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Mathematical Methods

2009

Trial Examination 2

SECTION 1 Multiple-choice questions

Instructions for Section 1

Answer **all** questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.

Question 1

The graph of $y = 20x^3 + 41x^2 + 28x + 25$ has

A. one inflection point and no stationary points.

B. only one stationary inflection point.

C. a local maximum point, a local minimum point and an inflection point.

D. a local maximum point, a local minimum point and no inflection points.

E. a stationary point and an inflection point only.

Question 2

In the equation $3(a-x)^2 e^x + 1 = 3(a-x)e^{\frac{x}{2}}$, a is a constant. When the equation is solved for x, the number of real distinct solutions is

A. zero B. one C. two D. three E. four

Question 3

Consider the function $f : D \to R$, $f(x) = \tan\left(\frac{x}{k}\right)$ and $D = \left(-\frac{k\pi}{4}, \frac{k\pi}{4}\right)$. Which one of the following statements is true?

A. $x = -\frac{k\pi}{2}$ and $x = \frac{k\pi}{2}$ are asymptotes of f.

B. $x = -\frac{k\pi}{4}$ and $x = \frac{k\pi}{4}$ are asymptotes of f.

C. f has no asymptotes.

D. $x = -\frac{k\pi}{8}$ and $x = \frac{k\pi}{8}$ are asymptotes of f.

E. $x = -\frac{2\pi}{k}$ and $x = \frac{2\pi}{k}$ are asymptotes of f.

Question 4

The inverse function of $x^2 - b$ for $x \geq 0$ is $\sqrt{x+b}$, where $b > 0$.
The solution of the equation $x^2 - b = \sqrt{x+b}$ for x is

A. $\sqrt{b+1}$ B. $\frac{\sqrt{b+1}}{2}$ C. $\frac{\sqrt{b+1}+1}{2}$ D. $\frac{\sqrt{2b+1}+1}{2}$ E. $\frac{\sqrt{4b+1}+1}{2}$

Question 5

Which one of the following is an alternative form of $10^{(\log_5 x)(\log_2 y)}$?

A. $\frac{(\log_e x^x)(\log_e y^y)}{\log_e 2 \log_e 5}$

B. $\frac{\log_e(x^x + y^y)}{\log_e 10}$

C. $\frac{\log_e x^x + \log_e y^y}{\log_e 2 \log_e 5}$

D. $\frac{\log_e(x^y + y^x)}{\log_e 10}$

E. $x^{\log_2 y} y^{\log_5 x}$

Question 6

If $1 - 3f(2-2x) = 4x^2$, then $f(x) =$

A. $\frac{(x-1)(3-x)}{3}$

B. $\frac{(1-x)(3-x)}{3}$

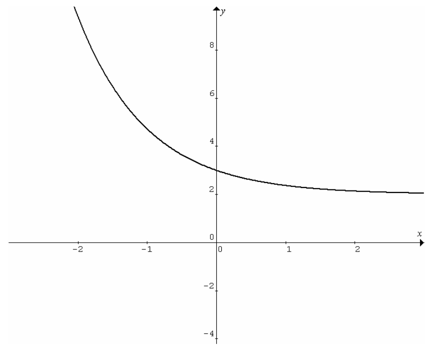
C. $\frac{(x+1)(x+3)}{3}$

D. $\frac{(x-1)(x+3)}{3}$

E. $\frac{(x+1)(x-3)}{3}$

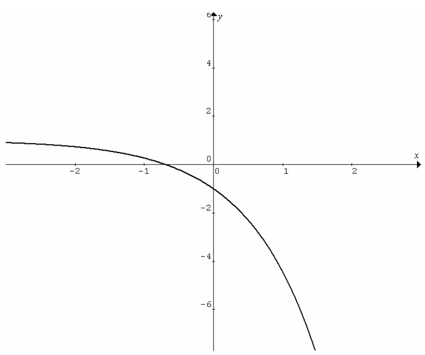
Question 7

After a downward translation of 3 units, and then reflection in the *x*-axis and in the *y*-axis, the resulting graph of the transformed function is shown below.

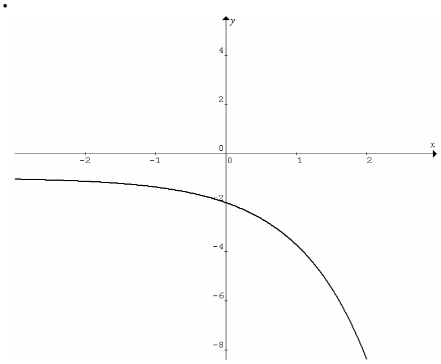


Which one of the following is the graph of the original function?

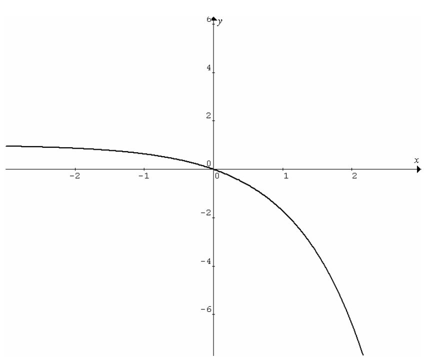
A.



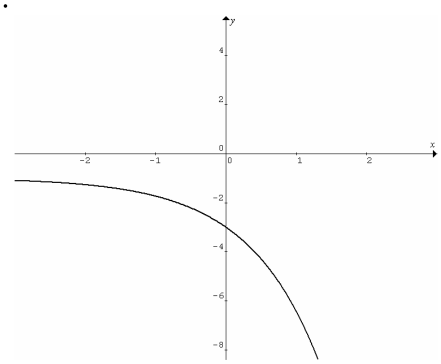
B.



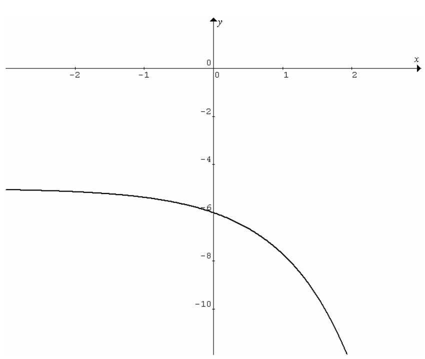
C.



D.



E.



Question 8

Consider the function $f(x)=\sqrt{\dfrac{ax+b}{cx-d}}$, where $a,b,c,d \in R^+$ and $a \neq b \neq c \neq d$.

The maximal domain of $f(x)$ is

A. $R \setminus \left\{x : -\dfrac{b}{a} \leq x < \dfrac{d}{c}\right\}$

B. $R \setminus \left\{x : -\dfrac{b}{a} < x \leq \dfrac{d}{c}\right\}$

C. $R \setminus \left\{\dfrac{d}{c}\right\}$

D. $\left\{x : x < \dfrac{d}{c}\right\}$

E. $\left\{x : x > -\dfrac{b}{a}\right\}$

Question 9

Given $(x+5)P(x)=x^4+c$, where c is a real constant, the constant term in polynomial $P(x)$ is

A. -5 B. 5 C. -25 D. 25 E. -125

Question 10

If $e^x+e^y=2$ and $e^x-e^y=1$, then

A. $x+y=\log_e 4$

B. $x+y=\log_e 3$

C. $x-y=\log_e 2$

D. $x-y=\log_e 1.5$

E. $x+y=\log_e 0.75$

Question 11

Given $f(x) = 1 + \log_e x$, which one of the following statements is **false** for $x, y \in R^+$?

A. $f(xy) = f(x) + f(y) - 1$

B. $f\left(\dfrac{x}{y}\right) = f(x) - f(y) + 1$

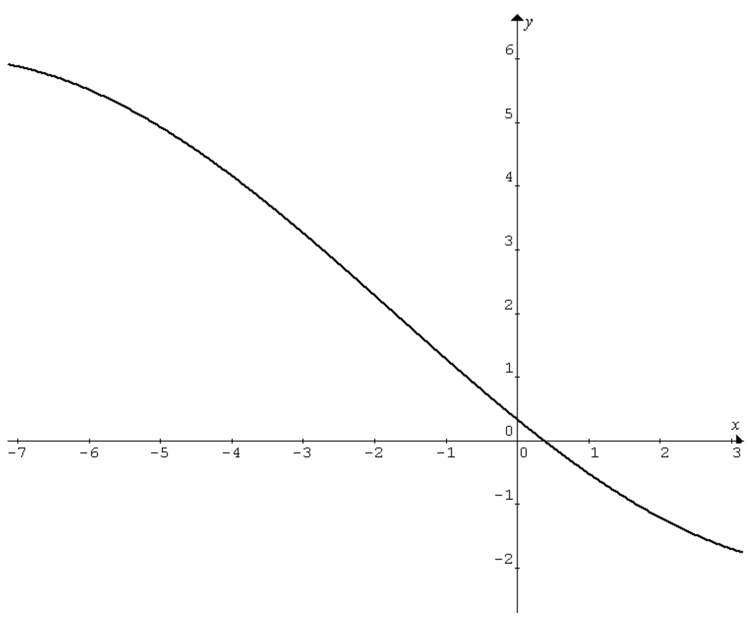
C. $f(xy) + f\left(\dfrac{x}{y}\right) = 2f(x)$

D. $f\left(\dfrac{x}{y}\right) - f\left(\dfrac{y}{x}\right) = 2[f(x) - f(y)]$

E. $f(x+y) = f(x) + f(y) - f(x)f(y)$

Question 12

The graph of $y = f(x)$ is shown below.



The value of $f'(-5)$ is closest to

A. 0.9 B. 0.7 C. -0.5 D. -0.7 E. -0.9

Question 13

Given $P(x)=\dfrac{g\left(\sqrt{x}\right)}{\sqrt{x}}$, $P'(x)=$

A. $\dfrac{g\left(\sqrt{x}\right)-\sqrt{x}g'\left(\sqrt{x}\right)}{2x\sqrt{x}}$

B. $\dfrac{xg'\left(\sqrt{x}\right)-\sqrt{x}g\left(\sqrt{x}\right)}{2x^2}$

C. $\dfrac{xg'\left(\sqrt{x}\right)-\sqrt{x}g\left(\sqrt{x}\right)}{2x}$

D. $\dfrac{g\left(\sqrt{x}\right)-\sqrt{x}g'\left(\sqrt{x}\right)}{2x}$

E. $\dfrac{g\left(\sqrt{x}\right)-\sqrt{x}g'\left(\sqrt{x}\right)}{2x\sqrt{x}}$

Question 14

$\displaystyle\int_{\frac{1}{3}}^{3}\left(\log_e(2x)-\frac{1}{2x}\right)dx=$

A. $\displaystyle -\int_{\frac{1}{3}}^{1}\log_e(2x)dx+\int_{1}^{3}\log_e(2x)dx+\log_e 9$

B. $\displaystyle -\int_{\frac{1}{3}}^{1}\log_e(2x)dx+\int_{1}^{3}\log_e(2x)dx-\log_e 3$

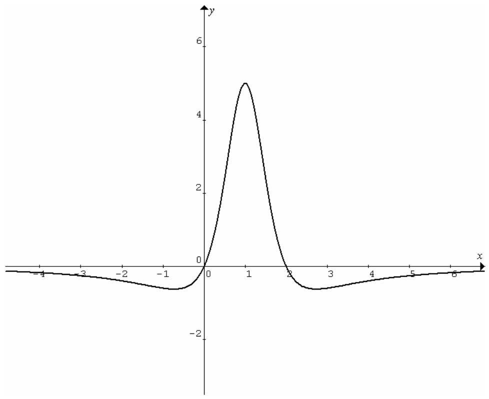
C. $\displaystyle \int_{\frac{1}{3}}^{3}\log_e(2x)dx+\log_e 9$

D. $\displaystyle \int_{\frac{1}{3}}^{3}\log_e(2x)dx-\log_e 3$

E. $-\dfrac{8}{3}+\log_e 3$

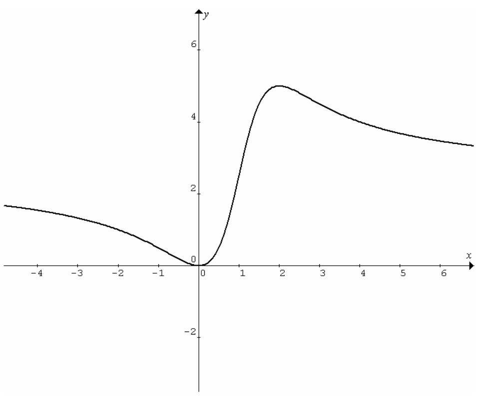
Question 15

The graph of $y = f'(x)$ is shown below.

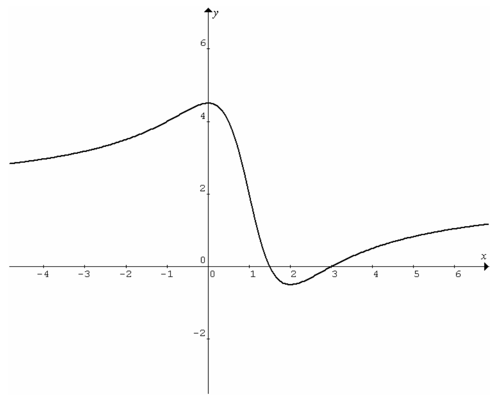


The graph of $y = f(x)$ is

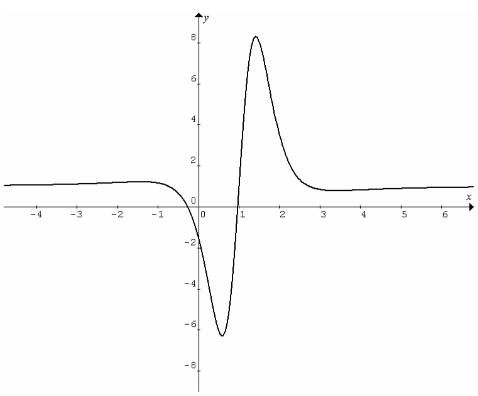
A.



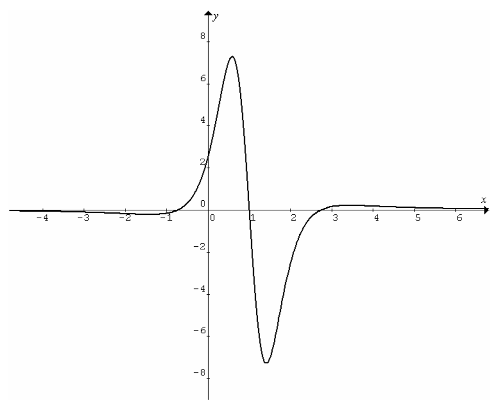
B.



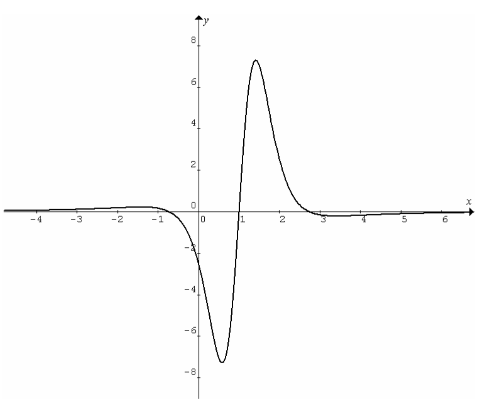
C.



D.



E.



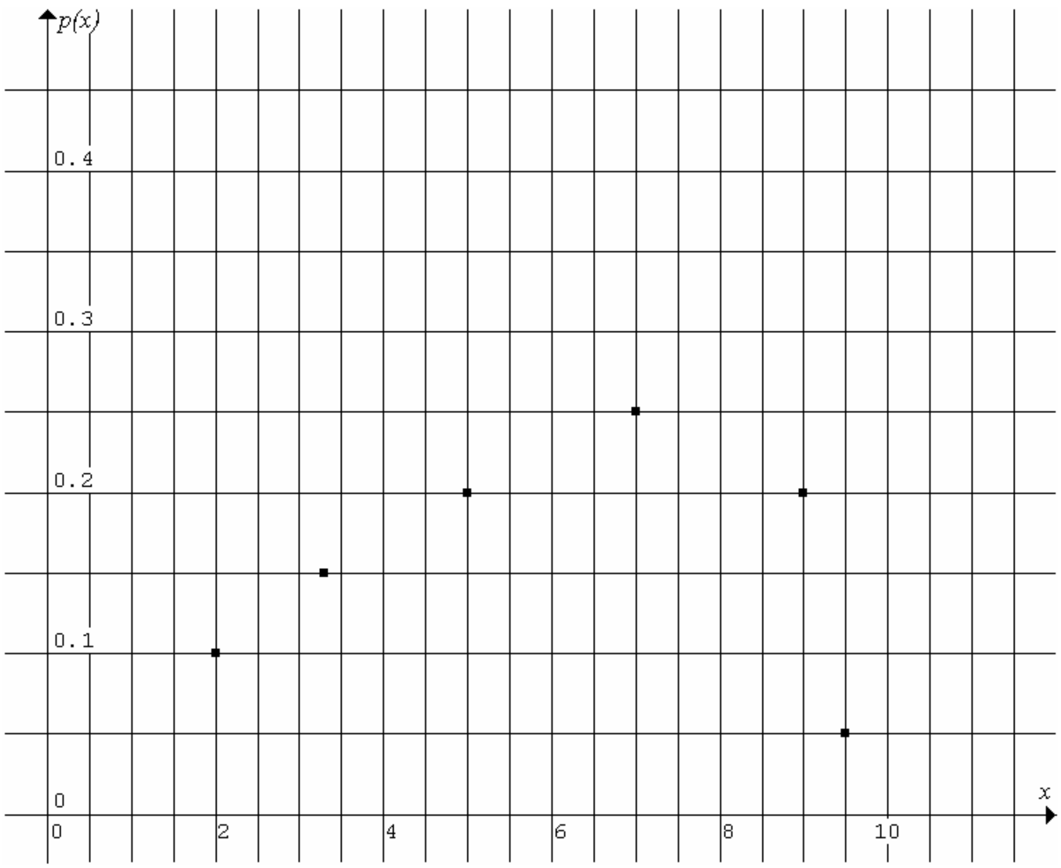
Question 16

The average rate of change of $f(x) = 2\sin\left(\frac{x}{2}\right) - \cos x$ with respect to x over $\left[\frac{\pi}{3}, \frac{4\pi}{3}\right]$ equals

A. $\frac{\sqrt{3}-2}{\pi}$ B. $\frac{\sqrt{3}+2}{\pi}$ C. $\frac{\sqrt{3}}{\pi}$ D. $\frac{\pi\sqrt{3}}{3}$ E. $\frac{2\pi}{\sqrt{3}-2}$

Question 17

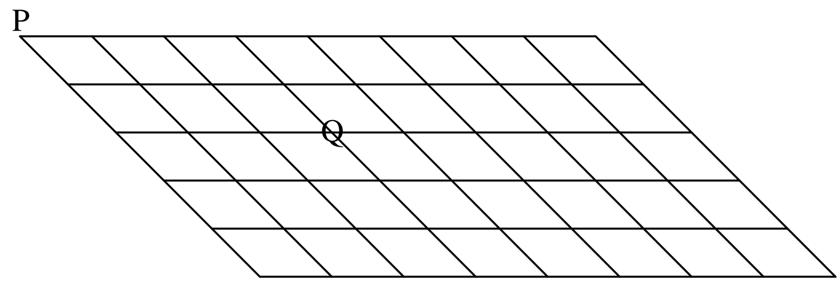
The discrete random variable X has seven values. The probability distribution of X is shown in the graph below with the missing point at $X = 8.2$ (one of the seven values of X).



The mean of X is closest to

A. 5.0 B. 5.7 C. 6.0 D. 6.8 E. 7.0

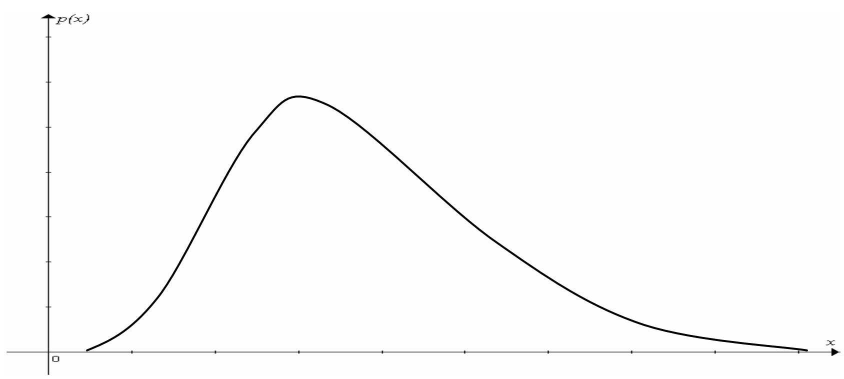
Question 18



A drunkard starts from point P and **always** travels **randomly away** from it, the probability that the drunkard passes through point Q is

A. $\frac{5}{8}$ B. $\frac{5}{16}$ C. $\frac{5}{32}$ D. $\frac{3}{32}$ E. $\frac{1}{32}$

Question 19



Consider the distribution of probability specified by the probability density function shown above.

A. Mode < median < mean

B. Mode < mean < median

C. Mode > median > mean

D. Median > mode > mean

E. Mean > mode > median

Question 20

Given $\Pr(A \mid B) = \frac{1}{3}$ and $\Pr(BAA) = \frac{1}{16}$, where BAA represents the occurrence of events B, A and A in that order **after the occurrence of *A***, a possible value of $\Pr(B \mid A)$ is

A. $\frac{1}{2}$ B. $\frac{1}{3}$ C. $\frac{1}{4}$ D. $\frac{1}{6}$ E. $\frac{1}{8}$

Question 21

The marks on an examination are normally distributed with mean $\mu = 70$ and standard deviation $\sigma = 8.5$. The probability of scoring less than 85 for a student passing the examination is 0.85. The pass mark p is

A. $45 < p < 55$

B. $55 < p < 65$

C. $65 < p < 70$

D. $70 < p < 75$

E. $75 < p < 80$

Question 22

Which one of the following is a possible random variable when three fair dice are rolled?

A. The uppermost numbers are all even.

B. The sum of the uppermost numbers is even.

C. Two of the uppermost numbers are greater than two.

D. One is not one of the uppermost numbers.

E. Number of equal uppermost numbers.

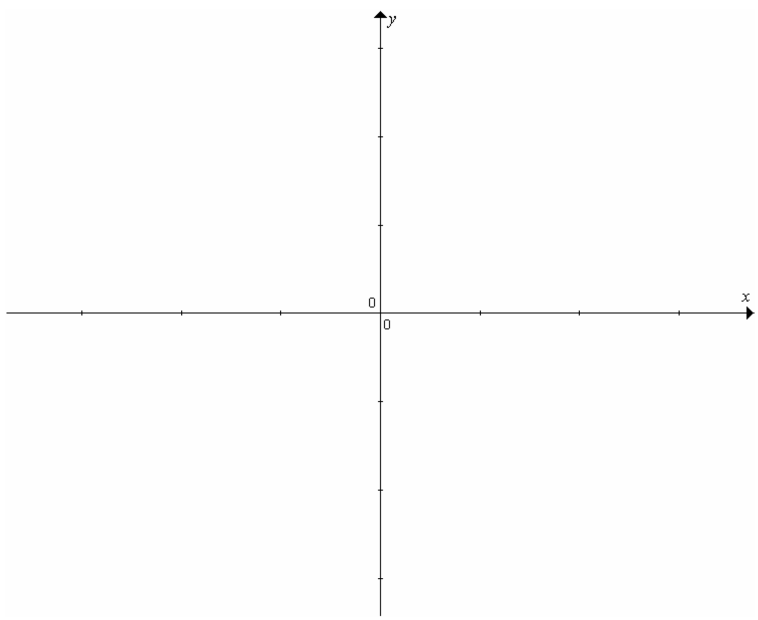
SECTION 2 Extended-answer questions

Instructions for Section 2

Answer **all** questions.
A decimal approximation will not be accepted if an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Where an instruction to **use calculus** is stated for a question, you must show an appropriate derivative or antiderivative.
Unless otherwise indicated, the diagrams in this exam are **not** drawn to scale.

Question 1
Consider the function $f(x)=\log_2(x+2)-2$.

a. Sketch the graph of $y=f(x)$. Label the asymptote(s) and intercept(s). 2 marks



b. Determine the equation of the inverse of $f(x)$. State the range of the inverse. 2 marks

c. On the graph of $y = f(x)$ there is a point which is closest to the origin. Find the coordinates (4 decimal places) of that point.

3 marks

d. Find the **exact** area of the region bounded by the *x*-axis, the *y*-axis and the curve $y = f(x)$. 4 marks

e. i. A rectangle with a vertex at the origin can be fitted in the region described in part **d**. Find the dimensions (4 decimal places) of the rectangle with the greatest area *A*. 2 marks

e. ii. If the base of the rectangle increases at a constant rate of $\log_e 2$ with respect to time *t*, write an **exact** expression for $\frac{dA}{dt}$.

2 marks

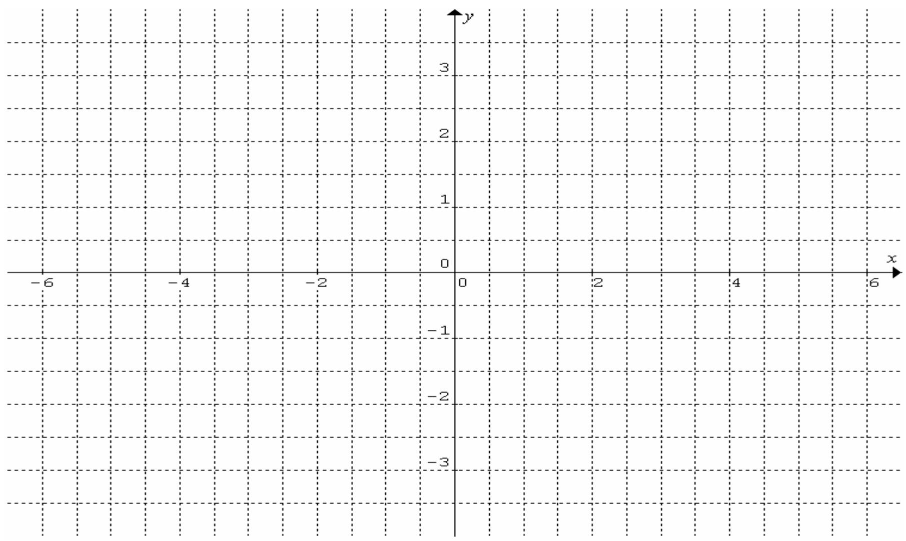
Total 15 marks

Question 2

Function $f(x)$ satisfies the following properties.

Property 1: $f(-x) = -f(x)$.
Property 2: $f(x-2) = -f(x)$.
Property 3: $f(x) = 2 - \cos\left(\frac{\pi}{2}x\right)$ when $x \in (-1, 0)$.

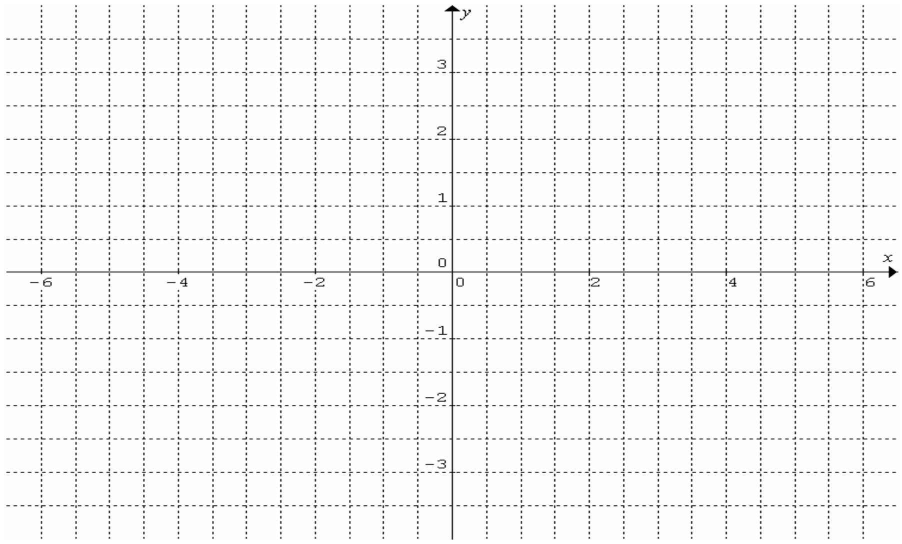
a. Sketch the graph of $y = f(x)$. You need to show from $x = -6$ to $x = 6$. 4 marks



b. Write down the domain and range of function $f(x)$. 2 marks

c. Determine the exact value of $f(-3.5)$. 2 marks

d. Function $f(x)$ is dilated by a factor of 1.5 parallel to the y-axis, then translated to the right by 1 unit. Sketch the graph of the transformed function. You need to show from $x = -6$ to $x = 6$. 2 marks

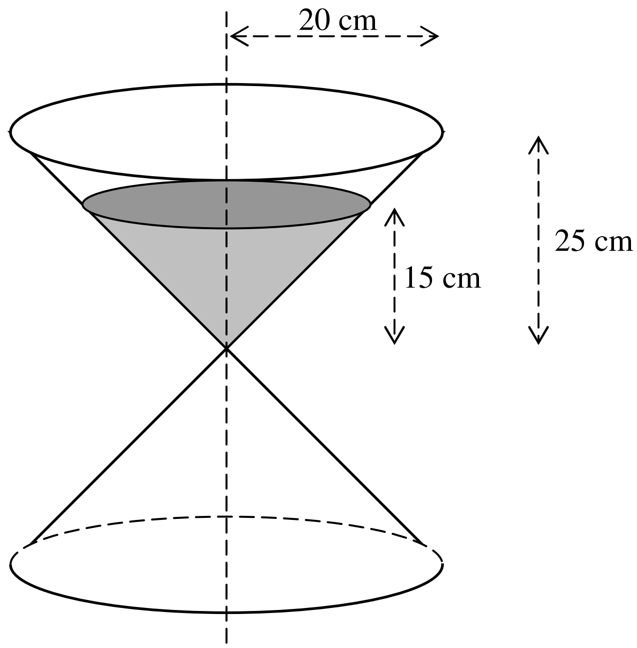


e. Modify properties 1, 2 and 3 (if necessary) to suit the transformed function in part d. 2 marks

Total 12 marks

Question 3

The following diagram shows an hourglass consisting of two identical cone-shape glass containers joined at their vertices. **Initially** the bottom one is empty and the top one is filled with a colour liquid to a depth of 15 cm. Assume that the liquid flows down to the bottom cone at a constant rate. Refer to the diagram for the other measurements.



a. Write an expression for the volume of liquid V (cm^3) remaining in the top cone in terms of the depth h (cm) at time t seconds from the start of flow.

2 marks

b. Show that the rate of flow of the colour liquid from the top cone to the bottom cone is $\frac{\pi}{5}$ cm^3 s^{-1}.

2 marks

c. Calculate the exact rate (cm per second) of decrease of the depth of liquid in the top cone when the depth of liquid in the top cone is 5 cm.

2 marks

d. Calculate the exact rate (cm per second) of increase of the depth of liquid in the bottom cone when the depth of liquid in the bottom cone is 5 cm.

2 marks

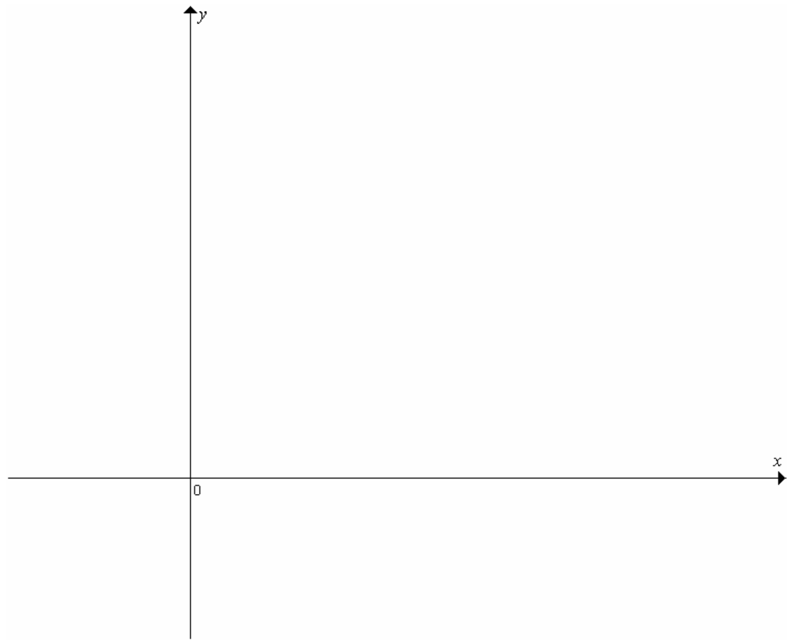
e. Given $\frac{dt}{dh} = \frac{1}{\frac{dh}{dt}}$, show that the time required to empty the top cone is exactly an hour. 2 marks

Total 10 marks

Question 4

Consider $f(x)=\left|ae^{-x}-1\right|$ and $g(x)=ae^{-x}$, where $1<a<2$.

a. Sketch the graphs of $y=f(x)$ and $y=g(x)$ on the same set of axes. Draw and label asymptotes with equations. Label intercepts in terms of a.



4 marks

b. Determine the coordinates of the intersection of the two curves in terms of a. 1 mark

c. Determine the exact area of the region bounded by the two curves and the y-axis in terms of a.
3 marks

Total 8 marks

Question 5

The lengths x (cm) of fish in a large pond are normally distributed according to the probability density function $f(x) = ke^{-\frac{1}{2}\left(\frac{x-25}{5}\right)^2}$, which can be changed to the **standard** normal probability density function $f(z) = he^{-\frac{1}{2}z^2}$, where k and h are constants. Assume that the fish population is very large.

a. Determine the value of k (2 decimal places). 1 mark

b. Find the percentage (2 decimal places) of fish longer than 20 cm. 1 mark

A fish is ready for the market if it is longer than 30 cm.

c. Five fish are randomly taken out of the pond. What is the probability (2 decimal places) that two of them are ready for the market?

2 marks

Now fish shorter than 20 cm are moved to a second pond.

d. Five of the fish left in the first pond are randomly taken out. What is the probability (2 decimal places) that two of them are ready for the market?

2 marks

e. i. Find a probability density function to describe the distribution of fish length in the first pond.

2 marks

e. ii. ***Hence*** find the probability (2 decimal places) that two of five fish randomly taken out of the first pond are ready for the market.

1 mark

f. Find the mean fish length (2 decimal places) in the first pond. Use your answer to explain why the probability distribution of fish length in the first pond is **no longer** a normal distribution.

2 marks

g. Fish that are ready for the market are to be sold at price $p = 0.01x^2$ dollars each if a fish is x cm long. There are 1000 fish ready for the market. Determine the total price (nearest thousand dollars) of the fish.

2 marks

Total 13 marks

End of exam 2