2009 VCAA Mathematical Methods (CAS) Exam 2 Solutions

SECTION 1: Multiple-choice questions

1	2	3	4	5	6	7	8	9	10	11
B	C	B	A	D	D	B	C	E	B	D

12	13	14	15	16	17	18	19	20	21	22
D	C	D	E	C	A	B	E	E	B	D

Q1 $kx-3y=0$, $5x-(k+2)y=0$.
$5kx-15y=0$, $5kx-k(k+2)y=0$.
A unique solution: $k(k+2)\neq 15$, $k^2+2k-15\neq 0$,
$(k+5)(k-3)\neq 0$, $k\neq -5$, 3. B

Q2 C

Q3 $2x+1>0$, $x>-\frac{1}{2}$ B

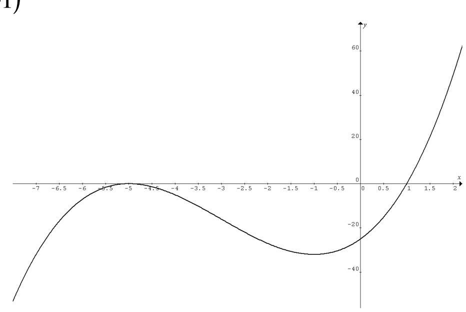
Q4 $\sin 2x=-1$, $2x=2n\pi+\frac{3\pi}{2}$ or $2n\pi-\frac{\pi}{2}$, where $n\in Z$.
$\therefore x=n\pi+\frac{3\pi}{4}$ or $n\pi-\frac{\pi}{4}$. A

Q5 $f(x-y)=(x-y)^2=x^2+y^2-2xy=f(x)+f(y)-2xy$
$\neq f(x)-f(y)$ D

Q6
$\Pr(X>17)=\Pr\left(Z>\frac{X-\mu}{\sigma}\right)=\Pr(Z>1.5)=\Pr(Z<-1.5)$ D

Q7 $y=e^{2x}\cos 3x$,
$\frac{dy}{dx}=e^{2x}(-3\sin 3x)+(2e^{2x})\cos 3x=e^{2x}(-3\sin 3x+2\cos 3x)$
When $x=0$, $\frac{dy}{dx}=2$ B

Q8 $(-5,-1)$ C



Q9 $(3,8)$ is the image of $(1,5)$ after the translations.
$\therefore$ the tangent at $(3,8)$ has the same gradient as $y=3+2x$, i.e. 2.
$y-8=2(x-3)$, $\therefore y=2x+2$ E

Alternatively, make the same translations to the original tangent,
$y-3=3+2(x-2)$, $y=2x+2$

Q10 B

Q11 $\int_a^{0.5}\pi\sin 2\pi x\,dx=0.2$, $\left[-0.5\cos 2\pi x\right]_a^{0.5}=0.2$,
$-0.5\cos\pi+0.5\cos 2a\pi=0.2$, $0.5\cos 2a\pi=-0.3$,
$\cos 2a\pi=-0.6$, $2a\pi\approx 2.2143$, $a\approx 0.35$ D

Q12 $y'=1-3\sin(2x'+\pi)$, $\frac{y'-1}{-3}=\sin(2x'+\pi)$,
$\therefore y=\frac{y'-1}{-3}$, i.e. $y'=-3y+1$, and $x=2x'+\pi$, i.e. $x'=\frac{x}{2}-\frac{\pi}{2}$.
$\therefore\begin{bmatrix}x'\\y'\end{bmatrix}=\begin{bmatrix}\frac{1}{2}&0\\0&-3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}-\frac{\pi}{2}\\1\end{bmatrix}$ D

Q13 Binomial: $n=12$, $p=0.5$.
By calculator, $\Pr(X\leq 4)\approx 0.1938$ C

Q14 $f(x)=\left|x^{\frac{3}{5}}\right|+2$, $f'(x)$ is undefined at $x=0$. D

Q15
$y=\sqrt{1-f(x)}$, $\frac{dy}{dx}=\frac{1}{2\sqrt{1-f(x)}}\times(-f'(x))=\frac{-f'(x)}{2\sqrt{1-f(x)}}$ E

Q16 Range of *f* is (e^3,∞) is the domain of f^{-1}.
Let $y=e^{2x+3}$, equation of f^{-1} is $x=e^{2y+3}$, $2y+3=\log_e x$,
$y=\frac{1}{2}\log_e x-\frac{3}{2}=\log_e\sqrt{x}-\frac{3}{2}$.
$f^{-1}(x)=\frac{1}{2}\log_e x-\frac{3}{2}=\log_e\sqrt{x}-\frac{3}{2}$ C

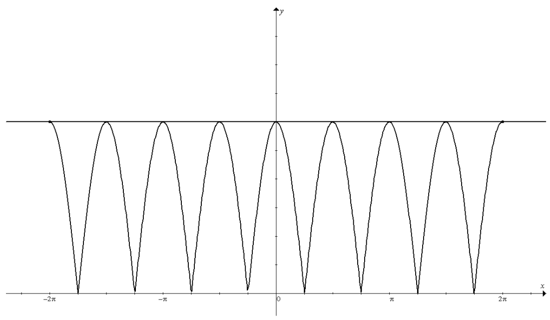
Q17 Let $X=\{1,3,5,7,9,11\}$ and $Y=\{1,4,7,10\}$
$\Pr(X\cap Y)=\Pr(\{1,7\})=\frac{2}{12}=\frac{1}{6}$
$\Pr(X)\Pr(Y)=\frac{6}{12}\times\frac{4}{12}=\frac{1}{6}$
$\therefore$ *X* and *Y* are independent. A

Q18 $\frac{1}{k}\int_0^k \frac{1}{2x+1}dx = \frac{1}{6}\log_e 7$, $\frac{1}{k}\left[\frac{1}{2}\log_e(2x+1)\right]_0^k = \frac{1}{6}\log_e 7$,

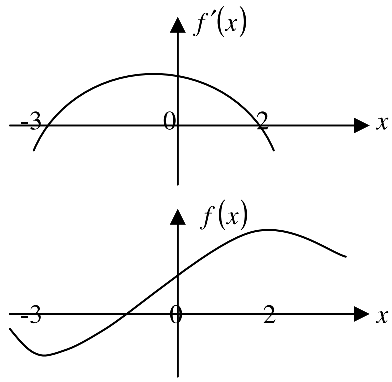
$\frac{1}{2k}\log_e(2k+1) = \frac{1}{6}\log_e 7$. $\therefore k = 3$. B

Q19 To obtain the graph of $1 - f(2x)$, dilate $f(x)$ horizontally by a factor of $\frac{1}{2}$, then reflect in the *x*-axis, and then translate upwards by 1 unit. E

Q20 9 solutions E



Q21 $f'(x) = a(x-2)(x+3)$ is a quadratic function. For it to have a maximum value, $a < 0$.



B

Q22 Inverse of $y = \log_e(x-1)$ is $y = e^x + 1$

Area $= \int_0^3 (e^x + 1)dx = [e^x + x]_0^3 = (e^3 + 3) - (1) = e^3 + 2$ D

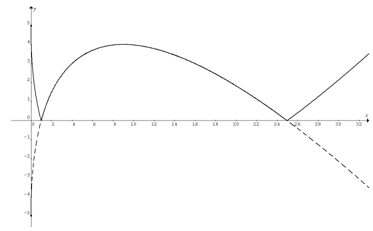
SECTION 2:

Q1a $f(x) = 6\sqrt{x} - x - 5$, $f'(x) = \frac{3}{\sqrt{x}} - 1$.

Let $\frac{3}{\sqrt{x}} - 1 = 0$, $\sqrt{x} = 3$, $x = 9$.

For $x \in (9, \infty)$, the graph of *f* is strictly decreasing.

Q1b



Q1c By calc. area $= 64$ square units.

$\therefore 24 \times AD = 64$, $AD = \frac{64}{24} = \frac{8}{3}$.

Q1di Gradient of chord $AB = m = \frac{0-3}{25-16} = -\frac{1}{3}$.

Q1dii $f'(x) = \frac{3}{\sqrt{x}} - 1$, $f'(a) = \frac{3}{\sqrt{a}} - 1 = -\frac{1}{3}$, $\frac{3}{\sqrt{a}} = \frac{2}{3}$,

$\sqrt{a} = \frac{9}{2}$, $a = \frac{81}{4}$.

Q1ei $f(x) = 6\sqrt{x} - x - 5$, $g(x) = x^2$,

$f(g(x)) = 6\sqrt{g(x)} - g(x) - 5 = 6\sqrt{x^2} - x^2 - 5$.

Q1eii $h'(x) = \frac{df(g(x))}{dg(x)} \times \frac{dg(x)}{dx}$

$= \left(\frac{3}{\sqrt{g(x)}} - 1\right)2x = \frac{6x}{\sqrt{x^2}} - 2x = \frac{6x}{|x|} - 2x$, $x \neq 0$.

For $x > 0$, $\frac{d}{dx}f(g(x)) = 6 - 2x$.

For $x < 0$, $\frac{d}{dx}f(g(x)) = -6 - 2x$.

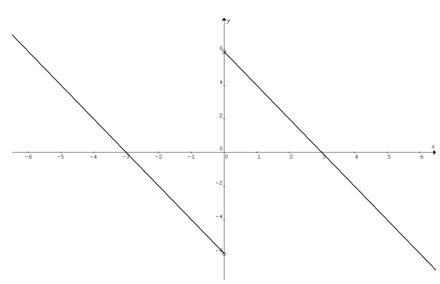
Alternatively,

$h(x) = 6\sqrt{g(x)} - g(x) - 5 = 6\sqrt{x^2} - x^2 - 5 = 6|x| - x^2 - 5$

$= \begin{cases} 6x - x^2 - 5, & x > 0 \\ -6x - x^2 - 5, & x < 0 \end{cases}$

$h'(x) = \begin{cases} 6 - 2x, & x > 0 \\ -6 - 2x, & x < 0 \end{cases}$

Q1eiii



Q2ai $y=\frac{1}{200}\left(ax^3+bx^2+c\right)$, $\frac{dy}{dx}=\frac{1}{200}\left(3ax^2+2bx\right)$.

Turning point at $x=4$, $\therefore \frac{1}{200}(48a+8b)=0$ ……(1)

Gradient $=-0.06$ at $(2,0)$, $\therefore \frac{1}{200}(12a+4b)=-0.06$ ……(2)

Passes through $(2,0)$, $\therefore \frac{1}{200}(8a+4b+c)=0$ ……(3)

Q2aii

$$\begin{bmatrix}48 & 8 & 0\\12 & 4 & 0\\8 & 4 & 1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}0\\-12\\0\end{bmatrix}$$

$$\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}48 & 8 & 0\\12 & 4 & 0\\8 & 4 & 1\end{bmatrix}^{-1}\begin{bmatrix}0\\-12\\0\end{bmatrix}=\begin{bmatrix}1\\-6\\16\end{bmatrix}$$ by calc.

Q2bi $y=\frac{1}{200}\left(x^3-6x^2+16\right)=\frac{1}{200}(x-2)\left(x^2-4x-8\right)=0$,

$\therefore x^2-4x-8=0$, $x=\frac{4\pm\sqrt{16+32}}{2}=\frac{4\pm4\sqrt{3}}{2}=2\pm2\sqrt{3}$.

M is $\left(2+2\sqrt{3},0\right)$ and P is $\left(2-2\sqrt{3},0\right)$.

Q2bii Length of tunnel $=NP=2-2+2\sqrt{3}=2\sqrt{3}$ km.

Q2biii $y=\frac{1}{200}\left(x^3-6x^2+16\right)$. Use calc. to find the local minimum $(4,-0.08)$. Maximum depth $=0.08$ km $=80$ m.

Q2c $PQ=6.2$ km. At P, $d=0$, $v=w$ km/h.

$v=k\log_e\frac{d+1}{7}$, $w=k\log_e\frac{1}{7}$, $w=-k\log_e 7$, $k=-\frac{w}{\log_e 7}$.

Q2d $v=\frac{120\log_e 2}{\log_e 7}$ when $d=2.5$,

$\therefore \frac{120\log_e 2}{\log_e 7}=k\log_e\frac{2.5+1}{7}$, $\therefore \frac{120\log_e 2}{\log_e 7}=k\log_e\frac{1}{2}$,

$\therefore \frac{120\log_e 2}{\log_e 7}=-k\log_e 2$, $\therefore k=-\frac{120}{\log_e 7}$

$\therefore w=120$ km/h

Q2e When $v=0$, $0=k\log_e\frac{d+1}{7}$, $\therefore \log_e\frac{d+1}{7}=0$,

$\frac{d+1}{7}=1$, $d=6$ km.

Distance $=6.2-6=0.2$ km $=200$ m.

Q3a $\Pr(X<68.5)=0.9332$ by calc.

Q3b $\Pr(65.6<X<68.4)=0.8385$ by calc.

Q3ci $\Pr(65.6<X<68.4 \mid X<68.5)$

$=\frac{\Pr(65.6<X<68.4)}{\Pr(X<68.5)}=\frac{0.8385}{0.9332}=0.8985$

Q3cii Binomial: $n=4$

Those in the tin outside $(65.6,68.4)$, $p=1-0.8985=0.1015$, $q=0.8985$.

$\Pr(at\_least\_one)=1-\Pr(none)=1-0.8985^4=0.3483$.

Q3d $\Pr(65.6<X<68.4)=0.99$,

$\Pr\left(\frac{65.6-67}{\sigma}<Z<\frac{68.4-67}{\sigma}\right)=0.99$

$\Pr\left(\frac{-1.4}{\sigma}<Z<\frac{1.4}{\sigma}\right)=0.99$, $\therefore \Pr\left(Z<\frac{1.4}{\sigma}\right)=0.995$

By calc. $\frac{1.4}{\sigma}\approx 2.5758$, $\therefore \sigma\approx 0.54$ mm.

Q3e $\Pr(buy\_buy\_buy)=0.8\times0.8\times0.8=0.512$

Q3f

$\Pr(buy\_buy\_buy')+\Pr(buy\_buy'\_buy)+\Pr(buy'\_buy\_buy)$

$=0.8\times0.8\times0.2+0.8\times0.2\times0.15+0.2\times0.15\times0.8=0.176$

Q3g $\begin{bmatrix}0.8 & 0.15\\0.2 & 0.85\end{bmatrix}^n=\begin{bmatrix}p & -\\- & -\end{bmatrix}$.

By calc. $p\le 0.45$ when $n\ge 8$. Smallest value of n is 8.

Q4ai $\frac{h}{r}=\frac{8}{4}$, $\therefore h=2r$.

Q4aii $V=\frac{1}{3}\pi r^2h=\frac{1}{3}\pi\left(\frac{h}{2}\right)^2h=\frac{1}{12}\pi h^3$.

Q4b $\frac{dV}{dh}\times\frac{dh}{dt}=\frac{dV}{dt}$, $\frac{dh}{dt}=\frac{\frac{dV}{dt}}{\frac{dV}{dh}}=\frac{\frac{9\pi}{4}}{\frac{\pi h^2}{4}}=\frac{9}{h^2}$ metres per hour.

Q4ci When $h=2$, $\frac{dh}{dt}=\frac{9}{2^2}=\frac{9}{4}$ metres per hour.

Q4cii When $\frac{dh}{dt}=\frac{1}{2}\times\frac{9}{4}=\frac{9}{8}$, $\frac{9}{h^2}=\frac{9}{8}$, $h=\sqrt{8}=2\sqrt{2}$ m.

Q4di $\frac{dh}{dt}=\frac{9}{h^2}$, $\frac{dt}{dh}=\frac{1}{\frac{dh}{dt}}=\frac{1}{\frac{9}{h^2}}$, $\therefore \frac{dt}{dh}=\frac{h^2}{9}$.

Q4dii $t=\int \frac{h^2}{9}\,dh=\frac{h^3}{27}+c$.

$h=0$ when $t=0$, $\therefore t=\frac{h^3}{27}$, $\therefore h=3t^{\frac{1}{3}}$.

Q4ei At time *t*, distance above ground level $=14-t$ metres.

Q4eii When the statue first touches the acid, $3t^{\frac{1}{3}}=14-t$.
By calc. $t=8$, i.e. 5.00 pm.

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