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MATHS METHODS (CAS) 3 & 4
TRIAL EXAMINATION 2
SOLUTIONS
2009

SECTION 1 – Multiple–choice answers

1. C	9. B	17. D
2. A	10. D	18. A
3. D	11. B	19. D
4. E	12. D	20. B
5. A	13. E	21. E
6. C	14. B	22. E
7. A	15. C	
8. A	16. A	

SECTION 1 – Multiple-choice solutions

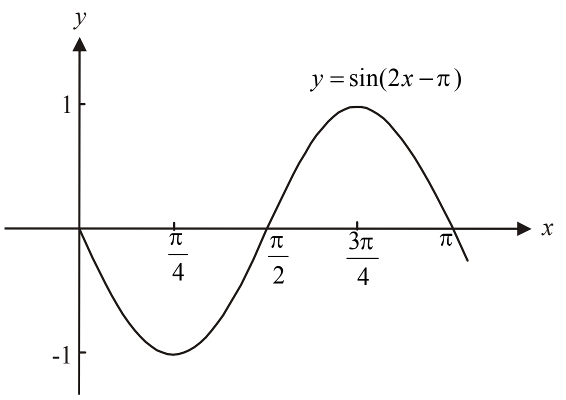
Question 1

$$\begin{aligned} E(X) &= \text{mean of } X \\ &= -2\times 0\cdot 3 + -1\times 0\cdot 2 + 0\times 0\cdot 4 + 1\times 0\cdot 1 \\ &= -0\cdot 7 \end{aligned}$$

The answer is C.

Question 2

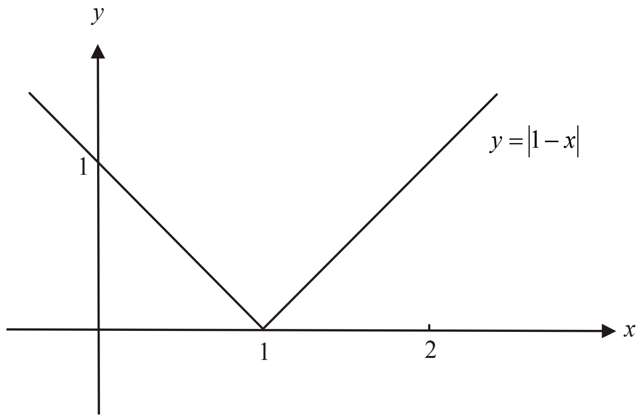
f^{-1} exists if f is a 1:1 function.
Sketch the graph of $y=\sin(2x-\pi)$ for $x \geq 0$.



If $a=\frac{\pi}{4}$, then f is a 1:1 function and f^{-1} exists.
The answer is A.

Question 3

Sketch the function $y=|1-x|$.



The rate of change is the gradient of the function. At $x=2$, the gradient $=1$.
The answer is D.

Question 4

$$5x+(a-3)y=1$$
$$ax+2y=a$$
$$\begin{bmatrix}5 & a-3\\ a & 2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}1\\ a\end{bmatrix}$$
For no solutions or an infinite number of solutions the determinant equals zero.

$$5\times 2-a(a-3)=0$$
$$10-a^2+3a=0$$
$$-(a^2-3a-10)=0$$
$$-(a-5)(a+2)=0$$
$$a=5 \text{ or } a=-2$$

If $a=5$, $\quad 5x+2y=1$
$\quad 5x+2y=5$
We have parallel lines, hence no solution.

If $a=-2$, $\quad 5x-5y=1$
So $\quad x-y=\frac{1}{5}$
$-2x+2y=-2$
So $\quad x-y=1$
We have parallel lines, hence no solution.
So $\quad a\in\{-2,5\}$ for no solutions.
The answer is E.

Question 5

This is a binominal distribution with $n = 8$ and $p = 0.1$.

$$\begin{aligned}\Pr(X \geq 2) &= 1 - \Pr(X < 2)\\ &= 1 - \{\Pr(X = 0) + \Pr(X = 1)\}\\ &= 1 - \{{}^8C_0(0\cdot 1)^0(0\cdot 9)^8 + {}^8C_1(0\cdot 1)^1(0\cdot 9)^7\}\\ &= 1 - (0\cdot 430467... + 0\cdot 382637...)\\ &= 0\cdot 186896...\\ &= 0\cdot 1869 \text{ (to four decimal places)}\end{aligned}$$

The answer is A.

Question 6

$$\begin{aligned}\text{Average value} &= \frac{1}{b-a}\int_a^b f(x)\,dx\\ &= \frac{1}{2}\int_1^3 \left(\frac{1}{x+2} - 3\right)dx\\ &= \frac{1}{2}\left(\log_e\left(\frac{5}{3}\right) - 6\right)\\ &= \frac{1}{2}\log_e\left(\frac{5}{3}\right) - 3\end{aligned}$$

The answer is C.

Question 7

$$\begin{aligned}\int_0^4 (2 - 5f(x))dx &= \int_0^4 2dx - 5\int_0^4 f(x)dx\\ &= [2x]_0^4 - 5\times 3\\ &= 8 - 0 - 15\\ &= -7\end{aligned}$$

The answer is A.

Question 8

Sketch the function.

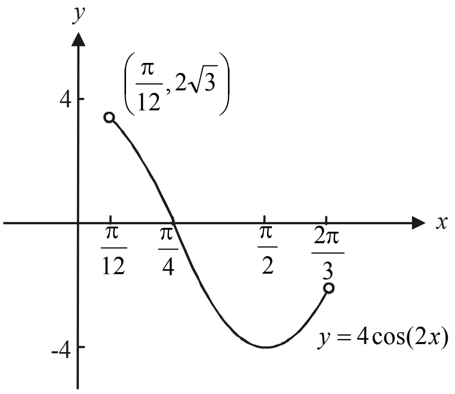
At $x = \frac{\pi}{12}$,

$$\begin{aligned}&4\cos(2x)\\ &= 4\cos\left(\frac{\pi}{6}\right)\\ &= 4\times\frac{\sqrt{3}}{2}\\ &= 2\sqrt{3}\end{aligned}$$

From the diagram,

$r_f = [-4, 2\sqrt{3})$

The answer is A.



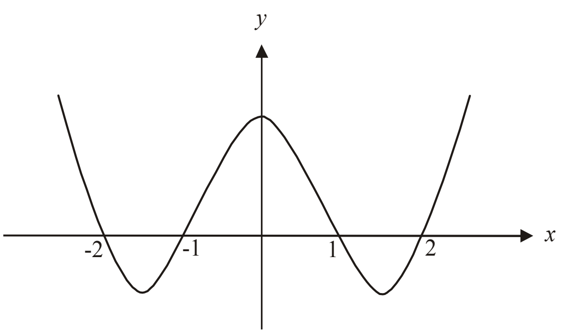
Question 9

$$\Pr(X<5 \mid X<10)=\frac{\Pr(X<5\cap X<10)}{\Pr(X<10)}$$
$$=\frac{\Pr(X<5)}{0\cdot 5}$$
$$=\frac{0\cdot 00621}{0\cdot 5}$$
$$=0\cdot 01242$$

The answer is B.

Question 10

$$y=x^4-5x^2+4$$
$$=(x^2-4)(x^2-1)$$
$$=(x-2)(x+2)(x-1)(x+1)$$



The graph is symmetrical about the y-axis.
Use your calculator to find the minimum turning points.
They occur at $(-1\cdot 58114,-2\cdot 25)$ and, by symmetry at $(1\cdot 58114,-2\cdot 25)$.
The gradient is positive for $x\in(-1\cdot 5811...,0)\cup(1\cdot 5811...,\infty)$
The closest answer is D.

Question 11

$$\text{Average rate of change}=\frac{f(2)-f(1)}{2-1}$$
$$=\log_e(5)-\log_e(3)$$
$$=\log_e\left(\frac{5}{3}\right)$$

The answer is B.

Question 12

Option A is incorrect because h is discontinuous at $x=1$ and $x=3$.
Option B is incorrect because h does not exist at $x=3$.
Option C is incorrect because $h(x)\le 0$ for $x\in[-3,0]$.
Option D is correct because $h(x)$ exists at $x=1$.
Option E is incorrect because $h'(1)$ does not exist. This is because the limits for $h'(x)$ from the left and right hand side of $x=1$ are not equal.

The answer is D.

Question 13

Let $\begin{bmatrix} x' \\ y' \end{bmatrix}$ represent an image point.

$$\begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} 2 \\ -3 \end{bmatrix}=\begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} x'-2 \\ y'+3 \end{bmatrix}$$

$$\begin{aligned}\begin{bmatrix} x \\ y \end{bmatrix}&=\frac{-1}{4}\begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x'-2 \\ y'+3 \end{bmatrix}\\ &=-\frac{1}{4}\begin{bmatrix} 4(x'-2) \\ -1(y'+3) \end{bmatrix}\\ &=\begin{bmatrix} 2-x' \\ \frac{y'+3}{4} \end{bmatrix}\end{aligned}$$

So $\quad y=x^2+1$

becomes $\dfrac{y'+3}{4}=(2-x')^2+1$

$$y'+3=4(2-x')^2+4$$

$$y'=4(2-x')^2+1$$

The equation of the image is $y=4(2-x)^2+1$.
The answer is E.

Question 14

$\frac{dV}{dt} = 5$

$V = \frac{4}{3}\pi r^3$ (from formula sheet)

$\frac{dV}{dr} = 4\pi r^2$

$\frac{dr}{dt} = \frac{dr}{dV} \cdot \frac{dV}{dt}$

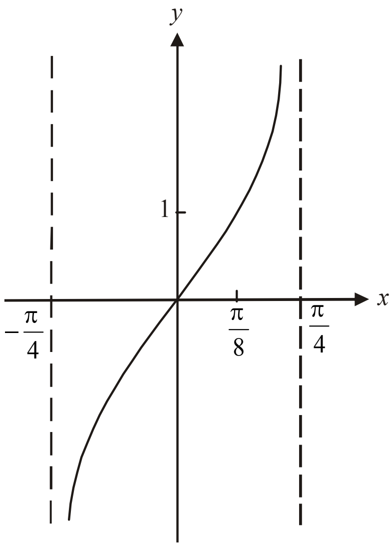
$= \frac{1}{4\pi r^2} \cdot 5$

$= \frac{5}{4\pi r^2}$

The answer is B.

Question 15

Do a quick sketch.



A dilation by a factor of 3 from the x-axis changes the rule $y = \tan(2x)$ to become

$\frac{y}{3} = \tan(2x)$

$y = 3\tan(2x)$

A reflection in the x-axis changes the rule to

$-y = 3\tan(2x)$

$y = -3\tan(2x)$

Note that the domain is not affected by these two transformations.

The answer is C.

Question 16

Since we have a probability density function,

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_0^1 \frac{x^2}{k}dx = 1$$

$$\frac{1}{k}\left[\frac{x^3}{3}\right]_0^1 = 1$$

$$\left[\frac{x^3}{3}\right]_0^1 = k$$

$$\frac{1}{3} - 0 = k$$

$$k = \frac{1}{3}$$

The answer is A.

Question 17

$$\begin{aligned} f(x) &= \log_e(3x) \\ 2f(x) &= 2\log_e(3x) \\ &= \log_e(3x)^2 \\ &= \log_e(9x^2) \end{aligned}$$

$$\begin{aligned} \text{So} \quad f(y) &= \log_e(9x^2) \\ &= \log_e(3 \times 3x^2) \end{aligned}$$

So $y = 3x^2$

The answer is D.

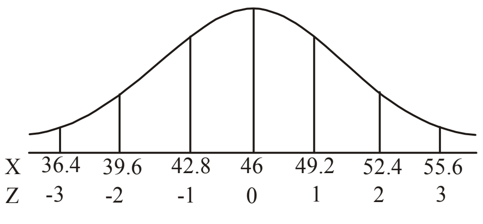
Question 18

Let w = minimum weight required.

$\Pr(X < w) = 0 \cdot 1$

$w = 41 \cdot 899$

The answer is A.



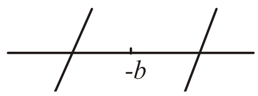
Question 19

If n is small, there will be few rectangles and the approximation would be not as accurate as it could be.
If a is small, b could be large so this does not guarantee an increase in accuracy.
Similarly if b is small a could be very much smaller and an increase in accuracy is not guaranteed.
If h is small then the width of the rectangles will be small which ensures a lot of rectangles, which increases the accuracy of the approximation.
The answer is D.

Question 20

For a stationary point of inflection to occur $f'(x)=0$. This only occurs at $x=-b$ and at $x=d$.

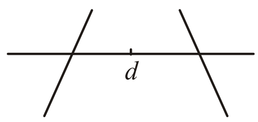
To the left of $x=-b,\ f'(x)>0$ and to the right of $x=-b,\ f'(x)>0$ so we have a stationary point of inflection at $x=-b$.



Note that to the left of $x=d,\ f'(x)>0$ and to the right of $x=d, f'(x)<0$

So we have a local maximum at $x=d$.

The answer is B.



Question 21

$f(x)=\dfrac{1}{\sqrt{x-1}}$, has a maximal domain.

That maximal domain is given by

$$x-1>0$$
$$x>1$$

So $d_f=(1,\infty)$.

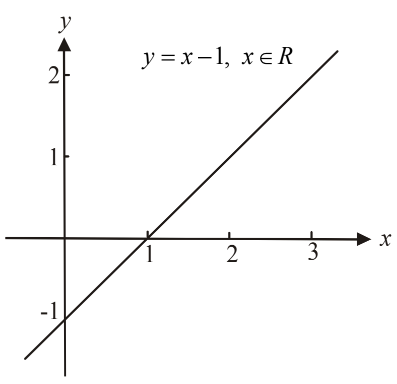
$f(g(x))$ exists iff $r_g\subseteq d_f$

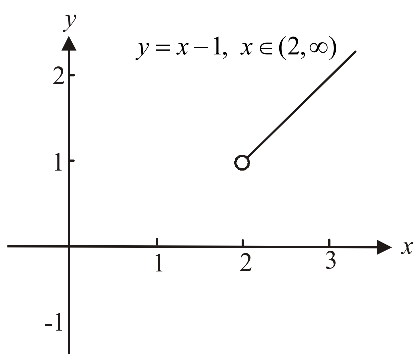
So we require $r_g\subseteq(1,\infty)$

<u>Method 1</u>

The graph of $y=x-1$ for $x\in R$ is shown in the diagram on the left below.

In order to restrict the range to $(1,\infty)$, we are going to have to restrict the domain to $x\in(2,\infty)$ as shown on the graph on the right below.





Since $r_g\subseteq(1,\infty),\ d_g\subseteq(2,\infty)$.

So $a\neq -2,-1,0$ or 1

So $a=2$ is the only possible answer.

The answer is E.

Method 2

We require $r_g \subseteq (1, \infty)$

$$r_g = (a-1, \infty)$$

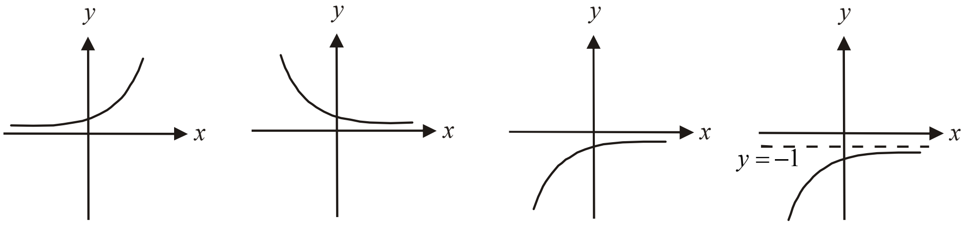
$$a - 1 \geq 1$$

$$a \geq 2$$

So $a = 2$ is the only possible answer.

The answer is E.

Question 22



$y = e^x$ $y = e^{-x}$ $y = -e^{-x}$ $y = -e^{-x} - 1$

The only possible graph is E.
The answer is E.

SECTION 2

Question 1

a. The graph of $y = f(x)$ passes through $(0,0)$

so, $0 = e^{a\times 0} - b$

$b = e^0$

$b = 1$ as required

(1 mark)

b. $f(x) = e^{ax} - 1$

$f'(x) = ae^{ax}$ **(1 mark)**

$f'(1) = ae^a$

Since $f'(1) = 2e^2$ (given)

then $a = 2$

(1 mark)

c. i. Note that the equation of the asymptote of $y = e^{2x} - 1$ is $y = -1$.

Method 1

The graph of $y = f(x)$ has undergone a reflection in the x-axis followed by a translation of 1 unit down. **(1 mark)** – reflection **(1 mark)** – translation

Method 2

The graph of $y = f(x)$ has undergone a translation of 1 unit up followed by a reflection in the x-axis. **(1 mark)** – reflection **(1 mark)** – translation

ii. Method 1

After the reflection in the x-axis the rule $y = e^{2x} - 1$ becomes $-y = e^{2x} - 1$ so $y = 1 - e^{2x}$. **(1 mark)**

After the translation of 1 unit down, $y = 1 - e^{2x}$ becomes $y = 1 - e^{2x} - 1 = -e^{2x}$. So $g(x) = -e^{2x}$ as required. **(1 mark)**

Method 2

After the translation of 1 unit up, the rule $y = e^{2x} - 1$ becomes $y = e^{2x} - 1 + 1$ so $y = e^{2x}$. **(1 mark)**

After the reflection in the x-axis, $y = e^{2x}$ becomes $-y = e^{2x}$ so $y = -e^{2x}$.

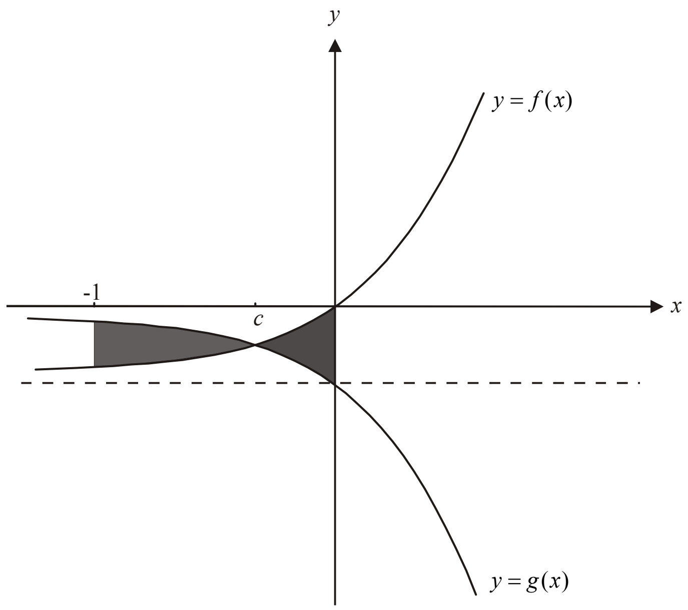
So $g(x) = -e^{2x}$ as required. **(1 mark)**

d. Solve $f(x) = g(x)$ **(1 mark)**

$x = -\frac{1}{2}\log_e(2)$

So $c = -\frac{1}{2}\log_e(2)$ as required. **(1 mark)**

e.



$$\text{Area required} = \int_{-1}^{c} (g(x) - f(x))dx + \int_{c}^{0} (f(x) - g(x))dx$$

OR

$$\int_{-1}^{-\frac{1}{2}\log_e(2)} (g(x) - f(x))dx + \int_{-\frac{1}{2}\log_e(2)}^{0} (f(x) - g(x))dx$$

(1 mark) for first integrand and terminals **(1 mark)** – for second integrand and terminals

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f. **i.**

$$f(x) = e^{2x} - 1$$

$$f\left(\frac{u+v}{2}\right) = e^{2\left(\frac{u+v}{2}\right)} - 1$$

$$= e^{(u+v)} - 1 \text{ as required}$$

(1 mark)

ii. To show

$$\left(f\left(\frac{u+v}{2}\right)\right)^2 = f(u+v) - 2f\left(\frac{u+v}{2}\right)$$

$$\left(f\left(\frac{u+v}{2}\right)\right)^2 = (e^{(u+v)} - 1)^2 \quad \text{from part i.}$$ **(1 mark)**

$$= e^{2(u+v)} - 2e^{(u+v)} + 1$$

$$= f(u+v) + 1 - 2\left\{f\left(\frac{u+v}{2}\right) + 1\right\} + 1$$ **(1 mark)**

$$= f(u+v) + 1 - 2f\left(\frac{u+v}{2}\right) - 2 + 1$$

$$= f(u+v) - 2f\left(\frac{u+v}{2}\right)$$

as required

(1 mark)

Total 15 marks

Question 2

a. **i.** Pr(rides on next 5 days)

$= 0 \cdot 6^5$

$= 0 \cdot 0778$ (correct to 4 decimal places)

(1 mark)

ii. This is a binomial distribution with $n = 5, x = 2$ and $p = 0 \cdot 6$.

Pr(rides 2 out of next 5 days)

$= \Pr(X = 2)$

$= {}^5C_2 (0 \cdot 6)^2 (0 \cdot 4)^3$ **(1 mark)**

$= 0 \cdot 2304$

(1 mark)

b. **i.** $\Pr(wwww) = 0 \cdot 4^4$

$= 0 \cdot 0256$

(1 mark)

ii. $\Pr(wcc) + \Pr(cwc) + \Pr(ccw)$ **(1 mark)**

$= 0 \cdot 4 \times 0 \cdot 6 \times 0 \cdot 7 + 0 \cdot 6 \times 0 \cdot 3 \times 0 \cdot 6 + 0 \cdot 6 \times 0 \cdot 7 \times 0 \cdot 3$ **(1 mark)**

$= 0 \cdot 402$

(1 mark)

iii. Use the transition matrix

one day

$$\begin{matrix} & w & c & \\ & \begin{bmatrix} 0 \cdot 4 & 0 \cdot 3 \\ 0 \cdot 6 & 0 \cdot 7 \end{bmatrix} & \begin{matrix} w \\ c \end{matrix} & \text{next day} \end{matrix}$$

(1 mark) – for transition matrix

Pr(walked on 10th day of term)

$= \Pr(\text{walks next 9 days})$

$= \begin{bmatrix} 0 \cdot 4 & 0 \cdot 3 \\ 0 \cdot 6 & 0 \cdot 7 \end{bmatrix}^9 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ **(1 mark)** – for giving matrix expression including the power of 9

$= \begin{bmatrix} 0 \cdot 3333... \\ 0 \cdot 6666... \end{bmatrix}$

So the probability that Jordan walked on the tenth day of term is 0.3333(correct to 4 decimal places).

(1 mark) – correct answer

iv. <u>Method 1</u>

Pr(walks over long term)

$= \dfrac{0 \cdot 3}{0 \cdot 6 + 0 \cdot 3}$

$= \dfrac{1}{3}$

Over the long term Jordan will walk on $\left(\dfrac{1}{3} \times \dfrac{100}{1}\right)\% = 33 \cdot 33\%$ (correct to 2 decimal places).

(1 mark)

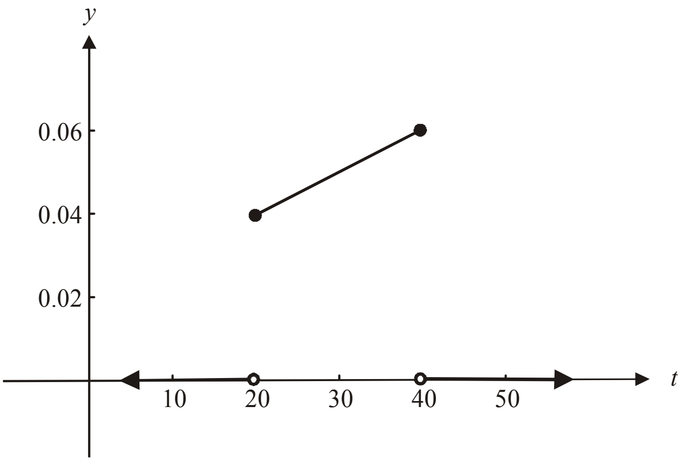
Method 2
Find the steady state.
Let $n = 20$

$$\begin{bmatrix} 0\cdot 4 & 0\cdot 3 \\ 0\cdot 6 & 0\cdot 7 \end{bmatrix}^{20} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0\cdot 3333... \\ 0\cdot 6666... \end{bmatrix}$$

Using our result from part **iii.,** we see that a steady state has been reached so the probability that Jordan will walk over the long term is 33.33% (correct to 2 decimal places). **(1 mark)**

c. **i.**



(1 mark) correct linear function and included endpoints for $20 \le t \le 40$
(1 mark) correct marking of function along t-axis

ii. From the graph, we see that the mode is 40; that is, the value of t with the highest probability; that is, the highest value of $f(t)$.
(1 mark)

iii. Let $m =$ median

Solve $\int_{20}^{m} f(t)\, dt = 0\cdot 5$ **(1 mark)**

$m = -70\cdot 9902$ **(1 mark)**
or $m = 30\cdot 9902$

Since $20 \le m \le 40$, $m = 31$ minutes (to the nearest minute).

(1 mark) correct answer

Total 17 marks

Question 3

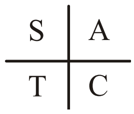
a. $f(x)=\cos(2x)$

$f'(x)=-2\sin(2x)$

(1 mark)

b. The gradient of the tangent is -1 when

$$f'(x)=-1$$
$$-2\sin(2x)=-1$$
$$\sin(2x)=\frac{1}{2}$$



$$2x=\frac{\pi}{6},\frac{5\pi}{6} \qquad x\in\left[0,\frac{\pi}{2}\right] \quad \text{so } 2x\in[0,\pi]$$
$$x=\frac{\pi}{12},\frac{5\pi}{12}$$

(1 mark)

$$f\left(\frac{\pi}{12}\right)=\cos\left(\frac{\pi}{6}\right) \qquad f\left(\frac{5\pi}{12}\right)=\cos\left(\frac{5\pi}{6}\right)$$
$$=\frac{\sqrt{3}}{2} \qquad\qquad =-\frac{\sqrt{3}}{2}$$

Required points are $\left(\frac{\pi}{12},\frac{\sqrt{3}}{2}\right)$ and $\left(\frac{5\pi}{12},-\frac{\sqrt{3}}{2}\right)$.

(1 mark)

c. **i.** Tangent passes through $\left(\frac{\pi+\sqrt{3}}{6},0\right)$ and $\left(0,\frac{\sqrt{3}\pi+3}{6}\right)$.

Method 1

$$m=\frac{y_2-y_1}{x_2-x_1}=\left(\frac{\sqrt{3}\pi+3}{6}-0\right)\div\left(0-\frac{\pi+\sqrt{3}}{6}\right)$$
$$=-\left(\frac{\sqrt{3}\left(\pi+\sqrt{3}\right)}{\pi+\sqrt{3}}\right)$$
$$=-\sqrt{3} \quad \text{as required}$$

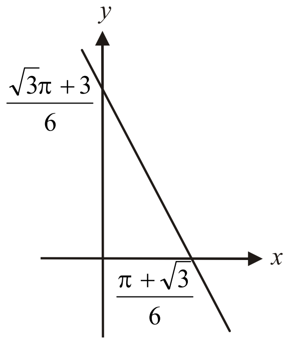
(1 mark)

Method 2

Note that the gradient must be negative since the tangent is sloping up to the left.

Gradient of tangent is given by

$$\frac{\text{rise}}{\text{run}}$$
$$=-\left(\frac{\sqrt{3}\pi+3}{6}\div\frac{\pi+\sqrt{3}}{6}\right)$$
$$=-\left(\frac{\sqrt{3}\pi+3}{6}\times\frac{6}{\pi+\sqrt{3}}\right)$$
$$=-\left(\frac{\sqrt{3}\left(\pi+\sqrt{3}\right)}{\pi+\sqrt{3}}\right)$$
$$=-\sqrt{3} \quad \text{as required}$$



(1 mark)

ii. $f'(x) = -2\sin(2x) = -\sqrt{3}$ $\quad x \in \left[0, \frac{\pi}{2}\right]$ **(1 mark)**

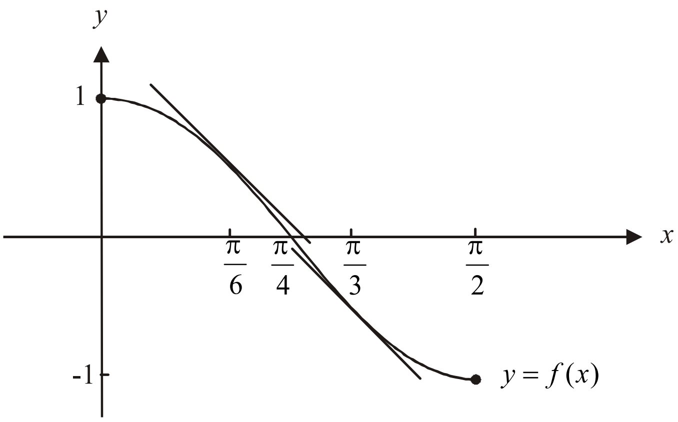
$$\sin(2x) = \frac{\sqrt{3}}{2} \qquad 2x \in [0, \pi]$$

$$2x = \frac{\pi}{3}, \frac{2\pi}{3}$$

$$x = \frac{\pi}{6}, \frac{\pi}{3}$$

(1 mark)

iii.



The x-intercept of the tangent we require is $\frac{\pi + \sqrt{3}}{6} = 0.8122...$

Now $\frac{\pi}{4} = 0.7853...$

From the diagram above, we can see that the tangent we require must pass through the point where $x = \frac{\pi}{6}$. The other possible tangent which passes through the point where $x = \frac{\pi}{3}$, has an x-intercept which is less than $\frac{\pi}{4}$ and therefore less than $\frac{\pi + \sqrt{3}}{6}$. **(1 mark)**

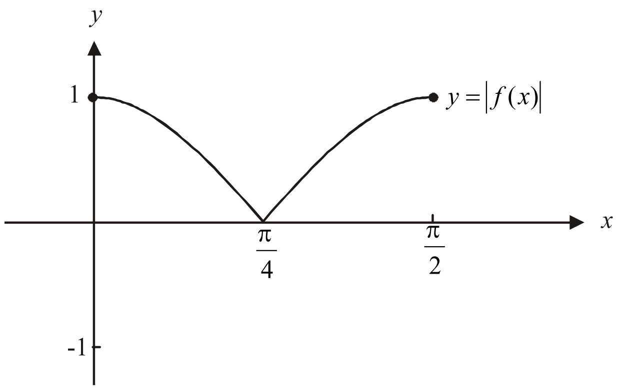
$$f(x) = \cos(2x)$$

$$f\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{3}\right)$$

$$= \frac{1}{2}$$

Point of tangency is $\left(\frac{\pi}{6}, \frac{1}{2}\right)$. **(1 mark)**

d. Do a quick sketch of the graph of $y=|f(x)|$.



The minimum value of $|f(x)|=0$. **(1 mark)**

The maximum value of $|f(x)|=1$. **(1 mark)**

e. $g: R \to R, g(x)=\cos(2x)$
$g(x)=0\cdot 5$

<u>Method 1</u> – using CAS
Solve $\cos(2x)=0\cdot 5$

$$x=\frac{(6n+1)\pi}{6} \text{ or } \frac{(6n-1)\pi}{6}$$

(1 mark)

When $n=0,\ x=\frac{\pi}{6},\frac{-\pi}{6}$

When $n=-1,\ x=\frac{-5\pi}{6},\frac{-7\pi}{6}$

When $n=1,\ x=\frac{7\pi}{6},\frac{5\pi}{6}$

When $n=-2,\ x=\frac{-11\pi}{6},\frac{-13\pi}{6}$

The values of x start to repeat.

$$x=\frac{\pi}{6}+n\pi \text{ or } x=\frac{5\pi}{6}+n\pi,\ n\in Z$$

Alternatively, $x=n\pi\pm\frac{\pi}{6},\quad n\in Z$.

(1 mark)

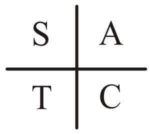
Method 2

Solve $\cos(2x) = 0 \cdot 5$ for $x \in [0, \pi]$ that is, for one period of the graph of $y = \cos(2x)$.

$\cos 2x = 0 \cdot 5 \quad x \in [0, \pi]$

$2x = \frac{\pi}{3}, \frac{5\pi}{3} \quad 2x \in [0, 2\pi]$

$x = \frac{\pi}{6}, \frac{5\pi}{6} \quad$ for $x \in [0, \pi]$ **(1 mark)**



For $x \in R$,

$$x = \frac{\pi}{6} + n\pi \text{ or } x = \frac{5\pi}{6} + n\pi, \quad n \in Z.$$

Alternatively,

$$x = n\pi \pm \frac{\pi}{6}, \quad n \in Z.$$

(1 mark)
Total 12 marks

Question 4

a. Solve $f(x) = 0$

$$5\log_e(x-10) = 0$$

$$e^0 = x - 10$$

$$1 = x - 10$$

$$x = 11$$

So $a = 11$ as required.

(1 mark)

b. Method 1 – using CAS

Find the inverse function of $y = 5\log_e(x-10)$

So $f^{-1}(x) = e^{\frac{x}{5}} + 10$ **(1 mark)** – correct rule

Method 2 – by hand

$f(x) = 5\log_e(x-10)$

Let $y = 5\log_e(x-10)$

Swap *x* and *y* for inverse.

$$x = 5\log_e(y-10)$$

$$\frac{x}{5} = \log_e(y-10)$$

$$e^{\frac{x}{5}} = y - 10$$

$$y = e^{\frac{x}{5}} + 10$$

$$f^{-1}(x) = e^{\frac{x}{5}} + 10$$

(1 mark) – correct rule

$$d_f = [11, 50]$$

$$r_f = [0, f(50)]$$

$$= [0,\ 18 \cdot 44...]$$

So $d_{f^{-1}} = r_f$

$$= [0,\ 18 \cdot 4] \quad \text{(correct to 1 decimal place)}$$

(1 mark) – correct domain

c. Since the graph of $y = f^{-1}(x)$ is a reflection of the graph of $y = f(x)$ in the line $y = x$, the floodlit areas to the north and to the east are the same.

$$\text{Total area} = 2\int_{11}^{50} f(x)\,dx$$

(1 mark)

$$= 10\int_{11}^{50} \log_e(x-10)dx$$

$$= 10\ 86\text{m}^2 \quad \text{(to nearest square metre)}$$

(1 mark)

d. Find the x-coordinates of the points of intersection between $y=15$ and $y=f(x)$ and between $y=15$ and $y=f^{-1}(x)$.

Method 1 – using CAS

$y=15$ and $y=5\log_e(x-10)$ intersect when $x=30\cdot 09$ (to 2 decimal places)

$y=15$ and $y=e^{\frac{x}{5}}+10$ intersect when $x=8\cdot 05$ (to 2 decimal places)

So $b\in(8\cdot 05, 30\cdot 09)$ or $8\cdot 05<b<30\cdot 09$

(1 mark) correct values
(1 mark) correct brackets or inequality signs

Method 2 – by hand

$$15=5\log_e(x-10) \qquad 15=e^{\frac{x}{5}}+10$$
$$3=\log_e(x-10) \qquad 5=e^{\frac{x}{5}}$$
$$e^3=x-10 \qquad \log_e(5)=\frac{x}{5}$$
$$x=10+e^3 \qquad x=5\log_e(5)$$
$$=30\cdot 09 \text{ (to 2 dec. places)} \qquad =8\cdot 05 \text{ (to 2 dec. places)}$$

So $b\in(8\cdot 05, 30\cdot 09)$ or $8\cdot 05<b<30\cdot 09$

(1 mark) correct values
(1 mark) correct bracket or inequality signs

e. **i.**

$$T=\sqrt{x^2+2500}+\frac{50-x}{2}$$

$$\frac{dT}{dx}=\frac{2x-\sqrt{x^2+2500}}{2\sqrt{x^2+2500}}$$

(1 mark)

$\frac{dT}{dx}=0$ for minimum.

$x=28\cdot 87$ (correct to 2 decimal places)

(1 mark)

ii. $T=68\cdot 3$ seconds correct to 1 decimal place.

(1 mark)

f. To take the quickest path, Victoria runs to $P(28\cdot 87, 50)$ from part **e. i.**

The straight line from $O(0,0)$ to $P(28\cdot 87, 50)$ is given by $y=\frac{50}{28\cdot 87}x$. **(1 mark)**

Check whether this intersects with the perimeter of the floodlit area to the north given by $f^{-1}(x)=e^{\frac{x}{5}}+10$.

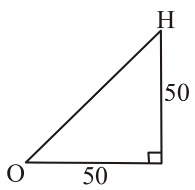
Solve $e^{\frac{x}{5}}+10=\frac{50}{28\cdot 87}x$.

There are two solutions $x=10\cdot 3476$ or $x=11\cdot 2262$.

So Victoria does enter the floodlit area.

(1 mark)

g. From part **e. ii.** the shortest time it takes is 68.3 seconds (to 1 decimal place).
The dogs have to run $\sqrt{50^2+50^2}=50\sqrt{2}\text{m}$.
They run at 7m/sec.
It takes the dogs



$50\sqrt{2}\text{m}\div\frac{7\text{m}}{\text{sec}}$

$=\frac{50\sqrt{2}}{7}\text{secs}$

$=10\cdot1015\text{...secs}$

to get to *H* from *O*. **(1 mark)**

The dogs leave 60 secs after Victoria so they arrive 70.1015…secs after she leaves $O(0,0)$.

So Victoria escapes the dogs; but only just!

(1 mark)

Total 14 marks