Mathematical Association of Victoria
Trial Exam 2009

MATHEMATICAL METHODS (CAS)

STUDENT NAME ____

Written Examination 2

Reading time: 15 minutes
Writing time: 2 hours

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
1	22	22	22
2	4	4	58
			Total 80

• Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved CAS calculator (memory DOES NOT need to be cleared) and, if desired, one scientific calculator. For approved computer based CAS, their full functionality may be used.
• Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied
• Question and answer book of 22 pages with a detachable sheet of miscellaneous formulas at the back
• Answer sheet for multiple-choice questions.

Instructions
• Detach the formula sheet from the back of this book during reading time.
• Write your **student number** in the space provided above on this page.
• All written responses must be in English.

At the end of the examination
• Place the answer sheet for multiple-choice questions inside the front cover of this book.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

SECTION 1

Instructions for Section 1

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.

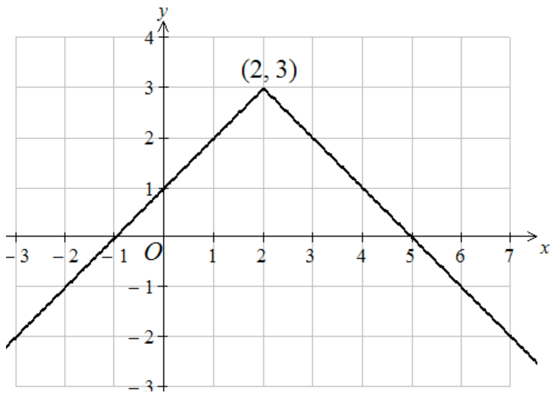
Question 1

Consider the function $f:[-4,2)\rightarrow R, f(x)=(x-1)^2$. The range of f is

A. (1, 25]
B. (0, 25]
C. [0, ∞)
D. (1, ∞)
E. [0, 25]

Question 2

Part of the graph of the function $g: R\rightarrow R$ is shown.



The rule of the function is

A. $g(x)=|2-x|-3$

B. $g(x)=-(|x|-2)+3$

C. $g(x)=|x+2|-3$

D. $g(x)=-|x-2|+3$

E. $g(x)=|x+3|-2$

Question 3

The transformation $T: R^2 \to R^2$ maps the curve with equation $y = f(x)$ to the curve with equation $y = -f\left(\frac{x}{2}\right) + 3$. The rule of T is

A. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \end{bmatrix}$

B. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0.5 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

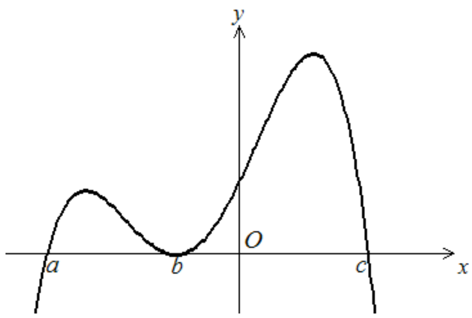
C. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \end{bmatrix}$

D. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

E. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0.5 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

Question 4

The graph of a quartic function f is shown below. The *x*-axes intercepts have coordinates $(a,0),(b,0)$ and $(c,0)$, where *a, b* and *c* are real constants.



The rule for f could be

A. $f(x) = (x-a)(x-c)(x-b)^2$

B. $f(x) = -(x-a)(x-c)(x-b)^2$

C. $f(x) = (x+a)(x-c)(x+b)^2$

D. $f(x) = -(x+a)(x-a)(x+b)^2$

E. $f(x) = -(x-a)(x-b)(x-c)^2$

Question 5

The power function $f:[m,\infty) \to R$, with rule $f(x) = (2x+1)^8$, will have an inverse function if

A. $m \le \frac{1}{2}$

B. $m < 0$

C. $m \ge -\frac{1}{2}$

D. $m \ge -1$

E. $m < 1$

Question 6

If $f(x)=\frac{1}{x^2}$ and $g(x)=\begin{cases}\sqrt{|x|} & \text{if } x\le 1\\ x^{\frac{1}{5}} & \text{if } x>1\end{cases}$, then the implied domain of $f-g$ is

A. R
B. $R\setminus\{0\}$
C. $(0,\infty)$
D. $(1,\infty)$
E. $(-\infty,1]$

Question 7

Consider the system of linear equations represented by the matrix equation $\begin{bmatrix}k+1 & 2\\ 4 & k-1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ k\end{bmatrix}$, where k is a real constant. The system of equations will have a **unique solution** when

A. $k\in R$
B. $k=3$
C. $k=-3$
D. $k\in\{-3,3\}$
E. $k\in R\setminus\{-3,3\}$

Question 8

The function $f:R\to R$ satisfies the functional equation $f(u+\pi)=f(u)$ for all $u\in R$.
The rule of f could be

A. $f(x)=\sin(x)$
B. $f(x)=\cos(x)$
C. $f(x)=\tan(x)$
D. $f(x)=\sec(x)$
E. $f(x)=\frac{1}{\sin(x)}$

Question 9

The general solution to $2\cos(3x)+1=0$ is

A. $x=\frac{2\pi(3k+1)}{9}$ or $x=\frac{2\pi(3k-1)}{9}$ where $k\in R$
B. $x=\frac{\pi(6k+1)}{9}$ or $x=\frac{\pi(6k+5)}{9}$ where $k\in Z$
C. $x=\frac{2\pi(3k+1)}{9}$ where $k\in Z$ only
D. $x=\frac{2\pi(3k+1)}{9}$ or $x=\frac{2\pi(3k-1)}{9}$ where $k\in Z$
E. $x=\frac{\pi(6k+1)}{9}$ where $k\in Z$ only

Question 10

If $g(x) = -\cos(f(x))$, then $g'(x)$ is equal to

A. $f'(x)\sin(f(x))$

B. $f'(x)\sin(x)$

C. $\sin(f'(x))$

D. $-f'(x)\cos(f(x))$

E. $-\cos(f'(x))$

Question 11

Consider the differentiable function $f : R \to R$ with the following properties

- $f'(x) > 0$ when $x \in (-1, 2)$ and
- $f'(x) < 0$ when $x \in (-\infty, -1) \cup (2, \infty)$.

The smooth and continuous graph of f has

A. a local maximum at $x = -1$ and a local minimum at $x = 2$

B. a stationary point of inflection at $x = -1$ and a local minimum at $x = 2$

C. a local maximum at $x = 2$ and a local minimum at $x = -1$

D. a stationary point of inflection at $x = 2$ and a local minimum at $x = -1$

E. local maximums at $x = -1$ and $x = 2$

Question 12

Which of the following is **false** for the function f where $f(x) = (x-1)^{\frac{2}{3}}$?

A. The graph has a cusp at $x = 1$.

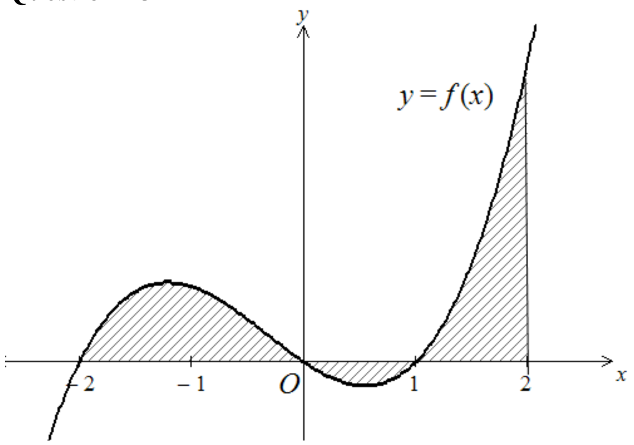
B. The graph has a vertical tangent at $x = 1$.

C. The derivative of f is not defined at $x = 1$.

D. f is continuous for $x \in R$.

E. f' exists for $x \in R$.

Question 13



The total area of the shaded regions in the diagram is given by

A. $\int_{-2}^{2} f(x)\,dx$

B. $\left|\int_{-2}^{2} f(x)\,dx\right|$

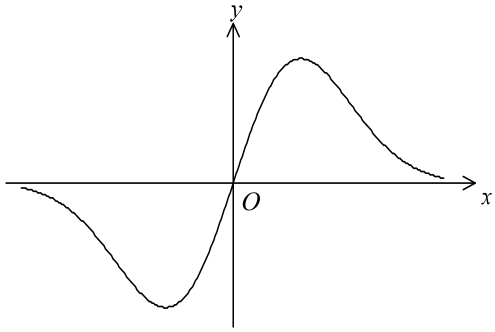
C. $\int_{-2}^{0} f(x)\,dx + \int_{1}^{0} f(x)\,dx + \int_{1}^{2} f(x)\,dx$

D. $\int_{-2}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx$

E. $\int_{0}^{-2} f(x)\,dx - \int_{1}^{0} f(x)\,dx + \int_{2}^{1} f(x)\,dx$

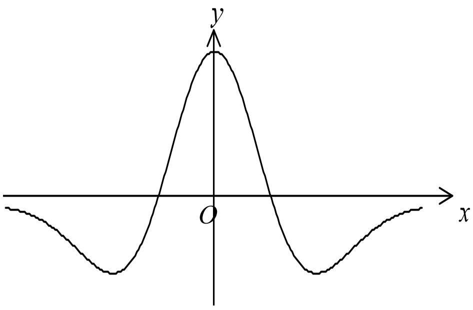
Question 14

The graph of the function $f: R \rightarrow R$, with rule $y = f(x)$ is shown below.

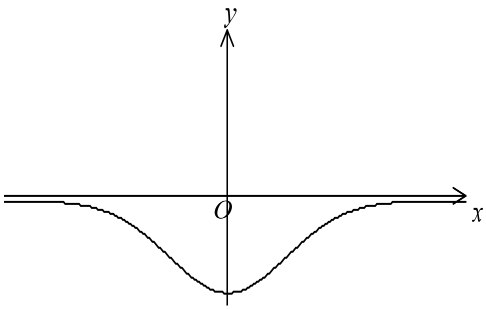


Which one of the following could be the graph of $y = f'(x)$?

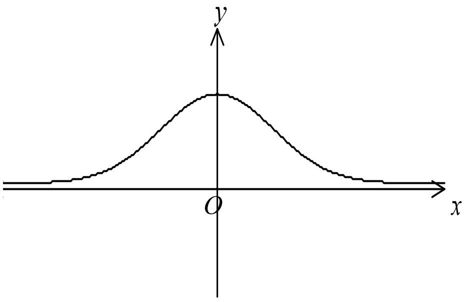
A.



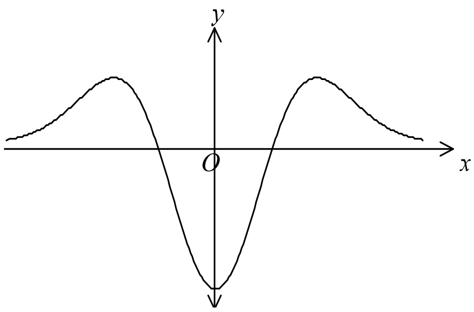
B.



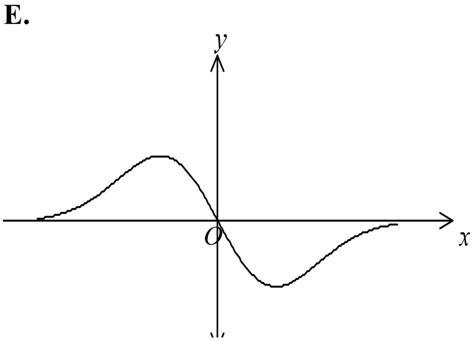
C.



D.



E.

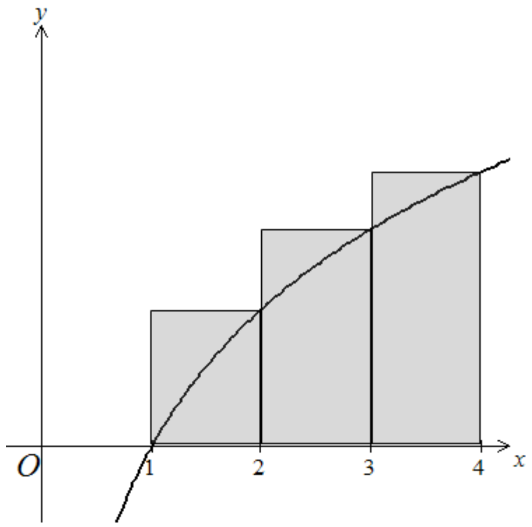


Question 15

If $\int_{-2}^{1} u(x)\,dx = 8$, then $\int_{-2}^{1} \left(2x - \frac{u(x)}{2}\right) dx$ is equal to

A. −7
B. −2
C. 1
D. 10
E. −4

Question 16
The area under the curve $y = \log_e(x)$ between $x = 1$ and $x = 4$ is approximated by three rectangles, as shown.



This approximation to the area is
A. $8\log_e(2) - 3$
B. $\log_e(24)$
C. $\log_e(6)$
D. $\log_e(7)$
E. $\log_e(16) - 3$

Question 17
The average value of the function with rule $n(t) = 20 - 12\cos\left(\frac{\pi}{12}t\right)$ in the interval $t \in [6,11]$, correct to the nearest integer, is
A. 20
B. 23
C. 27
D. 32
E. 134

Question 18
The probability that it will rain on a particular day given that it rained the day before is 0.35. The probability that it will rain on a particular day given that it did not rain the day before is 0.22. If it rains on Monday, then the probability that it will rain on the Wednesday is

A. 0.1225
B. 0.2655
C. 0.2435
D. 0.3500
E. 0.7345

Question 19
Consider the following probability distribution which has a mean value of 3.7.

x	1	2	4	6	7
$\Pr(X = x)$	0.1	a	b	0.2	0.2

The values of a and b can be found by evaluating which one of the following?

A. $\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1.0 \end{bmatrix}$

B. $\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1.0 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.5 \end{bmatrix}$

D. $\begin{bmatrix} 2 & 4 \\ 4 & 16 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

E. $\begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1.0 \\ 0.25 \end{bmatrix}$

Question 20
The results of a Mathematical Methods CAS examination are normally distributed with a mean of 30 and variance of 49. If the top 8% are awarded A+ then the cut off score, to the nearest integer, for an A+ would be

A. 10
B. 20
C. 35
D. 40
E. 50

Question 21
It is known that the probability of a person becoming ill after eating at a particular restaurant is 0.3. A random sample of 10 people who ate at the restaurant was taken and the people were monitored. The probability that at most two of them will **not** become ill, correct to four decimal places, is

A. 0.0016
B. 0.0015
C. 0.3545
D. 0.3827
E. 0.3828

Question 22
A continuous random variable X has the probability density function given by

$$f(x)=\begin{cases}\dfrac{x^2}{10} & 0\le x\le\sqrt[3]{30}\\ 0 & \text{elsewhere}\end{cases}.$$

The median value of X is

A. 2.33
B. 2.47
C. $0.75\times\sqrt[3]{30}$
D. $\sqrt[3]{30}$
E. $\sqrt[3]{15}$

END OF SECTION 1

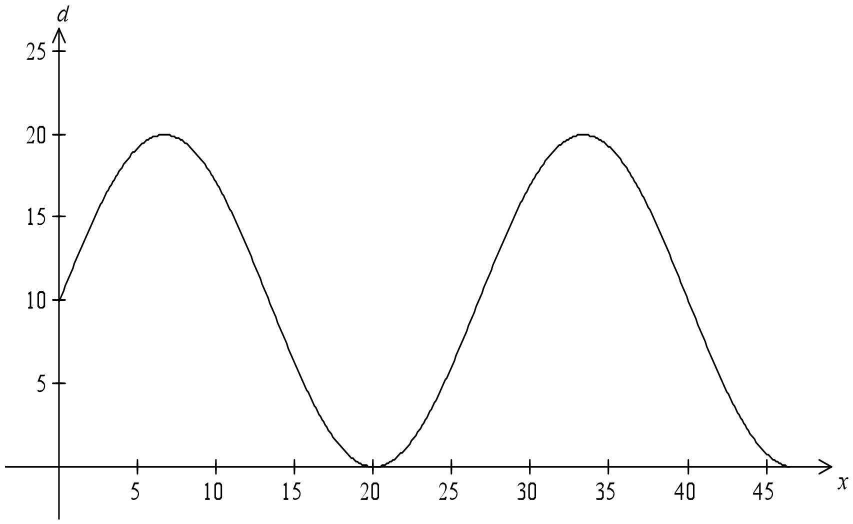
SECTION 2

Instructions for Section 2

Answer **all** questions in the spaces provided.
A decimal approximation will not be accepted if an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

The side view of the top of a roller coaster in a mouse's cage is shown below. d is the vertical height of the roller coaster above the bottom of the cage in cm and x is the horizontal distance in cm from the start of the roller coaster. The start of the roller coaster is at (0, 10) where it is attached to the side of the cage. The maximum height of the roller coaster is 20 cm and the roller coaster first reaches the bottom of the cage when $x = 20$ cm. The end of the roller coaster is where it touches the bottom of the cage for the second time.



The top section of the roller coaster can be modelled by the equation $d = A\sin(nx) + B$, where A, n and B are real constants.

a. Explain why $A = 10$, $B = 10$ and $n = \frac{3\pi}{40}$.

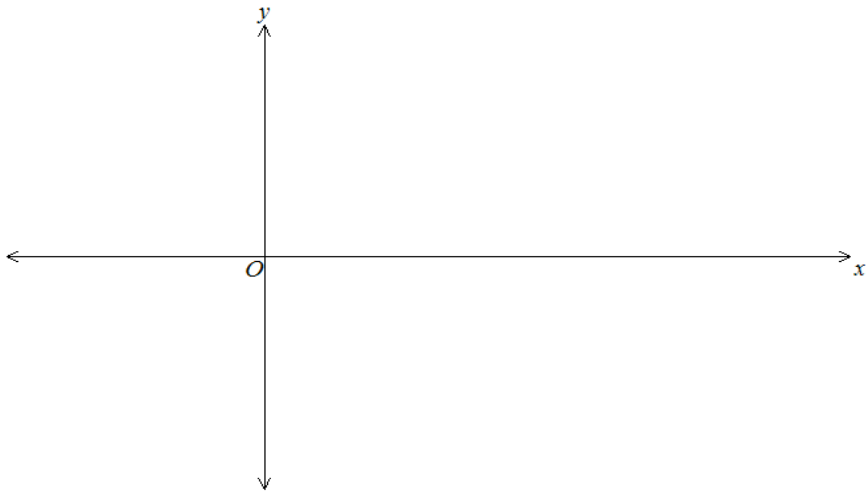
2 marks

b. Find the average value of d.

2 marks

The bottom section of the roller coaster, d_b cm, is a dilation of the curve of d by a factor of $\frac{1}{2}$ from the x-axis.

c. **i.** Sketch the graph of d_b on the set of axes below. Label the turning points and end points with their coordinates.



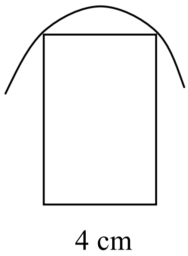
ii. Write down the rule for the bottom section of the roller coaster.

iii. What is the relationship between the area bounded by the curves of d and d_b and the line $x = 0$, and the area bounded by the curve of d_b, the x-axis and the line $x = 0$?

iv. If the roller coaster is 3 cm wide, what volume is taken up by the roller coaster? Give an exact answer.

2 + 1 + 1 + 2 = 6 marks

The roller coaster needs more support. A cuboid structure, with width 4 cm, is going to be put under each of the arches of d_b. The side view is shown below.



d. What is the height of the cuboid structure if it touches the edges of the arch? Give your answer correct to two decimal places.

2 marks

TOTAL 12 marks

Question 2

Consider the function $f: R \to R, f(x) = 2xe^{-\frac{x}{2}}$.

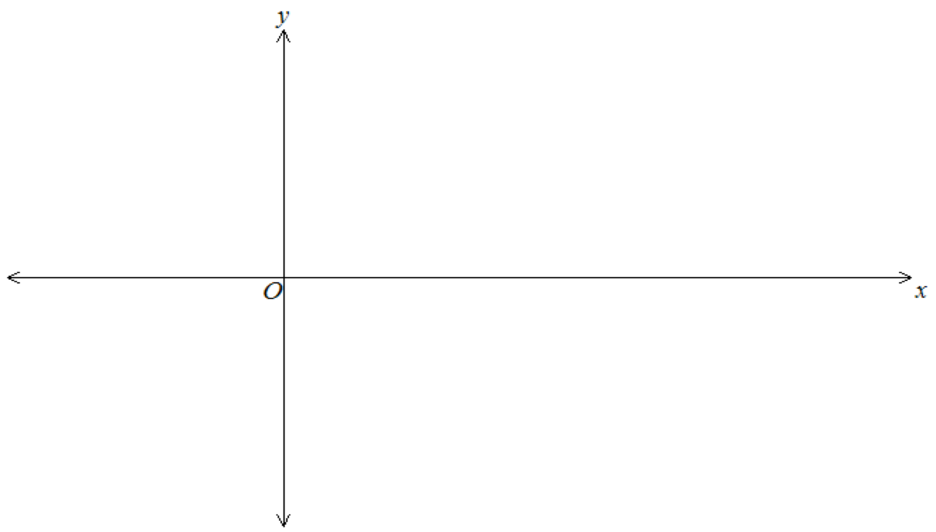
a. Find $f'(x)$

1 mark

b. Find the exact coordinates of the stationary point.

2 marks

c. Sketch the graph of *f* on the axes below. Label stationary points with their coordinates. Label any asymptote with its equation.



3 marks

d. i. Find the gradient of the tangent to the graph of *f* at the point where $x = 1$.

ii. Consider the line segment PQ with endpoints on the graph of f at $P(0,0)$ and $Q(a, f(a))$. Show that the gradient of PQ is equal to the gradient of the tangent to the graph of f at the point where $x=1$, when $a = 2\log_e(2)+1$.

iii. Find the area bounded by the graph of f and the line segment PQ, correct to four decimal places.

1 + 3 +2 = 6 marks

e. Find the equation of the normal to the graph of f at the point where $x=1$.

2 marks

f. Show that $2xy \times f(x+y) = (x+y)f(x)f(y)$ for $x \in R$ and $y \in R$.

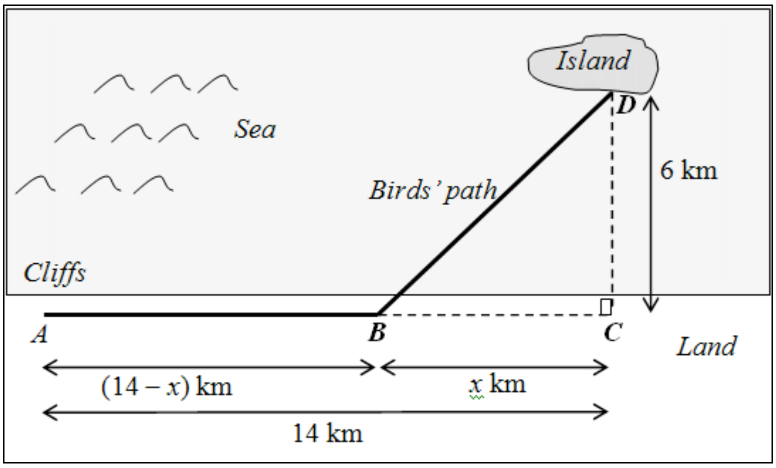
3 marks
TOTAL 17 marks

WORKING SPACE

Question 3

Miriam, a biologist, is studying the habits of a species of seabird. The diagram below shows the flight path taken by adult birds when flying from the cliffs at point A to the island at point D. Points A, B and C are on a straight shore and D is 6 km from the shore.

$AC = 14$ km, $BC = x$ km and $AB = (14 - x)$ km.



a. Find the exact length of BD.

1 mark

b. Suppose that a particular bird can travel at an average speed of 20 km/h over the land and at 12 km/h over the sea.

i. Write an expression, in terms of x, for the time, T hours, that it will take the bird to fly from A to B to D.

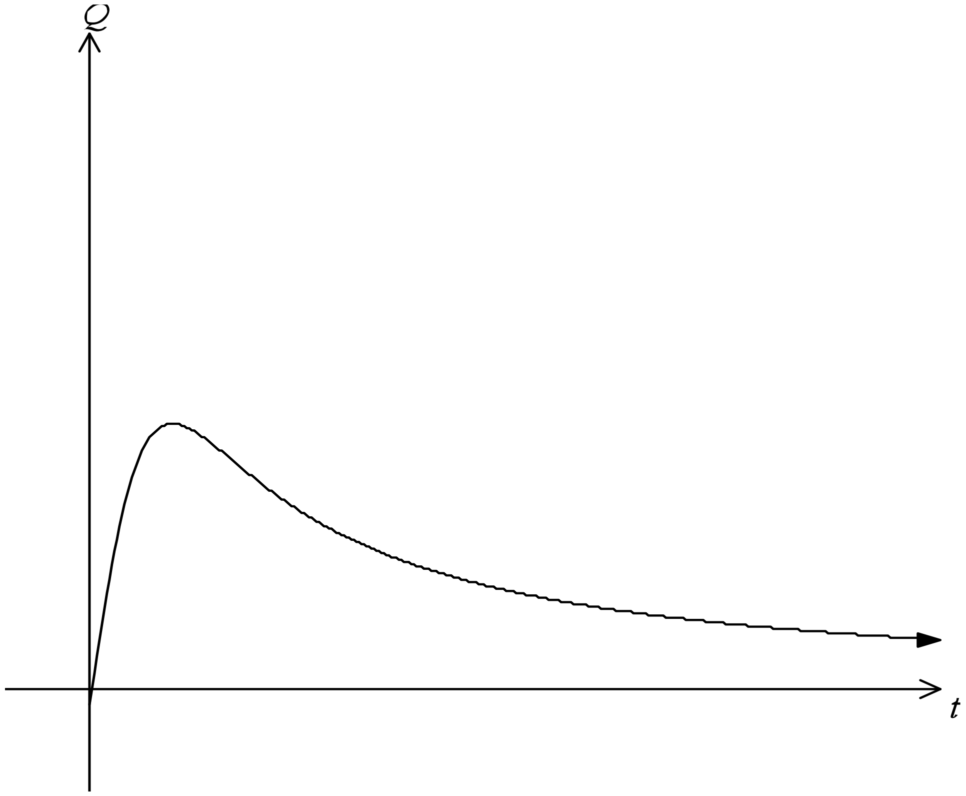
ii. Find the value of x for which the time taken by the bird to reach the island is a minimum.

iii. Find the minimum time, in hours, correct to one decimal place, required for the bird to fly to the island.

3 + 1 + 1 = 5 marks

Miriam is also treating an injured sea lion at the base of the cliffs. She administers a dose of an analgesic (pain reliever) at $t = 0$. The concentration, Q units/cm^3, of analgesic in the animal's bloodstream, t hours after it is administered, is modelled by the function

$Q:[0,\infty)\to R,\ Q(t)=\dfrac{6t}{t^2+1}$. The graph of this function is shown.



c. **i.** What is the maximum concentration of the analgesic, and how long after the dose is administered does the maximum occur?

ii. On the graph of Q above, label the stationary point with its exact coordinates.

2 + 1 = 3 marks

d. The analgesic will provide pain relief when the concentration is above 1.25 units/cm^3. For what length of time, in hours, correct to two decimal places, will the animal experience relief from the pain?

2 marks

$\frac{dQ}{dt}$ is a measure of the rate of at which the analgesic is being absorbed into the bloodstream (when the rate is positive) or expelled from the bloodstream (when the rate is negative).

Consider the function $S:(0,\infty)\to R, S(t)=\frac{dQ}{dt}$.

e. The set of axes above **part c. i**. show the graph of Q. On the same set of axes, sketch the graph of S. Label the t-axis intercept. Label the local minimum with its coordinates, correct to two decimal places. Label any asymptote with its equation.

3 marks
TOTAL 14 marks

WORKING SPACE

Question 4

If Grandma bottled tomatoes last year the probability that she will bottle tomatoes this year is 0.3. If she didn't bottle tomatoes last year the probability that she will not bottle tomatoes is 0.2. Grandma bottled tomatoes in 2008.

a. What is the probability that she will bottle tomatoes three years in a row from 2009 to 2011?

1 mark

b. Write down the transition matrix which represents this situation.

2 marks

c. What is the probability that Grandma will bottle tomatoes in 2011?

2 marks

d. What is the probability that Grandma will bottle tomatoes in the long term?

2 marks

On Grandma's farm tomatoes of the variety, Tom, have weights which are normally distributed. It is known that 5% of the tomatoes weigh more than 30 g and 10% weigh less than 15 g.

e. Find the mean and standard deviation of the weights of the Tom tomatoes in g correct to one decimal place.

2 marks

Bridget, her grand-daughter, randomly selects 10 Tom tomatoes from a large bin at the farm.

f. What is the probability that the first two and the last two she selects weigh between 15 and 30 g? Give your answer correct to four decimal places.

2 marks

g. What is the probability she will select at least 2 tomatoes which weigh more than 30 g? Give your answer correct to four decimal places.

2 marks

h. What is the least number of tomatoes that Bridget will need to select to ensure that the probability she gets at least two Tom tomatoes weighing more than 30 g is more than 0.95?

2 marks

TOTAL 15 marks

END OF QUESTION AND ANSWER BOOKLET

Mathematical Methods and Mathematical Methods (CAS) Formulas

Mensuration

area of a trapezium:	$\frac{1}{2}(a+b)h$	volume of a pyramid:	$\frac{1}{3}Ah$
curved surface area of a cylinder:	$2\pi rh$	volume of a sphere:	$\frac{4}{3}\pi r^3$
volume of a cylinder:	$\pi r^2 h$	area of a triangle:	$\frac{1}{2}bc\sin A$
volume of a cone:	$\frac{1}{3}\pi r^2 h$		

Calculus

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c, n \neq -1$$

$$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$$

$$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$$

$$\int \frac{1}{x}\,dx = \log_e|x| + c$$

$$\frac{d}{dx}\left(\sin(ax)\right) = a\,\cos(ax)$$

$$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + c$$

$$\frac{d}{dx}\left(\cos(ax)\right) = -a\,\sin(ax)$$

$$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + c$$

$$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\,\sec^2(ax)$$

product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

approximation: $f(x+h) \approx f(x) + hf'(x)$

Probability

$\Pr(A) = 1 - \Pr(A')$

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$

mean: $\mu = E(X)$

variance: $\mathrm{var}(X) = \sigma^2 = E((X-\mu)^2) = E(X^2) - \mu^2$

probability distribution		mean	variance
discrete	$\Pr(X = x) = p(x)$	$\mu = \sum x\,p(x)$	$\sigma^2 = \sum (x-\mu)^2\,p(x)$
continuous	$\Pr(a < X < b) = \int_a^b f(x)dx$	$\mu = \int_{-\infty}^{\infty} x\,f(x)dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$

END OF FORMULA SHEET