Student Name: ______________________________

MATHEMATICAL METHODS (CAS)

Units 3 & 4 – Written examination 1

2009 Trial Examination

Reading time: 15 minutes
Writing time: 1 hour

QUESTION AND ANSWER BOOK

Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
11	11	40
		Total 40

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners and rulers
- Students are NOT permitted to bring into the examination room: notes of any kind, blank sheets of paper, white out liquid/tape or a calculator of any type.

Materials supplied

- Question and answer book of 12 pages.
- Working space is provided throughout the book.

Instructions

- Print your name in the space provided on the top of this page.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic communication devices into the examination room.

Instructions

Answer **all** questions in the spaces provided.

A decimal approximation will not be accepted if an exact answer is required to a question.

In questions where more than one mark is available, appropriate working must be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

a. If $f(x)=\dfrac{3\cos(2x)}{\sin(2x)}$ show that: $f'(x)=\dfrac{-6}{\sin^2(2x)}$.

2 marks

b. Hence, find: $\displaystyle\int\left(\frac{2}{\sin^2(2x)}+1\right)dx$.

2 marks

Question 2

a. If: $f(x) = -3\log_e(x+2) - 1$, show that the graph of $y = f(x)$ has a y intercept at $\left(0, \log_e\left(\frac{1}{8e}\right)\right)$.

2 marks

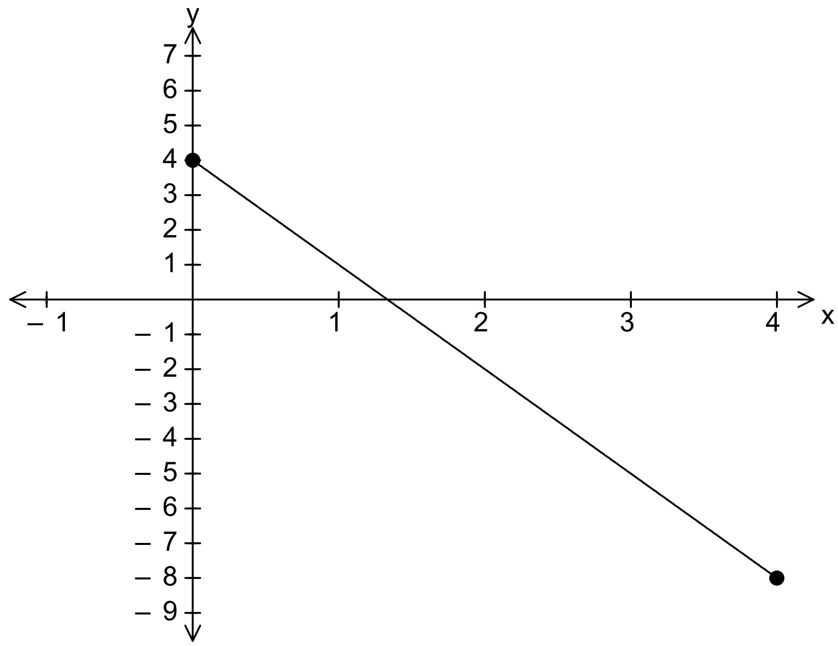
b. Find the inverse function: $f^{-1}(x)$.

2 marks

TURN OVER

Question 3

The graph of the function $g(x) = 4 - 3x, x \in [0,4]$ is shown on the axes below.



a. Sketch the graph of $h(x) = |g(x)| - 2$, for $x \in [0,4]$ on the same set of axes.

Show end points and the exact axial intercepts.

2 marks

b. State the domain for the derivative function $h'(x)$.

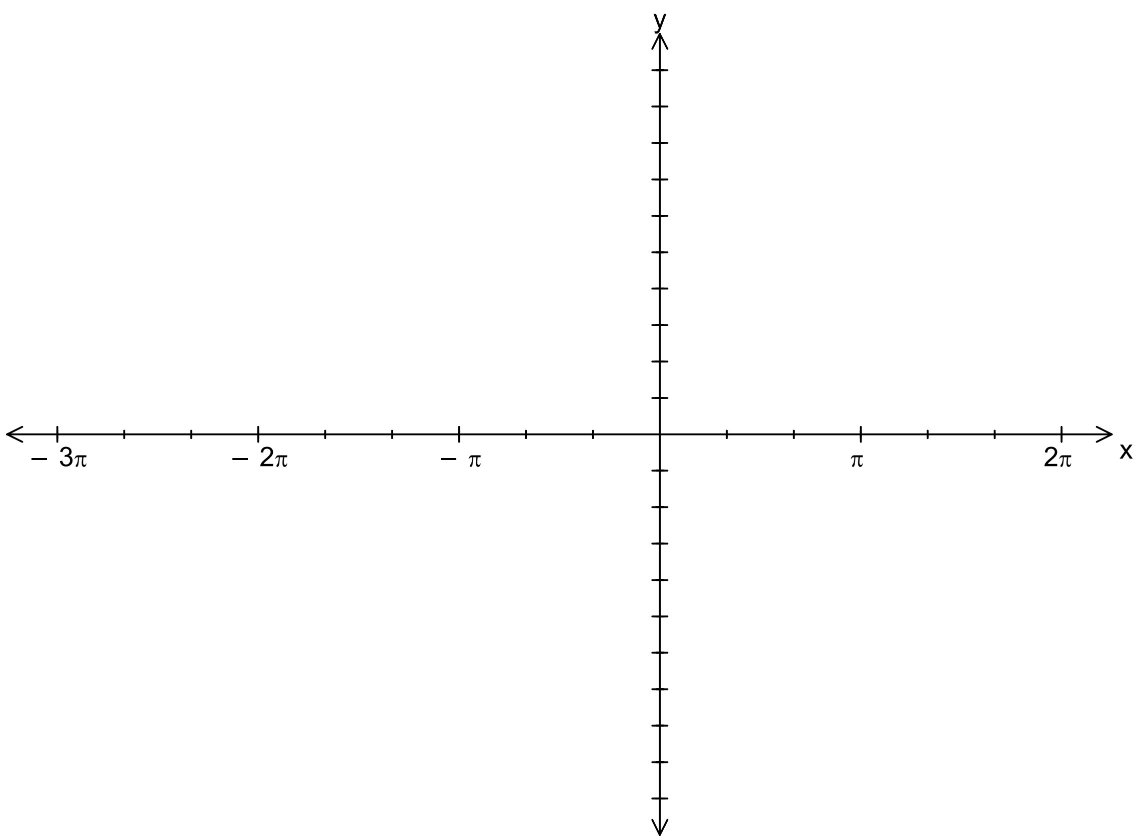
1 mark

Question 4

a. Solve the equation $\sqrt{3}\tan\left(\frac{x}{2}\right)+1=0$, for $[-3\pi, 2\pi]$.

2 marks

b. Hence sketch: $y=\sqrt{3}\tan\left(\frac{x}{2}\right)+1$, for $[-3\pi, 2\pi]$, showing all axial intercepts and asymptotes.

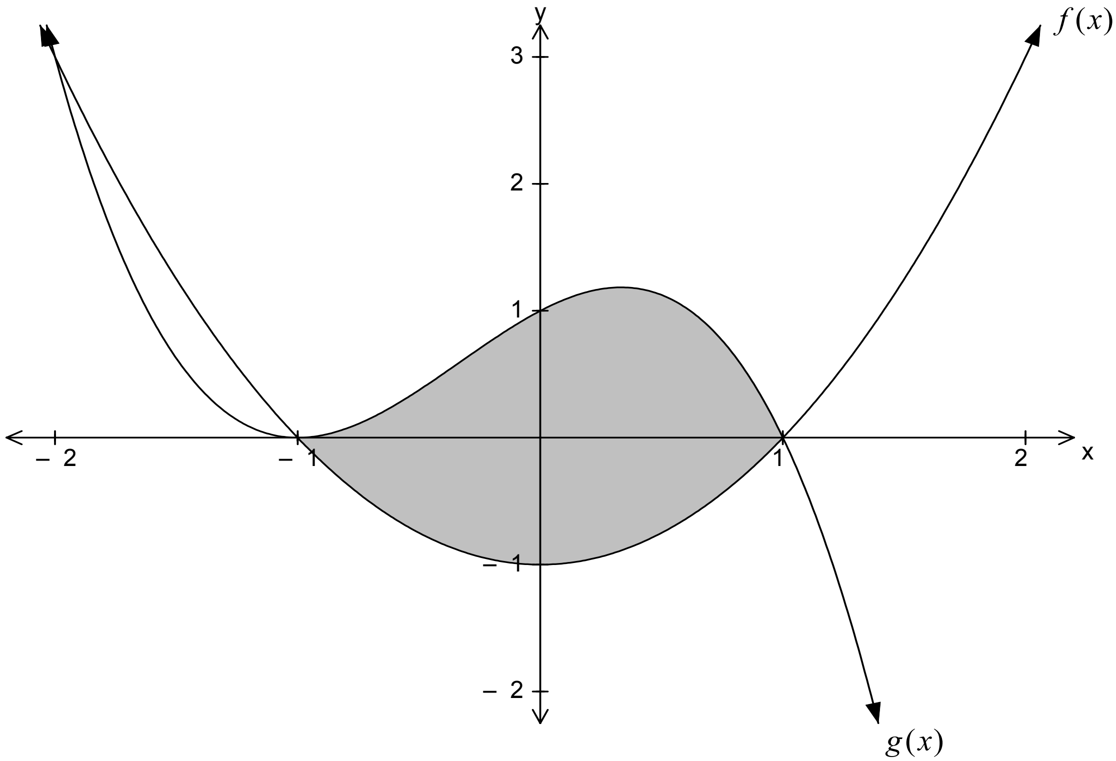


3 marks

TURN OVER

Question 5

The graphs of $f(x)=(x-1)(x+1)$ and $g(x)=(1-x)(x+1)^2$ are shown below.



Find the exact area enclosed by the curves $f(x)$ and $g(x)$ for $x \in [-1,1]$.

3 marks

Question 6

If $f(x)=\frac{1}{\sqrt{x}}$, use the approximation formula $f(x+h)\approx f(x)+hf'(x)$ to find the approximate value of $\frac{1}{\sqrt{4.1}}$. Give your answer as a **fraction**.

3 marks

Question 7

A spinner is numbered 1 to 5. The number X on the spinner is a discrete random variable, with a probability distribution given by:

x	1	2	3	4	5
Pr($X = x$)	0.35	0.25	0.1	0.15	0.15

a. What is the median of X?

1 mark

b. Find the probability of spinning 5, given that the number is greater than 2.

1 mark

TURN OVER

Question 8

Tom is a keen windsurfer and he has noticed that the conditions at Sandy Beach follow a pattern. If it is windy on one day, the probability that it will be windy on the next day is 0.8. If it is still on one day, the probability of it being windy on the next day is 0.5.

a. If it is windy on Thursday, find the probability that it will be windy on Saturday **and** Sunday, correct to three decimal places.

2 marks

b. If it is still on Friday, find the probability that it will be windy on **at least** one day over the weekend, correct to three decimal places.

2 marks

Question 9

A continuous random variable X has a probability density function defined by

$$f(x) = \begin{cases} \dfrac{x}{2}, & 1 \le x \le 2 \\ k, & 2 < x \le 4 \\ 0, & \text{elsewhere} \end{cases}$$

a. Use calculus to show that: $k = \dfrac{1}{8}$.

2 marks

b. Find $\Pr(X \le 3)$.

1 mark

TURN OVER

Question 10

The heights of boys in Year 12 are normally distributed with a mean of 175 cm and a standard deviation of 10 cm. It is known that Pr $(Z > 0.5) = 0.3$.

a. Using the information above, find the probability that a boy selected at random is greater than 170 cm.

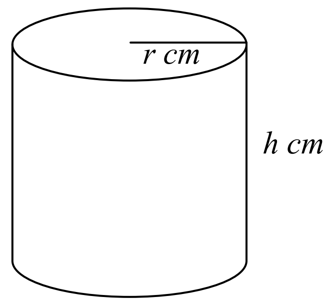
2 marks

b. Find the probability that a randomly selected boy is more than 180cm, given that his height is above the mean.

1 mark

Question 11

Joan is baking a nut loaf in a cylindrical tin that does **not** have a lid.



a. If she uses $200\pi\ cm^2$ of paper to line the base and sides of the tin, show that: $h = \frac{100}{r} - \frac{r}{2}$.

2 marks

b. Show that the volume of the tin is given by: $V = 100\pi r - \frac{\pi r^3}{2}$.

2 marks

TURN OVER

c. Use calculus to show that the maximum volume of the tin will be when $r = \frac{10\sqrt{6}}{3}\,cm$.

2 marks

END OF QUESTION AND ANSWER BOOK

MATHEMATICAL METHODS (CAS)

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SOLUTIONS

Question 1

a. $$f'(x) = \frac{-2(3\sin(2x))(\sin(2x)) - 3\cos(2x)(2\cos(2x))}{\sin^2(2x)}$$
$$= \frac{-6(\sin^2(2x) + \cos^2(2x))}{\sin^2(2x)}$$
$$= \frac{-6}{\sin^2(2x)}$$

M1 + A1
2 marks

b. $$\therefore \int \frac{-6}{\sin^2(2x)}\,dx = \frac{3\cos(2x)}{\sin(2x)} + c$$
$$-3\int \frac{2}{\sin^2(2x)}\,dx = \frac{3\cos(2x)}{\sin(2x)} + c$$
$$\int \frac{2}{\sin^2(2x)}\,dx = \frac{-\cos(2x)}{\sin(2x)} + c$$
$$\therefore \int \left(\frac{2}{\sin^2(2x)} + 1\right)dx = -\frac{\cos(2x)}{\sin(2x)} + x + c$$

M1 + A1
2 marks

Question 2

a. Let $x = 0$

$$y = -3\log_e(2) - 1$$
$$= \log_e(2^{-3}) - 1$$
$$= \log_e\left(\frac{1}{8}\right) - \log_e(e)$$
$$= \log_e\left(\frac{1}{8e}\right)$$

M1 + A1
2 marks

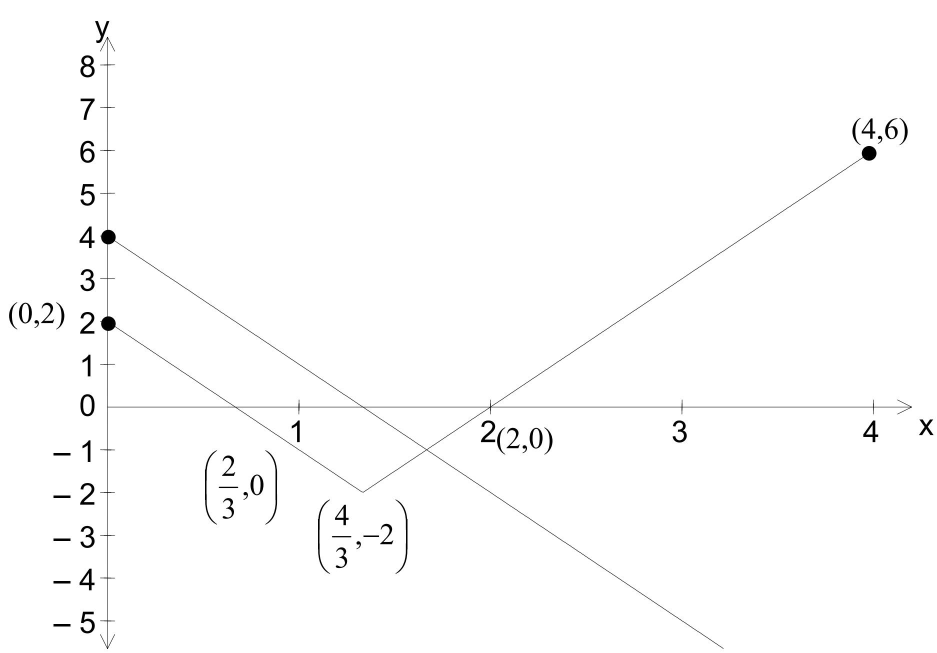
b. Let $x = -3\log_e(y+2) - 1$

$$x + 1 = -3\log_e(y+2)$$
$$y = e^{-\left(\frac{x+1}{3}\right)} - 2$$
$$\therefore f^{-1}(x) = e^{-\left(\frac{x+1}{3}\right)} - 2$$

M1 + A1
2 marks

Question 3

a.



A1 + M1
2 marks

b. $h'(x)$ has domain $(0,4) \setminus \left\{\frac{4}{3}\right\}$ or $\left(0, \frac{4}{3}\right) \cup \left(\frac{4}{3}, 4\right)$

A1
1 mark

Question 4

a.

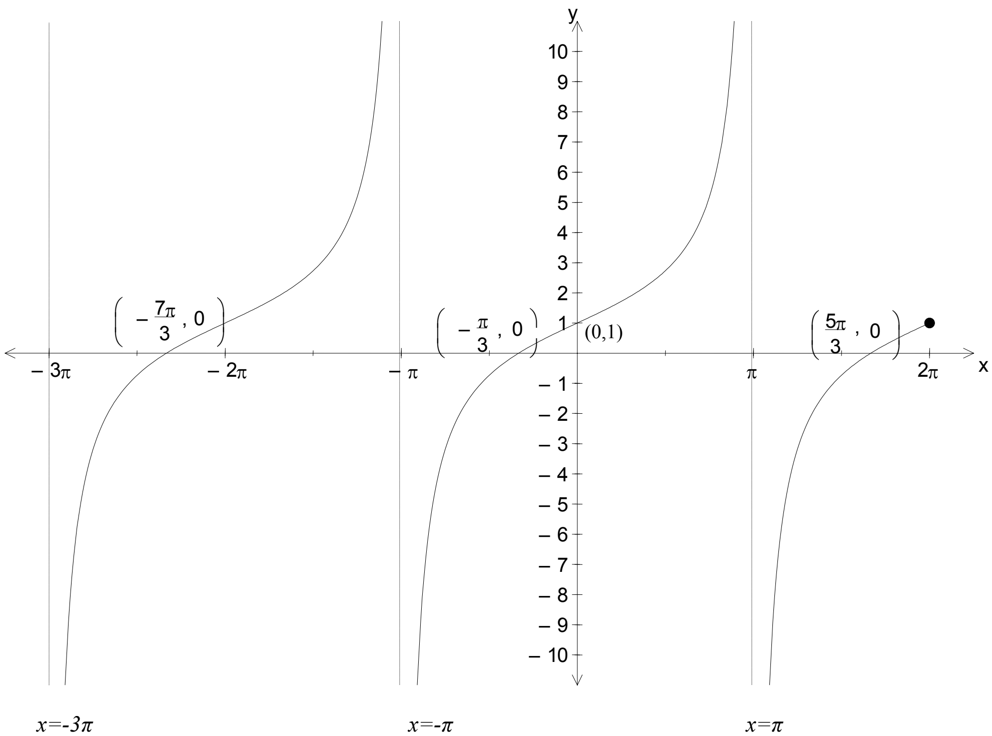
$$\tan\left(\frac{x}{2}\right) = -\frac{1}{\sqrt{3}}, x \in \left[-\frac{3\pi}{2}, \pi\right]$$

$$\frac{x}{2} = -\frac{7\pi}{6}, -\frac{\pi}{6}, \frac{5\pi}{6}$$

$$x = -\frac{7\pi}{3}, -\frac{\pi}{3}, \frac{5\pi}{3}$$

M1 + A1
2 marks

b.



A3
3 marks

Question 5

a. $\int_{-1}^{1}\left(g(x)-f(x)\right)dx=$

$$=\int_{-1}^{1}\left((-x^3-x^2-x+1)-(x^2-1)\right)dx$$

$$=\int_{-1}^{1}\left(-x^3-2x^2+x+2\right)dx$$

$$=\left[\frac{-x^4}{4}-\frac{2x^3}{3}+\frac{x^2}{2}+2x\right]_{-1}^{1}$$

$$=\frac{-1}{4}-\frac{2}{3}+\frac{1}{2}+2-\left[\frac{-1}{4}+\frac{2}{3}+\frac{1}{2}-2\right]$$

$$=4-\frac{4}{3}$$

$$=2\frac{2}{3}\text{ units}^2$$

M2 + A1
3 marks

Question 6

$$f(x)=x^{\frac{-1}{2}},\ f'(x)=-\frac{1}{2}x^{\frac{-3}{2}}$$

$$f(4.1)\approx f(4)+0.1\times f'(4)$$

$$\approx\frac{1}{\sqrt{4}}+0.1\left(-\frac{1}{2}\right)(4)^{-\frac{3}{2}}$$

$$\approx\frac{1}{2}-\frac{1}{10}\times\frac{1}{2}\times\frac{1}{8}$$

$$\approx\frac{1}{2}-\frac{1}{160}$$

$$\approx\frac{79}{160}$$

M2 + A1
3 marks

Question 7

a. Median divides the data into halves. Therefore, the median is 2

A1
1 mark

b. $\Pr(X = 5 \mid x > 2) = \dfrac{\Pr(X = 5)}{\Pr(X > 2)}$

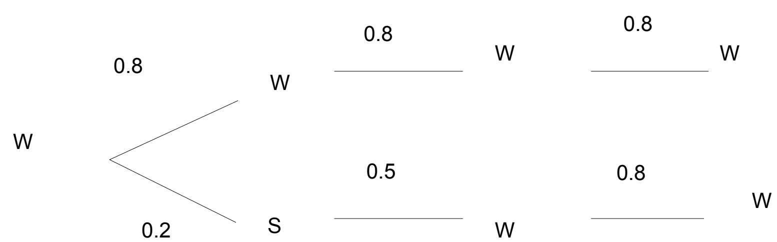
$= \dfrac{0.15}{0.4}$

$= \dfrac{3}{8}$

A1
1 mark

Question 8

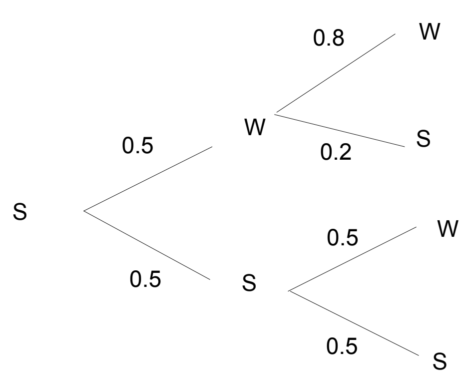
a.



$\Pr(\text{windy Sat and Sun} \setminus \text{windy Thurs}) = 0.512 + 0.080$
$= 0.592$

M1 + A1
2 marks

b.



$\Pr(\text{windy on at least Sat or Sun} \setminus \text{still on Fri}) = 0.5 \times 0.8 + 0.5 \times 0.2 + 0.5 \times 0.5$
$= 0.750$

M1 + A1
2 marks

Question 9

a. $1=\int_1^2\left(\frac{x}{2}\right)dx+\int_2^4(k)dx$

$=\left[\frac{x^2}{4}\right]_1^2+\left[kx\right]_2^4$

$=\frac{3}{4}+2k$

$\therefore k=\frac{1}{8}$

M1+A1
2 marks

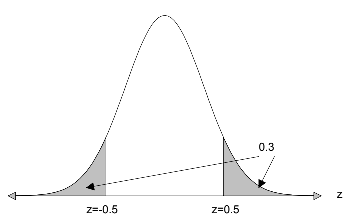
b. $\Pr(X\le 3)=\int_1^2\left(\frac{x}{2}\right)dx+\int_2^3\left(\frac{1}{8}\right)dx$

$=\frac{6+3-2}{8}=\frac{7}{8}$

A1
1 mark

Question 10

a. $0.5=\frac{x-175}{10}\ \therefore x=180$ this is 5 above the mean so $\Pr(X>170)=0.7$



M1+A1
2 marks

b. $\Pr(X>180\mid X>175)=\frac{\Pr(X>180)}{0.5}=\frac{0.3}{0.5}=\frac{3}{5}$

A1
1 mark

Question 11

a. $SA = \pi r^2 + 2\pi rh$

$200\pi = \pi(r^2 + 2rh)$

$\frac{200 - r^2}{2r} = h$

$\therefore h = \frac{100}{r} - \frac{r}{2}$

M1 + A1
2 marks

b. $V = \pi r^2 h$

$= \pi r^2 \left(\frac{200 - r^2}{2r} \right)$

$= 100\pi r - \frac{\pi r^3}{2}$

M1 + A1
2 marks

c. $\frac{dV}{dr} = 100\pi - \frac{3}{2}\pi r^2$

$0 = 100\pi - \frac{3}{2}\pi r^2$

$r^2 = \frac{200}{3} \Rightarrow r = \sqrt{\frac{200}{3}}$, rationalise

$r = \frac{10\sqrt{6}}{3} cm$

M1 + A1
2 marks