THE SCHOOL FOR EXCELLENCE
UNIT 3 & 4 MATHEMATICAL METHODS 2008
COMPLIMENTARY WRITTEN EXAMINATION 1 - SOLUTIONS

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QUESTION 1

Construct a Karnaugh Table:

	A	A'
B	a	b
B'	c	d
	$\frac{1}{2}$	$\frac{1}{2}$

	A	A'	
B	$\Pr(A \cap B)$	$\Pr(A' \cap B)$	$\Pr(B)$
B'	$\Pr(A \cap B')$	$\Pr(A' \cap B')$	$\Pr(B')$
	$\Pr(A)$	$\Pr(A')$	

Required to find: $\Pr(A \mid B') = \dfrac{\Pr(A \cap B')}{\Pr(B')} = \dfrac{c}{c+d}$ …. (1)

$\Pr(A' \mid B) = \dfrac{1}{3}$

$\dfrac{\Pr(A' \cap B)}{\Pr(B)} = \dfrac{b}{a+b} = \dfrac{1}{3}$

$\therefore 2b = a$ …. (2)

$\Pr(A \cup B) = \dfrac{3}{5}$

$1 - d = \dfrac{3}{5}$

$\therefore d = \dfrac{2}{5}$

From the Karnaugh Table: $d = \dfrac{2}{5}$

$b + \dfrac{2}{5} = \dfrac{1}{2}$

$\therefore b = \dfrac{1}{10}$

Substitute $b = \frac{1}{10}$ into (2): $a = \frac{1}{5}$

From the Karnaugh Table: $a = \frac{1}{5}$

$$\frac{1}{5} + c = \frac{1}{2}$$

$$\therefore c = \frac{3}{10}$$

Substitute $d = \frac{2}{5}$ and $c = \frac{3}{10}$ into (1):

$$\Pr(A \mid B') = \frac{\frac{3}{10}}{\frac{3}{10} + \frac{2}{5}} = \frac{3}{3+4} = \frac{3}{7}$$

4 marks

QUESTION 2

(a) $x(x^3 - 2x - 1) = 0$

$\Rightarrow x(x+1)(x^2 - x - 1) = 0$

$x = 0$ or $(x+1) = 0$ or $(x^2 - x - 1) = 0$

$\therefore x = 0$ $\qquad \therefore x = -1$ $\qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Therefore: $x = 0,\ -1,\ \frac{1 \pm \sqrt{5}}{2}$.

2 marks

(b) $f(x) = x^4 - 2x^2 - x$

$f'(x) = 4x^3 - 4x - 1$

$f'(1) = 4(1)^3 - 4(1) - 1 = -1$

$f'(-1) = 4(-1)^3 - 4(-1) - 1 = -1$

2 marks

(c) If the tangent is common to both points then the gradient to the curve has the same value at both points.

Part (b) suggests that the x-coordinates of the two points might be $x = 1$ and $x = -1$.

When $x = 1$**:** $y = f(1) = (1)^4 - 2(1)^2 - 1 = -2$

Equation of tangent at $(1, -2)$: $y - y_1 = m(x - x_1)$

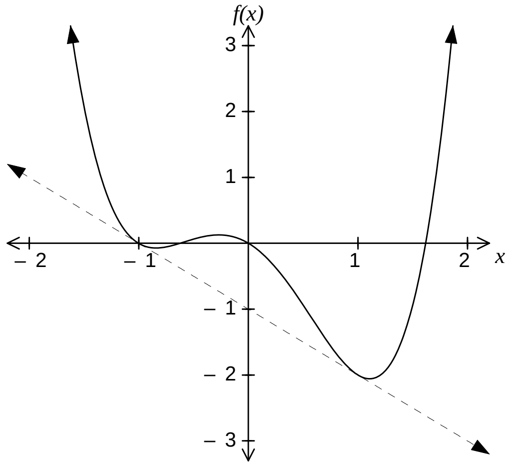
$$\Rightarrow y - (-2) = -1(x - 1)$$

$$\Rightarrow y = -x - 1$$

When $x = -1$**:** $y = f(-1) = (-1)^4 - 2(-1)^2 - (-1) = 0$

Equation of tangent at $(-1, 0)$: $y - y_1 = m(x - x_1)$

$$\Rightarrow y = -1(x - [-1]) = -x - 1$$



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2 marks

QUESTION 3

(a) $\log_b\left(\dfrac{5\sqrt{3}}{2}\right) = \log_b(5\sqrt{3}) - \log_b(2)$

$$= \log_b(5) + \log_b(\sqrt{3}) - \log_b(2)$$

$$= \log_b(5) + \log_b\left(3^{1/2}\right) - \log_b(2)$$

$$= \log_b(5) + \frac{1}{2}\log_b(3) - \log_b(2)$$

$$= r + \frac{1}{2}q - p$$

2 marks

(b) The $(r+1)^{th}$ term in the expansion of $(A+B)^n$ is $T_{r+1} = \binom{n}{r} A^{n-r} B^r$.

$A = 3x$, $B = -\dfrac{4}{x^2} = -4x^{-2}$ and $n = 14$.

$$T_{r+1} = \binom{14}{r}(3x)^{14-r}\left(-4x^{-2}\right)^r = \binom{14}{r}(3)^{14-r}(-4)^r x^{14-r}x^{-2r} = \binom{14}{r}(3)^{14-r}(-4)^r x^{14-3r}$$

Since the coefficient of x^2 is required:

$x^{14-3r} = x^2$
$14 - 3r = 2$
$\Rightarrow r = 4$

Substitute $r = 4$ into $\binom{14}{r}(3)^{14-r}(-4)^r$ to get the coefficient of x^2:

$$\binom{14}{4}(3)^{14-4}(-4)^4 = \binom{14}{4}(3)^{10}(-4)^4$$

$$= \binom{14}{4}\left(3^2\right)^5\left([-4]^2\right)^2 = \binom{14}{4}(9)^5(16)^2$$

2 marks

QUESTION 4

Multiply both sides of the equation by 2^x: $\left(2^x\right)^2 - k = 2^x$

$\Rightarrow \left(2^x\right)^2 - 2^x - k = 0$

Substitute $w = 2^x$: $w^2 - w - k = 0$

Solve for w using the quadratic formula: $w = 2^x = \dfrac{1 \pm \sqrt{1+4k}}{2}$

Case 1: $1 + 4k = 0$

$\Rightarrow k = -\dfrac{1}{4}$

Case 2: No real solution if $2^x < 0$.

$\Rightarrow 1 - \sqrt{1+4k} < 0$
$\Rightarrow -\sqrt{1+4k} < -1$
$\Rightarrow \sqrt{1+4k} > 1$
$\Rightarrow 1 + 4k > 1$
$\Rightarrow k > 0$

Case 3: $k = 0$

4 marks

QUESTION 5

(a) Let $y = x\log_e x$ and use the product rule: $\dfrac{dy}{dx} = \log_e(x) + 1$.

1 mark

(b) $I = \int_a^2 \left|\log_e x\right| dx = \int_a^1 -\log_e(x)\,dx + \int_1^2 \log_e(x)\,dx = -\int_a^1 \log_e(x)\,dx + \int_1^2 \log_e(x)\,dx$

Integration by recognition: $\dfrac{dy}{dx} = \log_e(x) + 1$

$\Rightarrow y = \int \log_e(x) + 1\,dx$

$\Rightarrow x\log_e(x) = \int \log_e(x)\,dx + \int 1\,dx$

$\Rightarrow \int \log_e(x)\,dx = x\log_e(x) - x$

Therefore: $I = -\left[x\log_e(x) - x\right]_a^1 + \left[x\log_e(x) - x\right]_1^2 = a\log_e(a) + 2\log_e(2) - a$

Compare $a\log_e(a) + 2\log_e(2) - a$ with $\log_e(2\sqrt{2}) - \dfrac{1}{2}$:

$\log_e(a)^a + \log_e(2)^2 - a = \log_e(2\sqrt{2}) - \dfrac{1}{2}$

$\log_e(a)^a(2)^2 - a = \log_e(2\sqrt{2}) - \dfrac{1}{2}$

$\therefore a = \dfrac{1}{2}$

Check: $\dfrac{1}{2}\log_e\left(\dfrac{1}{2}\right) + 2\log_e(2) = -\dfrac{1}{2}\log_e(2) + \log_e\left(2^2\right) = \log_e\left(2^{-1/2}\right) + \log_e\left(2^2\right)$

$= \log_e\left(2^2 \times 2^{-1/2}\right) = \log_e\left(2^{3/2}\right) = \log_e\left(2 \times 2^{1/2}\right) = \log_e\left(2\sqrt{2}\right)$, as required.

4 marks

QUESTION 6

$$\cos A \sin\left(\frac{\pi}{2} - A\right) + \cos\left(\frac{3\pi}{2} - A\right)\sin A$$

$$\begin{aligned}
&= \cos A \cos A + (-\sin A)\sin A \\
&= \cos^2 A - \sin^2 A \\
&= \cos^2 A - (1 - \cos^2 A) \\
&= \cos^2 A - 1 + \cos^2 A \\
&= 2\cos^2 A - 1 \\
&= a\cos^2 A + B
\end{aligned}$$

Where $a = 2$ and $b = -1$.

3 marks

QUESTION 7

Solve $t = \dfrac{2y - 6}{2y^2 - y + 3}$ for y where $y = f^{-1}(t)$:

$$t = \frac{2y - 6}{2y^2 - y + 3}$$

$$\Rightarrow 2ty^2 - ty + 3t = 2y - 6$$
$$\Rightarrow 2ty^2 - (t + 2)y + 3t + 6 = 0$$

Solve for y using the quadratic formula:

$$y = \frac{t + 2 \pm \sqrt{(t + 2)^2 - 4(2t)(3t + 6)}}{4t} = \frac{t + 2 \pm \sqrt{(-23t^2 - 44t + 4}}{4t}.$$

Decide which of the solutions to reject:

$\left(-1, -\dfrac{4}{3}\right)$ is a simple point on $y = f(t)$ therefore $\left(-\dfrac{4}{3}, -1\right)$ must satisfy $y = f^{-1}(t)$:

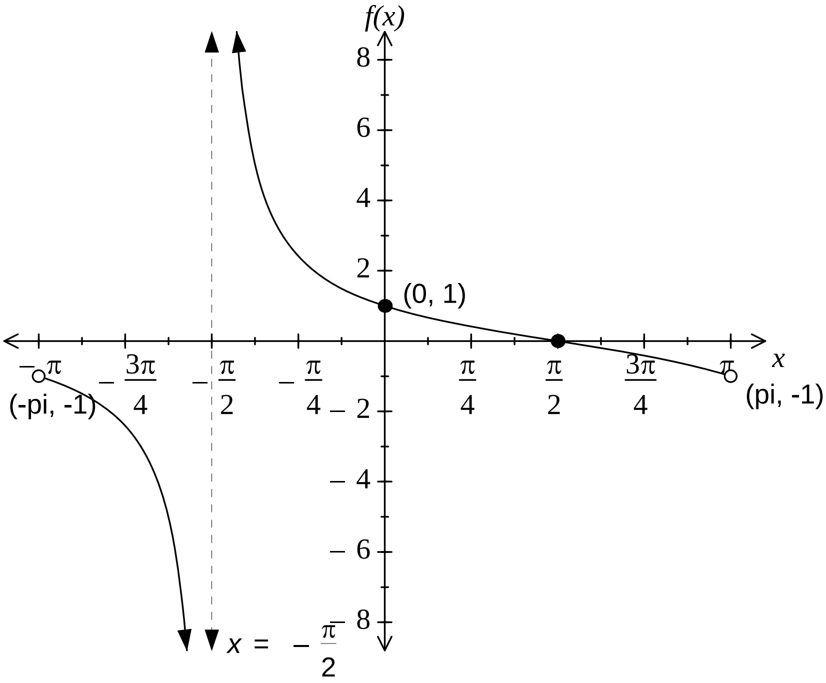
$$-1 = \frac{-\frac{4}{3} + 2 \pm \sqrt{(-23\left(-\frac{4}{3}\right)^2 - 44\left(-\frac{4}{3}\right) + 4}}{4\left(-\frac{4}{3}\right)} = \frac{\frac{2}{3} \pm \frac{14}{3}}{-\frac{16}{3}} = \frac{2 \pm 14}{-16}$$

Therefore the negative root solution is rejected and $y = \dfrac{t + 2 + \sqrt{(-23t^2 - 44t + 4}}{4t}$.

3 marks

QUESTION 8

Note that $f(x)=\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)=\tan\left(-\left[\frac{x}{2}-\frac{\pi}{4}\right]\right)=-\tan\left(\frac{x}{2}-\frac{\pi}{4}\right)$ which means that the reflection in the y-axis can be treated as a reflection in the x-axis.



3 marks

QUESTION 9

Find r when $\frac{dV}{dt}=75\pi\ cm^3/\text{min}$

Chain rule: $\frac{dV}{dt}=\frac{dV}{dr}\times\frac{dr}{dt}$

Substitute $V=\frac{4}{3}\pi r^3$: $\Rightarrow\frac{dV}{dr}=4\pi r^2$

$$\frac{dV}{dt}=4\pi r^2\times\frac{dr}{dt}$$

Substitute $\frac{dr}{dt}=0\cdot75$ and $\frac{dV}{dt}=75\pi$:

$$75\pi=4\pi r^2\times0\cdot75$$

$$\Rightarrow 4r^2=\frac{75}{0\cdot75}=100$$

$$\Rightarrow r^2=25$$

$$\Rightarrow r=5\,cm$$

3 marks

QUESTION 10

(a) $y = \dfrac{\cos x}{e^{2x}}$

Apply Quotient Rule:

$$\frac{dy}{dx} = \frac{\left(e^{2x} \times -\sin x\right) - (\cos x \times 2e^{2x})}{(e^{2x})^2} = \frac{-e^{2x}\sin x - 2\cos x e^{2x}}{(e^{2x})^2}$$

$$= \frac{-e^{2x}(\sin x + 2\cos x)}{(e^{2x})^2} = \frac{-(\sin x + 2\cos x)}{e^{2x}}$$

2 marks

(b) $f(x+h) \approx f(x) + hf'(x)$

$f(1) = \dfrac{\cos(1)}{e^{2(1)}} = \dfrac{\cos(1)}{e^2}$ **Note:** $\cos(1) \neq 0$

$f'(1) = -\dfrac{(\sin 1 - 2\cos 1)}{e^{2(1)}} = \dfrac{2\cos(1) - \sin(1)}{e^2}$

Let $x = 1$
Let $(x+h) = 0.9$
$\therefore h = -0.1$

$$f(0.9) \approx f(1) - 0.1f'(1)$$

$$\approx \frac{\cos(1)}{e^2} - 0.1\left[\frac{2\cos(1) - \sin(1)}{e^2}\right]$$

$$\approx \frac{\cos(1) - 0.2\cos(1) - 0.2\sin(1)}{e^2}$$

$$\approx \frac{0.8\cos(1) - 0.2\sin(1)}{e^2}$$

Change in f is given by: $f_{final} - f_{initial} = f(0.9) - f(1)$

$$\Delta f \approx \frac{0.8\cos(1) - 0.2\sin(1)}{e^2} - \frac{\cos(1)}{e^2}$$

$$\Delta f \approx \frac{0.2\cos(1) - 0.2\sin(1)}{e^2}$$

3 marks