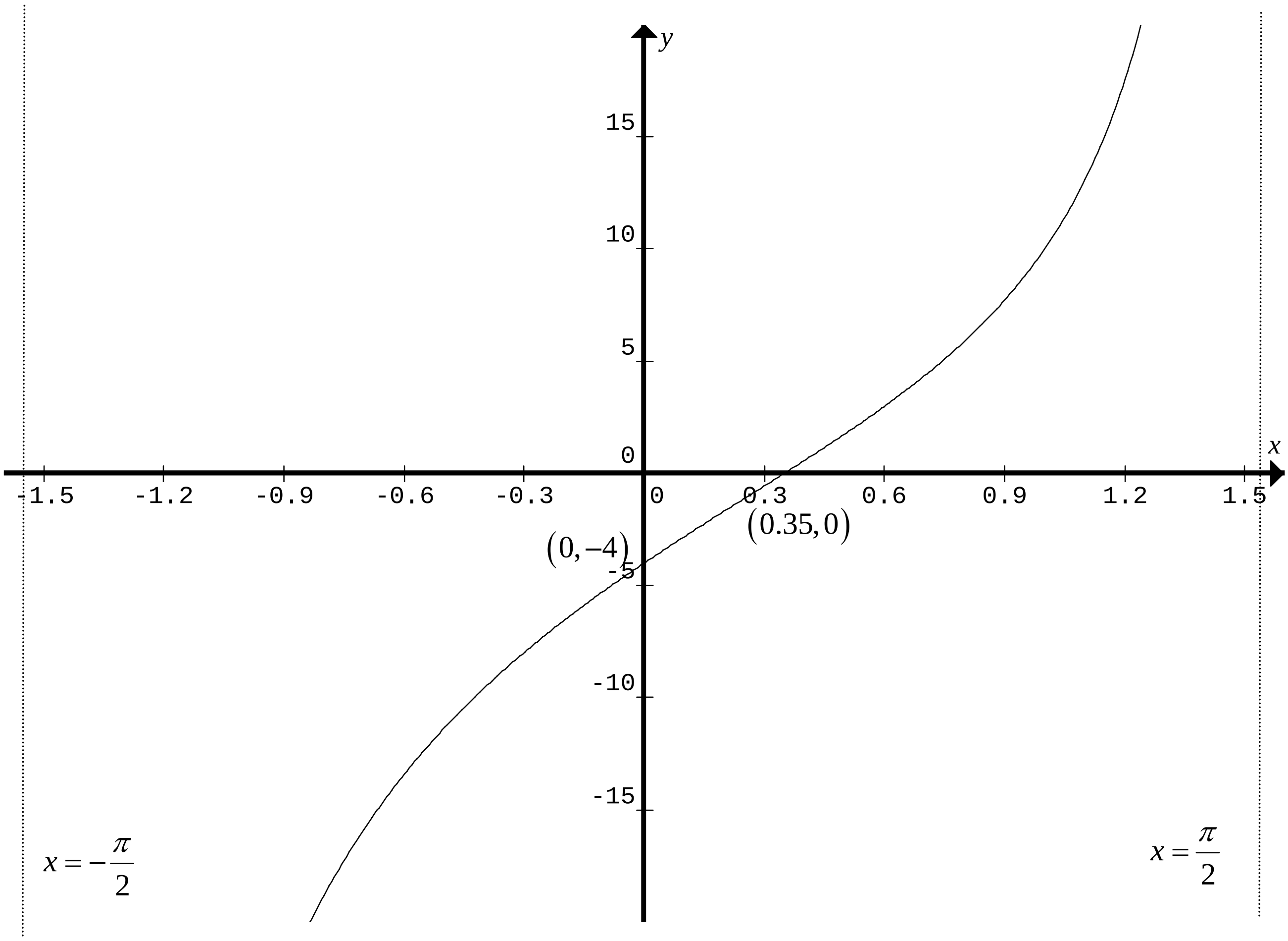
let $u = \sin(x)$ since $u \in [-1,1] \Rightarrow x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ from **a.iv**

so that $g'(x) \neq 0$ A1

so $g(x)$ has no stationary points it is a one-one function and hence A1
has an inverse which is a function.

iii. y-intercept $(0,-4)$, x-intercept $(0.35,0)$, no turning points, G1

$x=-\frac{\pi}{2}\ ,\ \frac{\pi}{2}$ vertical asymptotes, graph only for $x\in\left(-\frac{\pi}{2}\ ,\ \frac{\pi}{2}\right)$ A1



Question 2

a.i. $A(0, 18.5)$ $B(150, 15.5)$ $D(600, 18.5)$

the sine curve *ABCD*, has an amplitude of 3, and passes through *A* and *B*.

$x = 0 \quad y = 18.5 \quad \Rightarrow \quad a = 18.5$ and $b = -3$ A1

the period $T = \dfrac{2\pi}{n} = 600 \quad \Rightarrow \quad n = \dfrac{\pi}{300}$ A1

at the smooth join $\dfrac{dy}{dx} = nb\cos(nx) = -\dfrac{\pi}{100}\cos\left(\dfrac{\pi x}{300}\right)$ at $x = 600$ M1

$\left.\dfrac{dy}{dx}\right|_{x=600} = \dfrac{-\pi}{100}\cos(2\pi) = \dfrac{-\pi}{100} = m$ the gradient of the line A1

at $D \quad x = 600 \quad y = 18.5 \quad y = mx + c$

$18.5 = \dfrac{-\pi}{100}(600) + c \quad \Rightarrow c = 18.5 + 6\pi$ A1

ii. when $x = 1000$ $y = \left(-\dfrac{\pi}{100}\right)1000 + 6\pi + 18.5 = 18.5 - 4\pi$

$D(600, 18.5)$ $E(1000, 18.5 - 4\pi)$

$d(DE) = \sqrt{(600 - 1000)^2 + (18.5 - (18.5 - 4\pi))^2}$

$d(DE) = \sqrt{(-400)^2 + (4\pi)^2}$

$d(DE) = 4\sqrt{\pi^2 + 10000}$ A1

b.i. $y = 18.5 - 3\sin\left(\dfrac{\pi x}{300}\right) \quad \Rightarrow \quad \dfrac{dy}{dx} = -\dfrac{\pi}{100}\cos\left(\dfrac{\pi x}{300}\right)$

$$s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$s = \int_0^{600} \sqrt{1 + \left(\frac{\pi}{100}\cos\left(\frac{\pi x}{300}\right)\right)^2}\, dx + \int_{600}^{1000} \sqrt{1 + \frac{\pi^2}{10000}}\, dx$$ A2

ii. $s = 600.148 + d(DE) = 600.148 + 400.197$

$s = 1000.35$ bike track length A1

c.i. translate $f(x)$ 1.5 metres up parallel to the *y*-axis

$$f_n(x)=\begin{cases} 20-3\sin\left(\dfrac{\pi x}{300}\right) & \text{for } 0\le x\le 600 \\ -\dfrac{\pi x}{100}+20+6\pi & \text{for } 600\le x\le 1000 \end{cases}$$ A1

translate $f(x)$ 1.5 metres down parallel to the *y*-axis

$$f_s(x)=\begin{cases} 17-3\sin\left(\dfrac{\pi x}{300}\right) & \text{for } 0\le x\le 600 \\ -\dfrac{\pi x}{100}+17+6\pi & \text{for } 600\le x\le 1000 \end{cases}$$ A1

ii. $A=\int_0^{1000}\left(f_n(x)-f_s(x)\right)dx=\int_0^{1000}3\,dx$

area of asphalt is $3,000\ \text{m}^2$ A1

iii. $A=\int_0^{600}\left(17-3\sin\left(\frac{\pi x}{300}\right)\right)dx+\int_{600}^{1000}\left(-\frac{\pi x}{100}+17+6\pi\right)dx$ A1

$$A=\left[17x+\frac{900}{\pi}\cos\left(\frac{\pi x}{300}\right)\right]_0^{600}+\left[-\frac{\pi x^2}{200}+(17+6\pi)x\right]_{600}^{1000}$$

$$A=\left(17\text{x}600+\frac{900}{\pi}\cos(2\pi)\right)-\left(17\text{x}0+\frac{900}{\pi}\cos(0)\right)$$

$$+\left(-\frac{\pi(1000)^2}{200}+1000(17+6\pi)\right)-\left(-\frac{\pi(600)^2}{200}+600(17+6\pi)\right)$$

$A=10200+6800-800\pi$

$A=17000-800\pi\ \ \text{m}^2$ A1

Alternatively area of a rectangle + area of a trapezium

$=17\text{x}600+\frac{1}{2}(400)(17+17-4\pi)=17000-800\pi$

iv. $T\ (450,\ 23)$ A1

Question 3

a. X is the time of the quarter in minutes

$X \overset{d}{=} N\left(\mu = 27, \sigma^2 = 2^2\right)$

i. $\Pr\left(X > 30 \mid X > 25\right) = \dfrac{\Pr\left(X > 30\right)}{\Pr\left(X > 25\right)} = \dfrac{0.0668}{0.8413}$

$= 0.0794$ A1

ii. $\Pr\left(X > T\right) = 0.75$

$\dfrac{T - 27}{2} = -0.6745 \quad \Rightarrow \quad T = 25.65$ M1

longer than 25 minutes 39 seconds A1

iii. $Y \overset{d}{=} Bi\left(n = 4, p = 0.0668\right)$

$\Pr\left(Y \geq 1\right) = 1 - \Pr\left(Y = 0\right)$

$\Pr\left(Y \geq 1\right) = 1 - \left(1 - 0.0668\right)^4$ M1

$\Pr\left(Y \geq 1\right) = 0.2416$ A1

b.i. plays for the next three weeks

$\left(0.75\right)^3 = 0.4219$ A1

ii. $P \to P = 0.75$, $P \to N = 0.25$, $N \to P = 0.4$, $N \to N = 0.6$

$\Pr\left(2 \text{ matches}\right) = NPP + PNP + PPN$ M1

$= 0.25 \times 0.4 \times 0.75 + 0.75 \times 0.25 \times 0.4 + 0.75 \times 0.75 \times 0.25$

$= 0.2906$ A1

iii. $\dfrac{0.4}{0.4 + 0.25} = 0.6154$

or alternatively $\begin{array}{c} \\ N \\ P \end{array} \begin{array}{c} \begin{array}{cc} N & P \end{array} \\ \begin{bmatrix} 0.6 & 0.25 \\ 0.4 & 0.75 \end{bmatrix} \end{array}$

$\begin{bmatrix} 0.6 & 0.25 \\ 0.4 & 0.75 \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \end{bmatrix} \approx \dfrac{1}{13} \begin{bmatrix} 5 \\ 8 \end{bmatrix}$ for $n \geq 9$ $\quad \dfrac{8}{13} = 0.6154$ M1

in the long run, the percentage of games played 61.54% A1

c.i. the function is continuous at $s = 4 \Rightarrow f(4) = k\sqrt{4} = 2k = a\cos(0)$

$\Rightarrow \quad a = 2k$ A1

the total area under the curve is one.

$$k\int_0^4 \sqrt{s}\, ds + a\int_4^{10} \cos\left(\frac{\pi(s-4)}{12}\right) ds = 1$$ M1

$$k\left[\frac{2}{3}s^{\frac{3}{2}}\right]_0^4 + \frac{12a}{\pi}\left[\sin\left(\frac{\pi(s-4)}{12}\right)\right]_4^{10} = 1$$

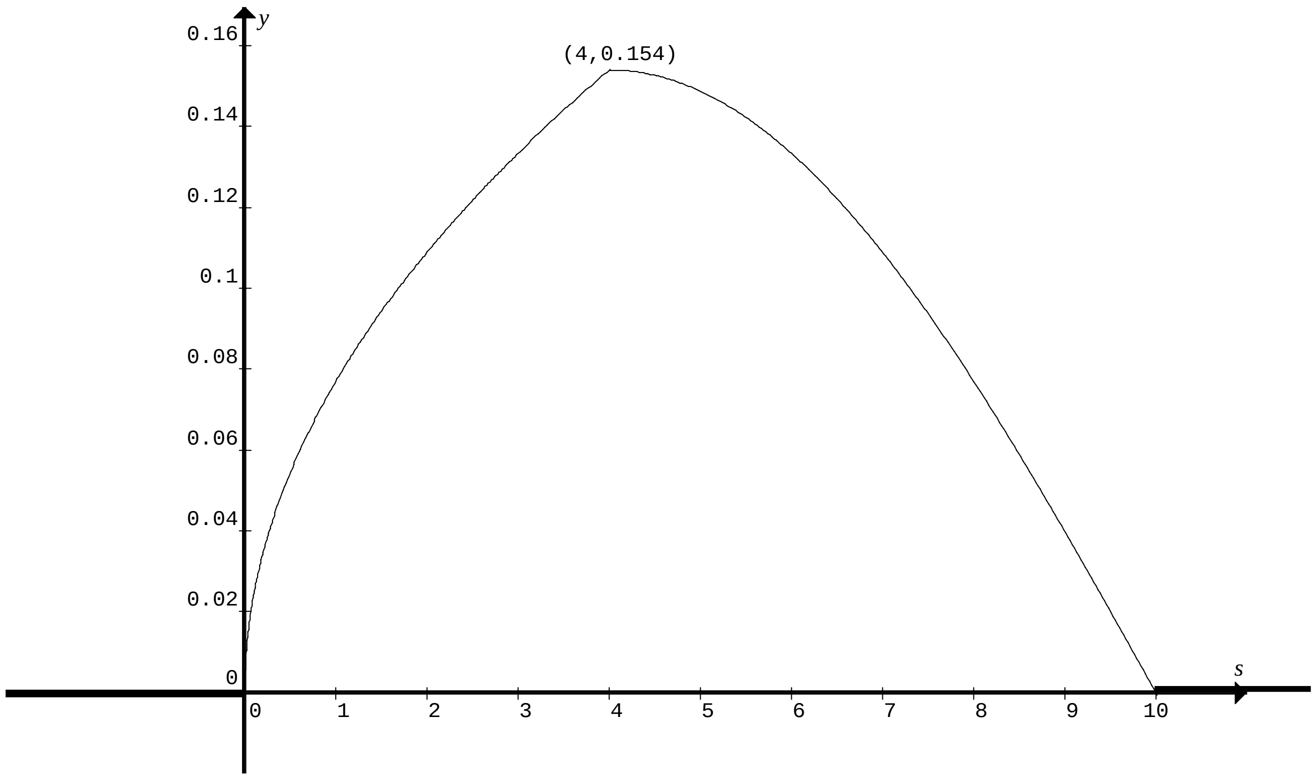
$\Rightarrow \quad \dfrac{16k}{3} + \dfrac{12a}{\pi} = 1$ A1

ii. $k = \dfrac{a}{2}$, $\dfrac{16k}{3} + \dfrac{12a}{\pi} = 1$ becomes $\dfrac{8a}{3} + \dfrac{12a}{\pi} = 1$

$\dfrac{8\pi a + 36a}{3\pi} = 1 \quad \Rightarrow \quad a = \dfrac{3\pi}{4(2\pi + 9)}$ A1

iii. must be continuous at $(4,\ 0.154)$ the maximum A1

and must show zero for $s \geq 10$ and $s \leq 0$ G1



iv. $\Pr(S<6)=1-\Pr(S>6)$

$$\Pr(S<6)=1-\frac{3\pi}{4(2\pi+9)}\int_6^{10}\cos\left(\frac{\pi(s-4)}{12}\right)ds$$ M1

$$\Pr(S<6)=1-\frac{3\pi}{4(2\pi+9)}\left[\frac{12}{\pi}\sin\left(\frac{\pi(s-4)}{12}\right)\right]_6^{10}$$

$$\Pr(S<6)=1-\frac{9}{2\pi+9}\left[\sin\left(\frac{6\pi}{12}\right)-\sin\left(\frac{2\pi}{12}\right)\right]$$

$$\Pr(S<6)=1-\frac{9}{2\pi+9}\left[1-\frac{1}{2}\right]$$

$$\Pr(S<6)=1-\frac{9}{2(2\pi+9)}=\frac{4\pi+9}{2(2\pi+9)}$$ A1

d.i. $P\overset{d}{=}\text{Bi}\left(n=8,\,p=\frac{2}{3}\right)$

$\Pr(P>4)=0.7414$ A1

ii. $T\overset{d}{=}\text{Bi}(n=22,\,p=0.74135)$

$\Pr(T\geq 16)=0.6651$ A1

Question 4

a.i. $y = f(x) = \frac{1}{x} \quad P\left(a, \frac{1}{a}\right)$

$\frac{dy}{dx} = f'(x) = -\frac{1}{x^2} \quad f'(a) = -\frac{1}{a^2}$ A1

equation of the tangent is

$y - \frac{1}{a} = -\frac{1}{a^2}(x-a) = -\frac{x}{a^2} + \frac{1}{a}$

$y = -\frac{x}{a^2} + \frac{2}{a}$ A1

ii. crosses the *y*-axis when $x = 0 \Rightarrow y_S = \frac{2}{a}$

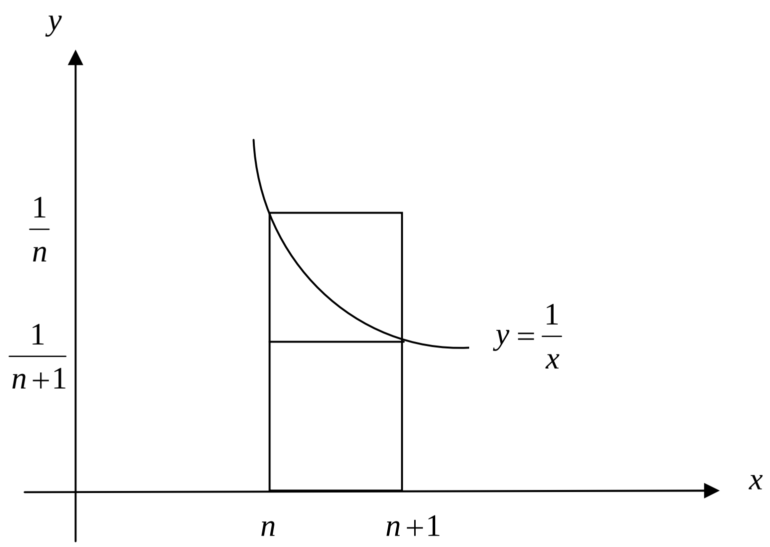
$S\left(0, \frac{2}{a}\right)$ A1

crosses the *x*-axis when $y = 0 \Rightarrow -\frac{x}{a^2} + \frac{2}{a} \Rightarrow x_Q = 2a$

$Q(2a, 0)$ A1

iii. area of the triangle *OQS* $\frac{1}{2}(2a)\left(\frac{2}{a}\right) = 2$ independent of *a* A1

b.i.



consider a rectangle, with width one unit, from x values $x = n$ to $n+1$ when

$x = n,\ y = \dfrac{1}{n}$ when $x = n+1,\ y = \dfrac{1}{n+1}$ M1

the area of the lower rectangle < true area < the area of the upper rectangle,

so that $\dfrac{1}{n+1} < \displaystyle\int_n^{n+1} \frac{1}{x}\,dx < \frac{1}{n}$ A1

ii. $\displaystyle\int_n^{n+1} \frac{1}{x}\,dx = \left[\log_e(x)\right]_n^{n+1}$

$= \log_e(n+1) - \log_e(n) = \log_e\left(\dfrac{n+1}{n}\right)$

$= \log_e\left(1+\dfrac{1}{n}\right)$ A1

from **b.i**

$\displaystyle\int_n^{n+1} \frac{1}{x}\,dx = \log_e\left(1+\frac{1}{n}\right) < \frac{1}{n}$

$\Rightarrow n\log_e\left(1+\dfrac{1}{n}\right) < 1$

$\Rightarrow \quad \log_e\left(1+\dfrac{1}{n}\right)^n < 1$

$\left(1+\dfrac{1}{n}\right)^n < e$ A1

also from **b.i**

$\dfrac{1}{n+1} < \log_e\left(1+\dfrac{1}{n}\right) = \displaystyle\int_n^{n+1} \frac{1}{x}\,dx$

$\Rightarrow 1 < (n+1)\log_e\left(1+\dfrac{1}{n}\right)$

$\Rightarrow \quad 1 < \log_e\left(1+\dfrac{1}{n}\right)^{n+1}$

$e < \left(1+\dfrac{1}{n}\right)^{n+1}$ A1

so that $\left(1+\dfrac{1}{n}\right)^n < e < \left(1+\dfrac{1}{n}\right)^{n+1}$ shown

END OF SECTION 2 SUGGESTED ANSWERS