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MATHS METHODS (CAS) 3 & 4
TRIAL EXAMINATION 1
SOLUTIONS
2009

Question 1

$f : R \to R,\ f(x) = x + 1$
$g : (0, \infty) \to R,\ g(x) = \log_e(2x)$

a. $f(g(x)) = f(\log_e(2x))$
$= \log_e(2x) + 1$

(1 mark)

b. $g(f(x))$ exists iff $r_f \subseteq d_g$.
Now $r_f = R$ and $d_g = (0, \infty)$
Since $R \not\subset (0, \infty)$, $g(f(x))$ does not exist.

(1 mark)

Question 2

a. $f(x) = x \log_e(x^2 + 5)$

$$f'(x) = x \times \frac{2x}{x^2 + 5} + \log_e(x^2 + 5)$$
$$= \frac{2x^2}{x^2 + 5} + \log_e(x^2 + 5)$$

(1 mark) – use of product rule
(1 mark) – correct derivative

b. $y = \dfrac{\tan(x)}{e^{2x}}$

$$\frac{dy}{dx} = \frac{e^{2x} \times \sec^2(x) - 2e^{2x}\tan(x)}{e^{4x}}$$

(1 mark) use of quotient rule

When $x = 0$,

(1 mark) substituting $x = 0$

$$\frac{dy}{dx} = \frac{e^0 \times \sec^2(0) - 2e^0 \tan(0)}{e^0}$$
$$= \frac{1 \times \frac{1}{\cos^2(0)} - 2 \times 1 \times 0}{1}$$
$$= 1$$

(1 mark) – correct answer

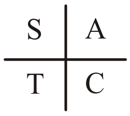
Question 3

$\sqrt{3}\tan(2x)=1 \qquad 0\le x\le 2\pi$

$\tan(2x)=\frac{1}{\sqrt{3}} \qquad 0\le 2x\le 4\pi$

$2x=\frac{\pi}{6},\frac{7\pi}{6},\frac{13\pi}{6},\frac{19\pi}{6}$

$x=\frac{\pi}{12},\frac{7\pi}{12},\frac{13\pi}{12},\frac{19\pi}{12}$



(1 mark) – for $\frac{\pi}{12}$

(1 mark) – for remaining 3 correct answers

Question 4

a. **i.** Method 1 – using a probability table or Karnaugh map.

This is what is given.

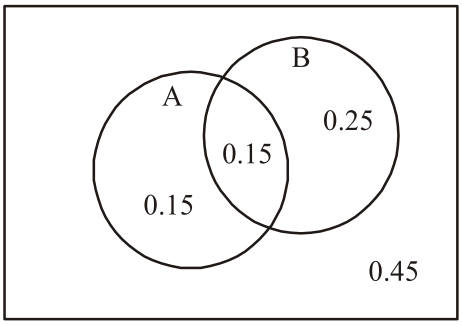
	A	A'	
B	0.15		0.4
B'			
	0.3		1

This is what we can work out.

	A	A'	
B	0.15	0.25	0.4
B'	0.15	0.45	0.6
	0.3	0.7	1

$\Pr(A'\cap B')=0\cdot 45$ **(1 mark)**

Method 2 – using a Venn Diagram



$\Pr(A'\cap B')=0\cdot 45$ **(1 mark)**

Method 3 –using Addition rule

$$
\begin{aligned}
\Pr(A\cup B)&=\Pr(A)+\Pr(B)-\Pr(A\cap B)\\
&=0.3+0.4-0.15\\
&=0.55\\
\Pr(A'\cap B')&=\Pr(A\cup B)'\\
&=1-\Pr A\cup B)\\
&=1-0.55\\
&=0.45
\end{aligned}
$$

(1 mark)

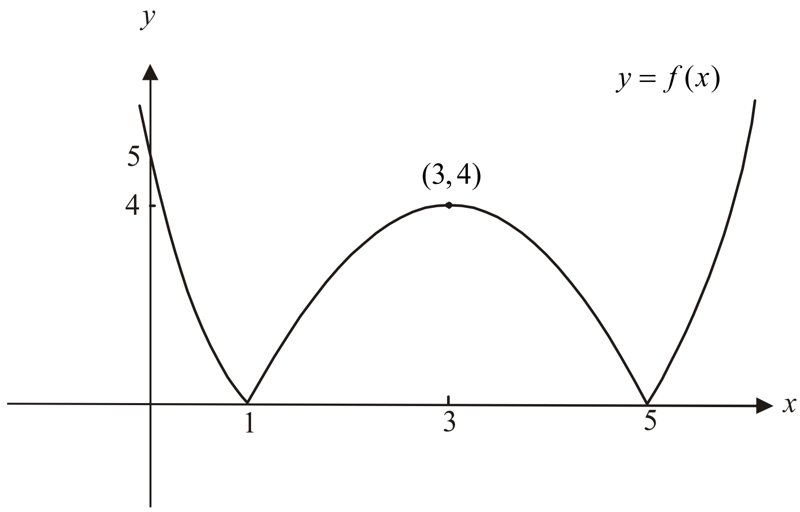
ii.

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$
$$= \frac{0.15}{0.4}$$
$$= \frac{3}{8}$$ **(1 mark)**

b. If A and B are mutually exclusive then $\Pr(A \cap B) = 0$ so $\Pr(A|B) = 0$. **(1 mark)**

Question 5

a.



(1 mark) – correct shape including cusps at $x = 1$ and $x = 5$
(1 mark) correct labelling of intercepts and turning point

b. $d_{f'} = R \setminus \{1, 5\}$ **(1 mark)**

c. $f'(x) > 0$ for $x \in (1,3) \cup (5, \infty)$ **(1 mark)**

Question 6

a. $\Pr(X < 2) = \int_1^2 \frac{1}{2\sqrt{x}} dx$ **(1 mark)**

$$= \frac{1}{2}\int_1^2 x^{-\frac{1}{2}} dx$$
$$= \frac{1}{2}\left[2x^{\frac{1}{2}}\right]_1^2$$
$$= \frac{1}{2}\left(2\sqrt{2} - 2\sqrt{1}\right)$$
$$= \sqrt{2} - \sqrt{1}$$
$$= \sqrt{2} - 1$$ **(1 mark)**

b.

$$\mu = \int_{-\infty}^{\infty} xf(x)dx$$

$$= \int_1^4 x \times \frac{1}{2\sqrt{x}}dx$$

(1 mark)

$$= \frac{1}{2}\int_1^4 x^{\frac{1}{2}}\,dx$$

$$= \frac{1}{2}\left[\frac{2x^{\frac{3}{2}}}{3}\right]_1^4$$

$$= \frac{1}{3}\left[x^{\frac{3}{2}}\right]_1^4$$

$$= \frac{1}{3}(4^{\frac{3}{2}} - 1^{\frac{3}{2}})$$

$$= \frac{1}{3}(2^3 - 1)$$

$$= \frac{1}{3}(8 - 1)$$

$$= \frac{7}{3}$$

(1 mark)

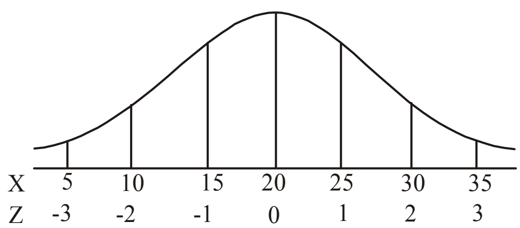
Question 7

a. From the diagram,

$$\Pr(X > 20) = \frac{1}{2}$$

$$\Pr(Z < 0) = \frac{1}{2}$$

$$m = 0$$



(1 mark)

b. $\Pr(X < 18) = \Pr(Z > n)$

Again from the diagram,

$\Pr(X < 18) = \Pr(X > 22)$ by symmetry

(1 mark)

Since $z = \dfrac{x - \mu}{\sigma}$

$$z = \frac{22 - 20}{5}$$

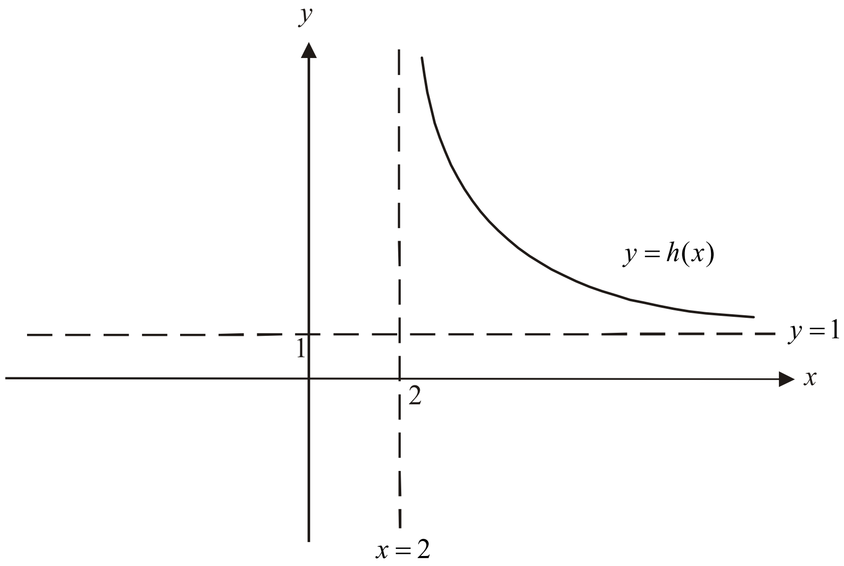
$$= 0 \cdot 4$$

So $n = 0 \cdot 4$

(1 mark)

Question 8

a.



(1 mark) – correct shape of graph
(1 mark) – correct asymptotes

b. $h(x)=\frac{1}{x-2}+1$

Let $y=\frac{1}{x-2}+1$

Swap x and y for inverse

$$x=\frac{1}{y-2}+1$$

Rearrange

$$x-1=\frac{1}{y-2}$$

$$(x-1)(y-2)=1$$

$$y-2=\frac{1}{x-1}$$

$$y=\frac{1}{x-1}+2$$

So $h^{-1}(x)=\frac{1}{x-1}+2$ **(1 mark)** – correct rule

$r_h=(1,\infty)$ (from graph)

So $d_{h^{-1}}=r_h=(1,\infty)$ **(1 mark)** – correct domain

Question 9

a.

$$y = \frac{2}{x}$$
$$= 2x^{-1}$$
$$\frac{dy}{dx} = -2x^{-2}$$
$$= \frac{-2}{x^2}$$

When $x = 2$

$$\frac{dy}{dx} = \frac{-2}{4}$$
$$= -\frac{1}{2}$$

The gradient of the tangent to f at $x = 2$ is $-\frac{1}{2}$.

Therefore the gradient of the normal to f at $x = 2$ is 2. **(1 mark)**

Also $f(2) = \frac{2}{2}$
$= 1$

The equation of the normal through $(2,1)$ with gradient of 2 is

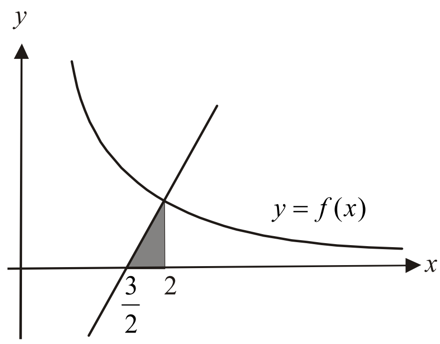
$$y - 1 = 2(x - 2)$$
$$y = 2x - 3$$

(1 mark)

b. The normal crosses the x-axis when $y = 0$.

$$y = 2x - 3$$
$$0 = 2x - 3$$
$$x = \frac{3}{2}$$

(1 mark)



Method 1

$$\text{Area} = \int_{\frac{3}{2}}^{2} (2x - 3)\,dx$$
$$= [x^2 - 3x]_{\frac{3}{2}}^{2}$$
$$= \left\{(4 - 6) - \left(\frac{9}{4} - \frac{9}{2}\right)\right\}$$
$$= -2 - \frac{-9}{4}$$
$$= \frac{-8}{4} + \frac{9}{4}$$
$$= \frac{1}{4} \text{ units}^2$$

(1 mark)

Method 2

$$\text{Area} = \frac{1}{2} \times \text{base} \times \text{height}$$
$$= \frac{1}{2} \times (2 - \frac{3}{2}) \times f(2)$$
$$= \frac{1}{2} \times \frac{1}{2} \times 1$$
$$= \frac{1}{4} \text{ units}^2$$

(1 mark)

Question 10

Method 1

Given $f(x) = x\sqrt{1-x}$ and $f'(x) = \dfrac{-x}{2\sqrt{1-x}} + \sqrt{1-x}$

Therefore $\displaystyle\int\left(\frac{-x}{2\sqrt{1-x}} + \sqrt{1-x}\right)dx = x\sqrt{1-x} + c$ **(1 mark)**

$$\text{so } -\frac{1}{2}\int\frac{x}{\sqrt{1-x}}\,dx + \int\sqrt{1-x}\ dx = x\sqrt{1-x} + c$$

$$-\frac{1}{2}\int\frac{x}{\sqrt{1-x}}\,dx = x\sqrt{1-x} - \int\sqrt{1-x}\ dx + c$$

$$\int\frac{x}{\sqrt{1-x}}\,dx = 2\int(1-x)^{\frac{1}{2}}\ dx - 2x\sqrt{1-x} - 2c$$

$$= 2\times\frac{1}{-1\times\frac{3}{2}}(1-x)^{\frac{3}{2}} - 2x\sqrt{1-x} - 2c$$

$$= \frac{-4}{3}\sqrt{(1-x)^3} - 2x\sqrt{1-x} \quad \text{where } c = 0 \text{ for an antiderivative}$$

$$\text{or } = \frac{-4}{3}(1-x)^{\frac{3}{2}} - 2x(1-x)^{\frac{1}{2}}$$

(1 mark) correct antiderivative of $\sqrt{1-x}$ **(1 mark)** correct answer

Method 2

Given $f(x) = x\sqrt{1-x}$ and $f'(x) = \dfrac{-x}{2\sqrt{1-x}} + \sqrt{1-x}$

$$\text{So, } \frac{x}{2\sqrt{1-x}} = \sqrt{1-x} - f'(x)$$

$$\frac{x}{\sqrt{1-x}} = 2\sqrt{1-x} - 2f'(x)$$

$$\int\frac{x}{\sqrt{1-x}}dx = 2\int\sqrt{1-x}\,dx - 2\int f'(x)dx$$ **(1 mark)**

$$= 2\times\frac{1}{-1\times\frac{3}{2}}(1-x)^{\frac{3}{2}} - 2\times x\sqrt{1-x} + c$$

$$= \frac{-4}{3}\sqrt{(1-x)^3} - 2x\sqrt{1-x} \quad \text{where } c = 0 \text{ for an antiderivative}$$

$$\text{or } = \frac{-4}{3}(1-x)^{\frac{3}{2}} - 2x(1-x)^{\frac{1}{2}}$$

(1 mark) correct antiderivative of $\sqrt{1-x}$ **(1 mark)** correct answer

Question 11

a. Perimeter $= 2x + 2y + \frac{1}{2} \times 2\pi x$ **(1 mark)**

So $100 = 2x + 2y + \pi x$

$$2y = 100 - 2x - \pi x$$

$$2y = 100 - x(\pi + 2)$$

$$y = \frac{100 - x(\pi + 2)}{2}$$

(1 mark)

b. Surface area $= 2xy + \frac{1}{2} \times \pi x^2$

$$= 2x\frac{(100 - x(\pi + 2))}{2} + \frac{\pi x^2}{2}$$ from part **a.**

$$= 100x - x^2(\pi + 2) + \frac{\pi x^2}{2}$$

$$= 100x - \pi x^2 - 2x^2 + \frac{\pi x^2}{2}$$

$$= 100x - \frac{\pi x^2}{2} - 2x^2$$

So $A = 100x - \frac{x^2}{2}(\pi + 4)$

(1 mark)

c. $A = 100x - \frac{x^2}{2}(\pi + 4)$

Max/min occur when $\frac{dA}{dx} = 0$. **(1 mark)**

$$\frac{dA}{dx} = 100 - x(\pi + 4) = 0$$

$$100 = x(\pi + 4)$$

$$x = \frac{100}{\pi + 4} \text{ m}$$

(1 mark)

d. Method 1 – using part **b.**

$A = 100x - \frac{x^2}{2}(\pi + 4)$

This is the equation of an inverted parabola which has a local maximum and hence there will be a maximum rather than a minimum value to be found.

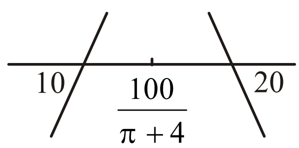
(1 mark) for reference to inverted parabola

Method 2 – using a sign diagram

$x = \frac{100}{\pi + 4} = 14.0...$

At $x = 10$, $\frac{dA}{dx} = 28.5... > 0$.

At $x = 20$, $\frac{dA}{dx} = -42.8... < 0$



From the sign diagram we see that we have a maximum surface area.

(1 mark)

Total 40 marks