MATHEMATICAL METHODS (CAS)

Units 3 & 4 – Written examination 2

2008 Trial Examination

SOLUTIONS

SECTION 1: Multiple-choice questions (1 mark each)

Question 1

Answer: D

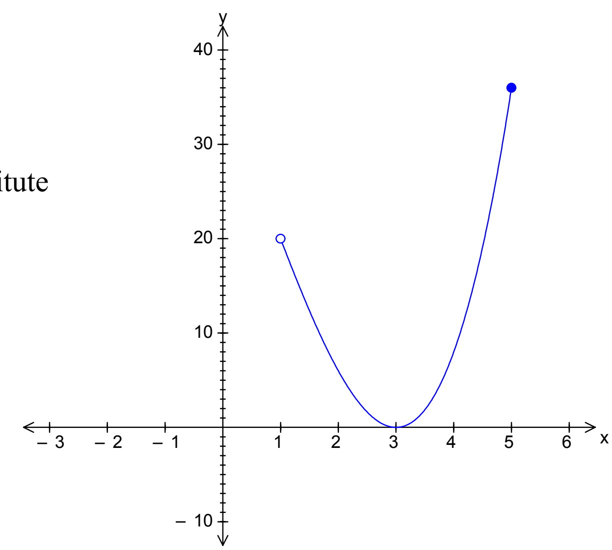
Explanation:

Factorised $y = x^2(4 - x^2) = x^2(2 - x)(2 + x)$ giving TP at $x = 0$, intercepts at $x = -2, 2$ curve reflected in x axis

Question 2

Answer: B

Explanation: From graph min range = 0 and substitute endpoint $x = 5$ for max = 36



Question 3

Answer: B

Explanation:

Solving this equation yields the two general solutions:

$x = \frac{\sqrt{3}\pi(6n+1)}{9}$ **(1)** and $x = \frac{\sqrt{3}\pi(6n-1)}{9}$ **(2)**, where $n \in Z$.

Therefore, using **(1)**: $x = \frac{\sqrt{3}\pi}{9}, \frac{7\sqrt{3}\pi}{9}, \frac{13\sqrt{3}\pi}{9}, \frac{19\sqrt{3}\pi}{9}$

Also, using **(2)**: $x = \frac{5\sqrt{3}\pi}{9}, \frac{11\sqrt{3}\pi}{9}, \frac{17\sqrt{3}\pi}{9}, \frac{23\sqrt{3}\pi}{9}$.

Arranging the solutions in order, the seventh solution must be $\frac{19\sqrt{3}\pi}{9}$.

Question 4

Answer: D

Explanation:

To solve, set up the matrix equation:

$$\begin{bmatrix} 2 & 1 & -3 & 0.5 & -0.2 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & -3 & 2 & -0.3 & 0.1 \\ 0.3 & -2 & 7 & -1 & 0.1 \\ 1 & -3 & -4 & -5 & -2 \end{bmatrix} \times \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} -14.24 \\ 9.3 \\ 12.39 \\ 32.3 \\ -3.3 \end{bmatrix}.$$

This yields the solution: $\begin{aligned} a &= 0.1 \\ b &= -1.2 \\ c &= 3.9 \\ d &= -2.4 \\ e &= 1.7 \end{aligned}$. Therefore, $c = 39a$ is correct.

Question 5

Answer: B

Explanation:

When $f(x) = 3x^2 - 7x - 0.5$ and $g(x) = 3x^2 + 0.25x - \sqrt{2}$:
$g(x-1) \approx 3x^2 - 5.75x + 1.34$, so $\sqrt{3}g(x-1) \approx 5.2x^2 - 10.0x + 2.31$.
Therefore, $f\left(\sqrt{3}g(x-1)\right) \approx 81x^4 - 310.5x^3 + 333.32x^2 - 68.54x - 0.64$

Question 6

Answer: A

Explanation:

Rearrange equation $y = e^{2(x-2)} - 2$ makes dilation factor and translations more obvious

Question 7

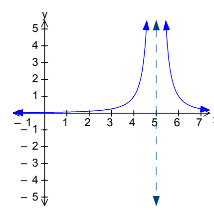
Answer: B

Explanation:

$$x = 0, y = \log_e(3) - 1 \therefore (0, \log_e(3) - 1)$$
$$y = 0, 1 = \log_e(x+3)$$
$$\text{.........}e = x + 3$$
$$\text{........}x = e - 3 \therefore (e - 3, 0)$$

Question 8

Answer: C



Explanation:

Need to restrict domain to make function one-to-one
From options given $x < 5$ *or* $x \in (-\infty, 5)$

Question 9

Answer: D

Explanation:

Reflected in x axis, amplitude 4 therefore $a = -4$, period = 4 therefore $n = \frac{\pi}{2}$, translated down 3 units so $c = -3$

Question 10

Answer: E

Explanation:

Using CAS technology (or the product rule) you find that $f'(x) = \frac{e^{4hx}x^{\frac{1}{2}}}{3}\left(\frac{3}{2} + 4hx\right)$.

Question 11

Answer: D

Explanation:

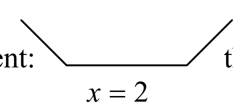
Solving for $\int_{\frac{\pi}{2}}^{\sqrt{7}} \left(x^3 - 7x^2 + xe^x - 2\right)dx$, we get (approximately) -5.143.

Therefore, the average value is found by $\frac{1}{\sqrt{7} - \frac{\pi}{2}} \times -5.143 \approx -4.784$.

Question 12

Answer: E

Explanation:

(2, 6) co-ordinate where gradient: 

therefore a local minimum

Question 13

Answer: C

Explanation:

Graph not smooth at $x = 0$, $x = 3$ gradient negative $x < 0$, gradient changes from positive to zero to negative in $0 < x < 3$, gradient positive $x > 3$ therefore must be C.

Question 14

Answer: A

Explanation:

$$(6x+1) \div (2x-3)$$

$$(2x-3)\overline{)6x+1} \quad 3$$

$$............6x-9$$

$$.................10$$

$$f(x) = \int (3 + \frac{10}{2x-3})dx$$

$$..........3x + \frac{10}{2}\log_e(2x-3) + c$$

$$\therefore 3x + 5\log_e(2x-3) + c$$

Question 15

Answer: E

Explanation:

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$$

$$\frac{dV}{dr} = \frac{4(3)\pi r^2}{3}$$

$$........ = 4\pi r^2$$

$$\frac{dr}{dV} = \frac{1}{4\pi r^2}$$

$$\therefore \frac{dr}{dt} = -\frac{3}{4\pi r^2}$$

Question 16

Answer: B

Explanation:

The only correct option provided for dealing with negative areas. Must integrate between $\boldsymbol{x}$ values and graphs swap at $x = 0$

Question 17

Answer: D

Explanation:

	A	***A'***	
B	*0.35*	*0.3*	*0.65*
B'	*0.25*	***0.1***	*0.35*
	0.6	*0.4*	*1*

The following is also suggested:

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$
$$..........0.9 = 0.6 + 0.65 - \Pr(A \cap B)$$
$$..........0.9 = 1.25 - \Pr(A \cap B)$$
$$........0.35 = \Pr(A \cap B)$$

Question 18

Answer: B

Explanation: (1) $np = 12$, (2) $npq = 4$

$$\frac{npq}{np} = \frac{4}{12}$$

$$(2) \div (1) = q = \frac{1}{3} \therefore p = \frac{2}{3}$$

$$n\left(\frac{2}{3}\right) = 12 \therefore n = 18$$

Question 19

Answer: C

Explanation:

The graph of the derivative has three x-intercepts, so the original function's graph must have three stationary points. Also, $f'(x) < 0$ for $x < -1$, so the graph of $f(x)$ must have a negative gradient for $x < -1$. Therefore, **C** is the appropriate option.

Question 20

Answer: C

Explanation:

T ~ Bi(150, k) and q =1 − k $\therefore\ {}^{150}C_{15}(k)^{15}(1-k)^{135}$

Question 21

Answer: A

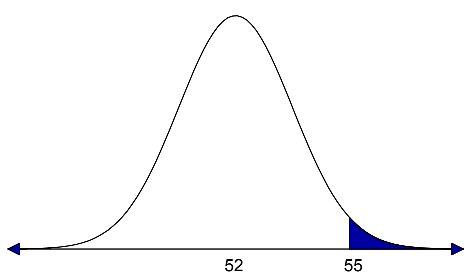
Explanation:

$$\int_0^{\frac{\pi}{4}} (x^2\sqrt{2}\cos(x))dx - \left(\int_0^{\frac{\pi}{4}} (x\sqrt{2}\cos(x))dx\right)^2 = 0.0499$$

Question 22

Answer: B

Explanation:



normalcdf(55, 10^{99},52, 2) = 0.0668
Therefore 7%

SECTION 2: Analysis Questions

Question 1

a. From the y-intercept, $d = 20$. M1

Set up three simultaneous equations with the three x-intercepts:

$x = -5.550 \Rightarrow -170.954a + 30.803b - 5.550c = -20$

When $x = 3.097 \Rightarrow 29.705a + 9.591b + 3.097c = -20$. Therefore, set up the

$x = 17.453 \Rightarrow 5316.310a + 304.607b + 17.453c = -20$

matrix equation: $\begin{bmatrix} -170.954 & 30.803 & -5.550 \\ 29.705 & 9.591 & 3.097 \\ 5316.310 & 304.607 & 17.453 \end{bmatrix} \times \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -20 \\ -20 \\ -20 \end{bmatrix}$ and solve, yielding

$a = 0.067 \approx \frac{1}{15}$, $b = -1$ and $c = -4$. M1

Therefore, $f(x) = \frac{x^3}{15} - x^2 - 4x + 20$ A1

b. (-5.802, −3.478), (3.246, −1.239), (17.941, 11.348)

A3

c. $A = \int_{-5.802}^{3.246} \left(\frac{x^3}{15} - x^2 - x + 20 - (e^{0.1(x+10)} - 5)\right)dx - \int_{3.246}^{17.941} \left(\frac{x^3}{15} - x^2 - x + 20 - (e^{0.1(x+10)} - 5)\right)dx$

$= \left[\frac{x^4}{60} - \frac{x^3}{3} - \frac{4x^2}{2} + 25x - 10e^{0.1(x+10)}\right]_{-5.802}^{3.246} - \left[\frac{x^4}{60} - \frac{x^3}{3} - \frac{4x^2}{2} + 25x - 10e^{0.1(x+10)}\right]_{3.246}^{17.941}$

$= 156.521 + 569.806$

$= 726.327 km^2$

M2 + A1

d.

$V = 156.521 \times 0.025$

$\ldots = 3.913025 \ km^3$

$\ldots = 3.913025 \times 10^6$

$= 3\,913\,025 \ ML$

A1

e. Find point on curve $f(5)=\frac{125}{15}-25-20+20=-16\frac{2}{3}$

$$f'(x)=\frac{3x^2}{15}-2x-4$$

Find gradient $f'(5)=\frac{25}{5}-10-4$

$$........=-9 \therefore m_n=\frac{1}{9}$$

Find equation of line:
$$y+16\frac{2}{3}=\frac{1}{9}(x-5)$$
$$y=\frac{1}{9}x-17\frac{2}{9} or..9y=x-155$$

M2 + A1

f. points of intersection on calculator (5, −16.67), (16.69, −15.37)

$$d=\sqrt{1.3^2+11.69^2}$$

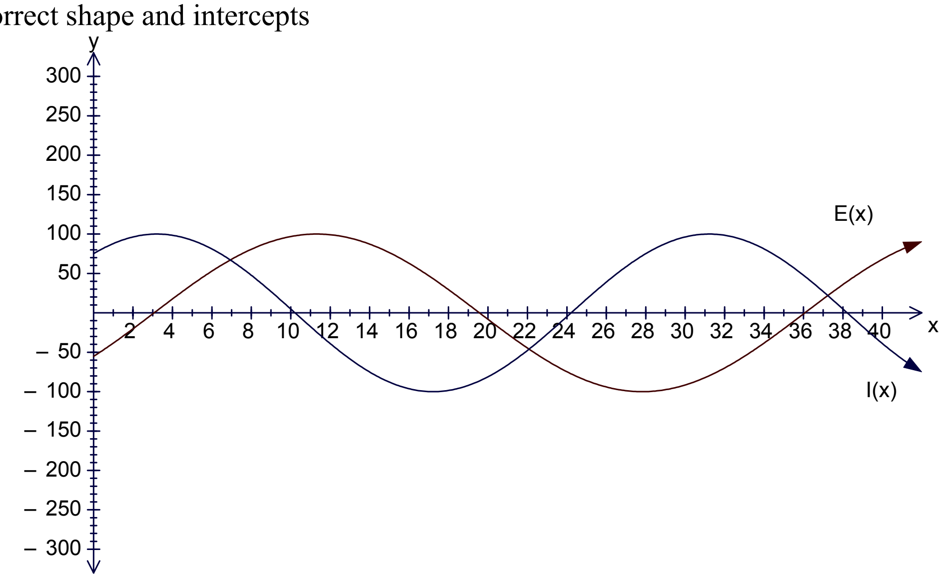
distance between $..=\sqrt{138.35}$

$$...=11.76km$$

M1 + A1
Total 15 marks

Question 2

a. Correct shape and intercepts

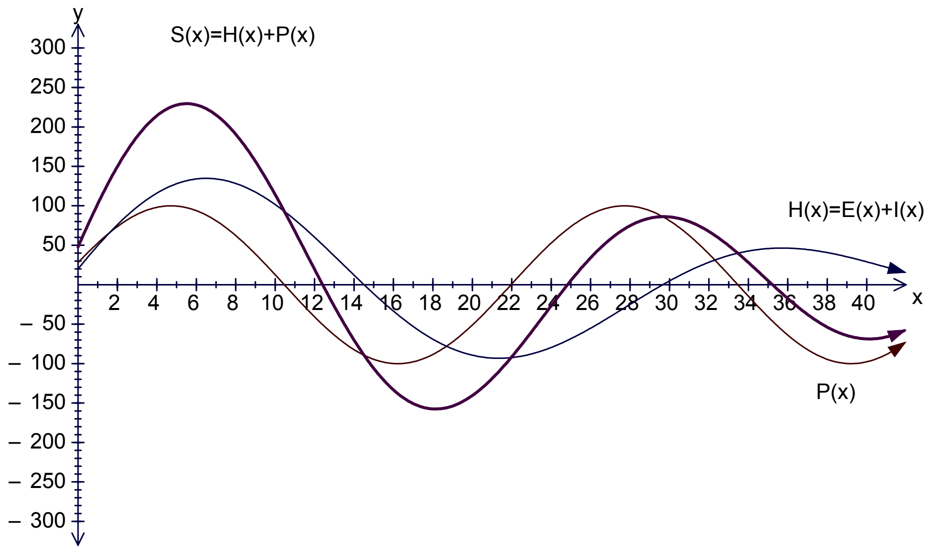


A2

b. Critical days $I(x)=0$, $x=3, 19, 36$ therefore 3rd Oct, 19th Oct
High days $I(x)=100$, $x=11$ therefore 11th October
Low days $I(x)=-100$, $x=27$ therefore 27th October
May be slight discrepancies if values read from graphs

A3

c. Correct shape, show important points, $x = 0$, points of intersection



A2

d. Best Date maximum of *S(x)* 5th October, Worst Date minimum of *S(x)* 18th October
Again, may vary if read from graphs

A2

e. Find point of intersection between $y = S(x)$ and $y = 105$ giving (1.051, 105) and (10.205, 105) therefore best dates are between 1st and 10th of October.

M1 + A1
Total 11 marks

Question 3

a.

$$9.2500 = Ae^{0} \therefore A = 9.2500$$
$$5.9724 = 9.2500e^{-9k}$$
$$0.64566 = e^{-9k}$$
$$-0.4375 = -9k$$
$$\therefore k = 0.0486 \,\&\, A = 9.2500$$

M2 + A1

b.

$$R = 9.2500e^{-0.0486\frac{1}{24}}$$
$$... = 9.2313$$
$$\therefore amount..decayed = 9.2500 - 9.2313 = 0.0187g$$

M1 + A1

c.

$$4.625 = 9.2500e^{-0.0486t}$$
$$0.5 = e^{-0.0486t}$$
$$-0.6931 = -0.0486t$$
$$t = 14.2622 \therefore 14days..6hours$$

M1 + A1

d. $D_0 = 5.2700g$

$D = 5.2700e^{-0.0309t}$

Find point of intersection of $y = D(t)$ and $y = R(t)$, giving (31.7849, 1.9736). Therefore,…

- **i.** After 31 days 19 hours
- **ii.** 1.9736g of both Raybon and Decabon

M1 + A1

Total 9 marks

Question 4

a. $\Pr(NNNWW) = \frac{7}{8} \times \frac{7}{8} \times \frac{7}{8} \times \frac{1}{8} \times \frac{1}{8} = 0.0105$

M1 + A1

b. $C \sim Bi(50, \frac{1}{8})$

$\Pr(X \geq 1) = 1 - \Pr(X < 0)$

$= 1\text{-binompdf}(50, \frac{1}{8}, 0)$

$= 0.9987$

M1 + A1

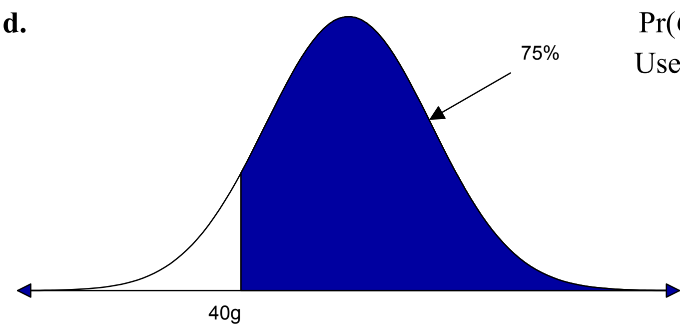
$\Pr(W \leq 10 | W \geq 1) = \frac{\Pr(1 \leq W \leq 10)}{\Pr(W \geq 1)}$

c. $... = binomcdf(50, \frac{1}{8}, 10) - binompdf(50, \frac{1}{8}, 0)$

$... = 0.9579$

M2 + A1

d.



$\Pr(C < 40) = 0.25$

Use standard normal Invnorm(0.25, 0, 1)

$z = -0.6745$

$-0.6745 = \frac{40 - \mu}{0.15}$

$\mu = 40 + 0.1011$

$\therefore \mu = 40.10g$

M2 + M1

e. $\Pr(C < 40) = normalcdf(-1EXP(99), 40, 40.1, 0.15)$

$................ = 0.2525$

M1 + A1

Total 12 marks

Question 5

a.

$$\int_0^2 (k(x^2+1))dx = 1$$

$$k\left[\frac{x^3}{3}+x)\right]_0^2 = 1$$

$$k\left[\frac{8}{3}+2\right] = 1$$

$$k\left(\frac{14}{3}\right) = 1$$

$$k = \frac{3}{14}$$

M1 + A1

b.

$$\int_0^m (\frac{3}{14}(x^2+1))dx = 0.5$$

$$\frac{m^3}{3}+m = 2\frac{1}{3}$$

Graph $y = 2\frac{1}{3}$ and $y = \frac{m^3}{3}+m$ find the points of intersection (1.4063, 2.3333). Therefore, the median is 1.4063

M1 + A1

c. mode is max of graph at $x = 2$ A1

d. $E(X) = \int_0^2 (\frac{3}{14}x(x^2+1))dx$

.... $= 1.2857$

M1 + A1

e. $\Pr(X < 1.2857) = \int_0^{1.2857} (\frac{3}{14}(x^2+1))dx$

$= 0.4273$ A1

f. $\text{var}(\mathbf{X}) = \int_0^2 \left(\frac{3}{14}x^2\left(x^2+1\right)\right)dx - \left(1.2857\right)^2$ M1

$= 1.9429 - 1.6530$

$= 0.2899$ A1

g. $\sigma_x = \sqrt{\text{var}(\mathbf{X})} = \sqrt{0.2899} \approx 0.5384$ A1

Total 11 marks