## 2010 VCAA Math. Methods (CAS) Exam 2 Solutions



### SECTION 1

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|
| D | E | A | D | B | C | D | E | A | C | C |

| 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 |
|---|---|---|---|---|---|---|---|---|---|---|
| B | B | A | C | D | B | D | D | D | C | E |

Q1 $Period = \frac{\pi}{\frac{1}{3}} = 3\pi$ D

Q2 $f(x) = x^3 + 2x$, $[1,5]$

When $x = 1$, $y = 3$; when $x = 5$, $y = 135$

$Av.rate = \frac{135-3}{5-1} = \frac{132}{4} = 33$ E

Q3 For $f(x) = |x^2 - 9|$, the range is $[0, \infty)$.

For $f(x) = |x^2 - 9| + 3$, the range is $[3, \infty)$. A

Q4 $f(x) = \frac{1}{2}e^{3x}$, $g(x) = \log_e(2x) + 3$

$g(f(x)) = \log_e(2f(x)) + 3 = \log_e(e^{3x}) + 3 = 3x + 3 = 3(x+1)$ D

Q5

$1x + 0y + 0z = 5$

$0x + 1y + 1z = 10$

$0x - 1y + 1z = 6$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 6 \end{bmatrix}$ B

Q6 $g'(x) = x^2 - 2x$, $(1,0)$

$g(x) = \frac{x^3}{3} - x^2 + c$

$g(1) = \frac{1^3}{3} - 1^2 + c = 0$, .: $c = \frac{2}{3}$

.: $g(x) = \frac{x^3}{3} - x^2 + \frac{2}{3}$ C

Q7 $(m-1)x + 5y = 7$ and $3x + (m-3)y = 0.7m$

Change to standard form:

$y = -\frac{m-1}{5}x + \frac{7}{5}$ and $y = -\frac{3}{m-3}x + \frac{0.7m}{m-3}$

Equate coefficients: $-\frac{m-1}{5} = -\frac{3}{m-3}$ and $\frac{7}{5} = \frac{0.7m}{m-3}$

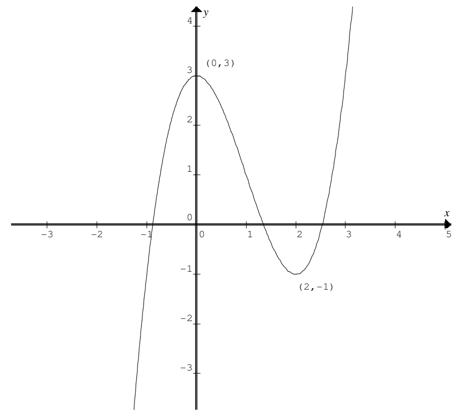
.: $m = 6$ D

Q8 $f(x) = 3\log_e(2x)$

$f(5x) = 3\log_e(2(5x)) = \log_e((10x)^3) = \log_e(y)$

.: $y = (10x)^3 = 1000x^3$ E

Q9 The graph of $f(x) = x^3 - 3x^2 + 3$:




The graph is 1 to 1 in the interval $(-\infty, 0]$, .: $a \le 0$ A

Q10 $Average = \frac{\int_0^{\pi} e^{2x}\cos(3x)dx}{\pi} \approx -26.3$ C

Q11 $\Pr(x < a) = \int_{\frac{3\pi}{4}}^{a} \cos(2x)dx = 0.25$

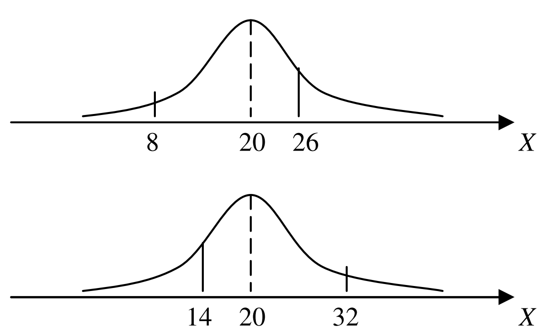
By CAS or by checking each alternative, $a \approx 2.88$ C

Q12 Binomial: $n = 15$, $p = \frac{3}{5}$

$\Pr(X < 7) = \Pr(X \le 6) = 0.0950$ B

Q13 $Z = \frac{X-\mu}{\sigma}$, .: $X = \mu + \sigma Z$

$\Pr(-2 < Z < 1) = \Pr(8 < X < 26) = \Pr(14 < X < 32)$ B



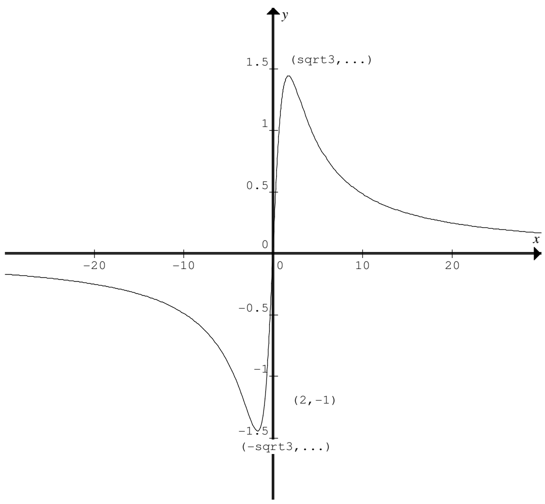

Q14 $\Pr(all.three.are.black) = \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8} = \frac{1}{6}$ A

Q15 $\sum \Pr(X=x)=a+b+0.4=1$ and

$\mu = 0\times a + 1\times b + 2\times 0.4 = 1$

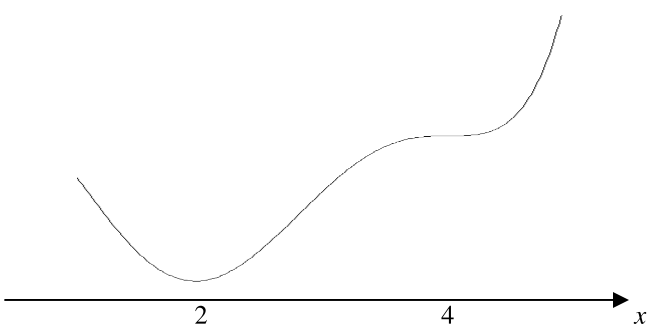
Solve the simultaneous equations: $a=0.4$ and $b=0.2$ C

Q16 The graph of $f(x)=\dfrac{5x}{x^2+3}$:

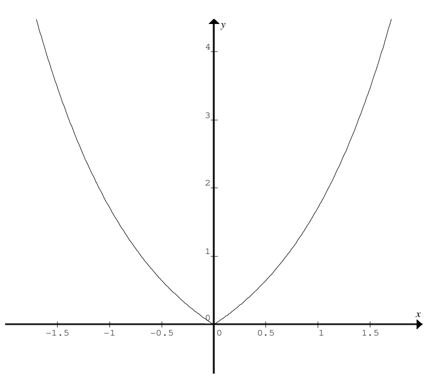



$f'(x)$ is negative for $x<-\sqrt{3}$ OR $x>\sqrt{3}$. D

Q17 Sketch according to the given conditions: B




Q18 The graph of $f(x)=e^{|x|}-1$:




$f(x)$ is not differentiable at $x=0$. D

Q19 Gradient function $f'(x)$ has three $x$-intercepts. They are at $x<0$, $x=0$ and $x>0$. .: function $f(x)$ has stationary points at those locations. On the right of the third $x$-intercept, $f'(x)>0$ .: $f(x)$ has a positive slope. D

Q20 $2\int_0^{5a}\left(f\left(\frac{x}{5}\right)+3\right)dx = 2\int_0^{5a} f\left(\frac{x}{5}\right)dx + 2\int_0^{5a} 3dx$

$= 2\int_0^{5a} 5f(u)\frac{du}{dx}dx + 2\int_0^{5a} 3dx$ (SM)

Let $u=\frac{x}{5}$, $5\frac{du}{dx}=1$

$=10\int_0^{a} f(u)du + 2[3x]_0^{5a}$

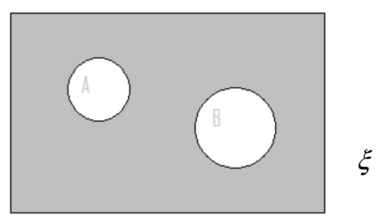
$=10a+30a=40a$ D

Alternatively:

$f\left(\frac{x}{5}\right)$ is the transformation (horizontal dilation by a factor of 5) of $f(x)$, .: the area under the graph of $f\left(\frac{x}{5}\right)$ is 5 times that of $f(x)$, i.e. $\int_0^{5a} f\left(\frac{x}{5}\right)dx = 5\int_0^{a} f(x)dx = 5a$ (FM)

$2\int_0^{5a}\left(f\left(\frac{x}{5}\right)+3\right)dx = 2\int_0^{5a} f\left(\frac{x}{5}\right)dx + 2\int_0^{5a} 3dx = 40a$

Q21 The Venn diagram shows mutually exclusive $A$ and $B$.




The shaded region represents $A'\cap B'$.

$\Pr(A'\cap B') = \Pr(\xi) - (\Pr(A)+\Pr(B)) = 1-(p+q)$ C

Q22 For $a>1$ and $b>1$, the range of the interval $[3, ab+2]$ is greater than the sum of the range of the interval $[3, a+2]$ and the range of the interval $[3, b+2]$.

Proof: $a>1$ and $b>1$, $(a-1)b>(a-1)$, $ab-b>a-1$, $ab>a+b-1$, $ab-1>a+b-2$, $ab-1>a-1+b-1$, .: $(ab+2)-3>(a+2)-3+(b+2)-3$.

.: $f(x)$ must be a decreasing function for $\int_3^{ab+2} f(x)dx = \int_3^{a+2} f(x)dx + \int_3^{b+2} f(x)dx$ to hold.

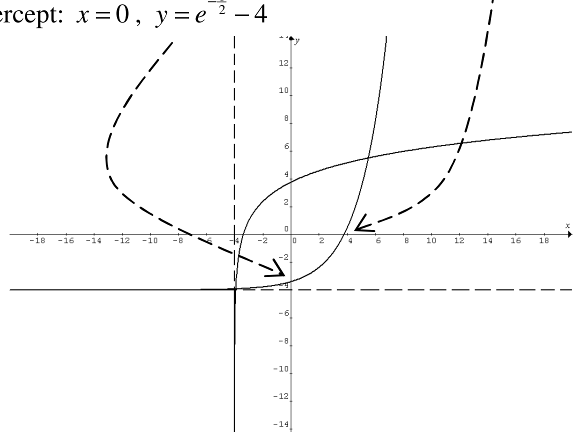
Only $f(x)=\dfrac{1}{x-2}$ in the given choices is a decreasing function.

E

## SECTION 2

Q1ai The domain of $g^{-1}$ is the range of $g$, i.e. $R$.
The equation of the inverse of $g$ is $x = 2\log_e(y+4)+1$.
Express $y$ in terms of $x$: $y = e^{\frac{x-1}{2}} - 4$. .: $g^{-1}(x) = e^{\frac{x-1}{2}} - 4$

Q1aii $x$-intercept, $y = 0$, $x = 2\log_e 4 + 1 = 4\log_e 2 + 1$
$y$-intercept: $x = 0$, $y = e^{-\frac{1}{2}} - 4$



Q1aiii Let $x = 2\log_e(x+4)+1$. By CAS calculator, $x \approx -3.914$ or $x \approx 5.503$.

Q1aiv $Area = \int_{-3.914}^{5.503} \left( (2\log_e(x+4)+1) - \left( e^{\frac{x-1}{2}} - 4 \right) \right) dx \approx 52.63$ unit squares, by CAS calculator.

Q1bi $f(x) = k\log_e(x+a)+c$, $a = 1$

Q1bii $y = 1$ when $x = 0$, .: $1 = k\log_e(1)+c$, .: $c = 1$

Q1biii From the results of Q1bi and Q1bii, and given $P(p,10)$,
$10 = k\log_e(p+1)+1$, .: $k = \frac{9}{\log_e(p+1)}$.

Q1biv $f(x) = k\log_e(x+1)+1$, $f'(x) = \frac{k}{x+1}$.
At $x = p$, $f'(p) = \frac{k}{p+1} = \frac{9}{(p+1)\log_e(p+1)}$

Q1bv Equation of the tangent at $P(p,10)$:
$y = \frac{9}{(p+1)\log_e(p+1)}(x-p)+10$
At $(-1,0)$, $0 = \frac{9}{(p+1)\log_e(p+1)}(-1-p)+10$,
$0 = \frac{-9}{\log_e(p+1)} + 10$, $\log_e(p+1) = \frac{9}{10}$, $p = e^{0.9} - 1$.

Q2a $\Pr(third.is.R) = \Pr(SSR) + \Pr(SRR)$
$= 1(1-p)p + 1(1-p)(1-(p-0.2)) = 0.12$ when $p = 0.9$

Q2b $\Pr(SSSS) = 1 \times p^3 = 0.9^3 = 0.729$

Q2c
$\begin{bmatrix} 0.9 & 0.7 \\ 0.1 & 0.3 \end{bmatrix}^n \to \begin{bmatrix} 0.875 & 0.875 \\ 0.125 & 0.125 \end{bmatrix}$ as $n \to \infty$
.: steady state $\Pr(S) = 0.875$

Q2di
The transition matrix is $\begin{bmatrix} p & p-0.2 \\ 1-p & 1-(p-0.2) \end{bmatrix} = \begin{bmatrix} p & p-0.2 \\ 1-p & 1.2-p \end{bmatrix}$.
State matrix for the first statue is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
State matrix for the second statue is
$\begin{bmatrix} p & p-0.2 \\ 1-p & 1.2-p \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} p \\ 1-p \end{bmatrix}$.
State matrix for the third statue is
$\begin{bmatrix} p & p-0.2 \\ 1-p & 1.2-p \end{bmatrix} \begin{bmatrix} p \\ 1-p \end{bmatrix} = \begin{bmatrix} 1.2p-0.2 \\ ... \end{bmatrix} = \begin{bmatrix} 0.7 \\ ... \end{bmatrix}$.
.: $1.2p - 0.2 = 0.7$, $p = 0.75$

Q2dii $\Pr(S \mid S) = p = 0.75$, $\Pr(R \mid S) = 0.25$,
$\Pr(S \mid R) = p - 0.2 = 0.55$, $\Pr(R \mid R) = 0.45$

$\Pr(noS's) = \Pr(RRR) = 1 \times 0.45 \times 0.45 = 0.2025$
$\Pr(1S) = \Pr(RSR) + \Pr(RRS)$
$= 1 \times 0.55 \times 0.25 + 1 \times 0.45 \times 0.55 = 0.385$
$\Pr(2S's) = \Pr(RSS) = 1 \times 0.55 \times 0.75 = 0.4125$

| $x$ | 0 | 1 | 2 |
|---|---|---|---|
| $\Pr(X = x)$ | 0.2025 | 0.385 | 0.4125 |

$E(x) = 0 \times 0.2025 + 1 \times 0.385 + 2 \times 0.4125 = 1.21$

Q2e Binomial: $n$ statues, $p_S = 0.2$
$\Pr(X \ge 2) \ge 0.9$, $1 - \Pr(X \le 1) \ge 0.9$
.: $\Pr(X \le 1) \le 0.1$, $binomcdf(n,0.2,1) \le 0.1$
Use CAS calculator to find $n$, , $n \ge 18$ .: minimum $n$ is 18.

Q3ai $\frac{\overline{AZ}}{10} = \cos(x)$, $\overline{AZ} = 10\cos(x)$, .: $\overline{AB} = 20\cos(x)$

Q3aii $\frac{\overline{WZ}}{10} = \sin(x)$, $\overline{WZ} = 10\sin(x)$

Q3b

*Total surface area* $S=(20\cos(x))^2+4\left(\frac{1}{2}(20\cos(x))(10\sin(x))\right)$

$=400(\cos^2(x)+\cos(x)\sin(x))$

Q3c $\overline{WY}=\sqrt{\overline{WZ}^2+\overline{ZY}^2}=\sqrt{100\sin^2(x)-100\cos^2(x)}$

$=10\sqrt{\sin^2(x)-\cos^2(x)}=10\sqrt{1-2\cos^2(x)}$

Q3d *Volume* $T=\frac{1}{3}\times(20\cos(x))^2\times 10\sqrt{1-2\cos^2(x)}$

$=\frac{4000}{3}\cos^2(x)\sqrt{1-2\cos^2(x)}=\frac{4000}{3}\sqrt{\cos^4(x)(1-2\cos^2(x))}$

$=\frac{4000}{3}\sqrt{(\cos^4(x)-2\cos^6(x))}$

Q3e $\frac{dT}{dx}=\frac{4000}{3}\times\frac{-4\cos^3(x)\sin(x)+12\cos^5(x)\sin(x)}{2\sqrt{(\cos^4(x)-2\cos^6(x))}}$

$=\frac{8000}{3}\times\frac{-\cos^3(x)\sin(x)+3\cos^5(x)\sin(x)}{\sqrt{(\cos^4(x)-2\cos^6(x))}}$

Let $\frac{dT}{dx}=0$, .: $-\cos^3(x)\sin(x)+3\cos^5(x)\sin(x)=0$

.: $\cos^3(x)\sin(x)(3\cos^2(x)-1)=0$

Since $\frac{\pi}{4}<x<\frac{\pi}{2}$, .: $3\cos^2(x)-1=0$, $\cos(x)=\frac{1}{\sqrt{3}}$,

$x=\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

.: maximum $T=\frac{4000}{3}\sqrt{\frac{1}{9}-\frac{2}{27}}=\frac{4000}{3\sqrt{27}}=\frac{4000}{9\sqrt{3}}=\frac{4000\sqrt{3}}{27}$ m$^3$

Q3f Let $\frac{4000}{3}\sqrt{(\cos^4(x)-2\cos^6(x))}=\frac{1}{2}\times\frac{4000}{3\sqrt{27}}$

.: $\sqrt{(\cos^4(x)-2\cos^6(x))}=\frac{1}{2\sqrt{27}}$

$\cos^4(x)-2\cos^6(x)=\frac{1}{4\times 27}=\frac{1}{108}$

.: $x\approx 0.81$ or 1.23 radians by CAS calculator.

Q4a $f(x)=\frac{1}{27}(2x-1)^3(6-3x)+1=-\frac{1}{9}(2x-1)^3(x-2)+1$

$f'(x)=-\frac{1}{9}\left((2x-1)^3+6(2x-1)^2(x-2)\right)=-\frac{1}{9}(2x-1)^2(8x-13)$

.: stationary points are at $x=\frac{1}{2}$ (inflection) and $x=\frac{13}{8}$ (maximum). The nature of each point is determined by sketching $f(x)$.

Q4b $f(x)=\frac{1}{27}(ax-1)^3(b-3x)+1$

$f'(x)=\frac{1}{9}\left(a(ax-1)^2(b-3x)-(ax-1)^3\right)$

$=\frac{1}{9}(ax-1)^2(a(b-3x)-(ax-1))$

$=\frac{1}{9}(ax-1)^2(ab+1-4ax)$

Stationary points are at $x=\frac{1}{a}$ and $x=\frac{ab+1}{4a}$.

Q4c $x=\frac{1}{a}$ and $x=\frac{ab+1}{4a}$ are undefined when $a=0$, i.e. no stationary points when $a=0$.

Q4d One stationary point when $\frac{1}{a}=\frac{ab+1}{4a}$, i.e. $a=\frac{3}{b}$.

Q4e The maximum number of stationary points is 3 for quartic polynomial functions. They are either local max. or min. In this case, quartic $f(x)$ is in the form of $\frac{1}{27}(ax-1)^3(b-3x)+1$, it has a stationary inflection point and a maximum (or minimum). .: the maximum number of stationary points is 2.

Q4f $f'(x)=\frac{1}{9}(ax-1)^2(ab+1-4ax)$

Stationary points are at $x=\frac{1}{a}$ and $x=\frac{ab+1}{4a}$.

Given two stationary points $(1,1)$ and $(p,p)$.

Consider the possibility: $p=\frac{1}{a}$ and $1=\frac{ab+1}{4a}$

.: $f(p)=\frac{1}{27}(ap-1)^3(b-3p)+1=1$, .: $f(p)\neq p$ and $p\neq\frac{1}{a}$.

Consider the possibility: $1=\frac{1}{a}$ and $p=\frac{ab+1}{4a}$

.: $a=1$ and $b=4p-1$

.: $f(1)=1$

.: $f(p)=\frac{1}{27}(p-1)^3(4p-1-3p)+1=\frac{1}{27}(p-1)^4+1=p$

.: $\frac{1}{27}(p-1)^4-(p-1)=0$, $(p-1)\left(\frac{1}{27}(p-1)^3-1\right)=0$

Since $p\neq 1$, i.e. $p-1\neq 0$, .: $\frac{1}{27}(p-1)^3-1=0$

Hence $(p-1)^3=27$, $p-1=3$, $p=4$.

*Please inform mathline@itute.com re conceptual, mathematical and/or typing errors*