INSIGHT

Trial Exam Paper

Insight Publications

2008

MATHEMATICAL METHODS (CAS)

Written examination 2

STUDENT NAME:

QUESTION AND ANSWER BOOK

Reading time: 15 minutes
Writing time: 2 hours

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
1	22	22	22
2	4	4	58
		Total	80

- Students are permitted to bring the following items into the examination: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one approved **graphics** calculator (memory DOES NOT need to be cleared) and, if desired, one scientific calculator, one bound reference.
- Students are NOT permitted to bring the following items into the examination: blank sheets of paper and/or white out liquid/tape.

Materials provided

- The question and answer book of 23 pages, with a separate sheet of miscellaneous formulas.
- An answer sheet for multiple-choice questions.

Instructions

- Write your **name** in the box provided and on the answer sheet for multiple-choice questions.
- Remove the formula sheet during reading time.
- You must answer the questions in English.

At the end of the exam

- Place the answer sheet for multiple-choice question inside the front cover of this question book.

Students are NOT permitted to bring mobile phones or any other electronic devices into the examination.

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SECTION 1

Instructions for Section 1

Answer **all** questions in pencil on the multiple-choice answer sheet.
Select the response that is **correct** for the question.
A correct answer scores 1 mark, an incorrect answer scores 0.
Marks will not be deducted for incorrect answers.
If more than one answer is selected no marks will be awarded.

Question 1

The range of the function $f(x) = x^3 - 5x^2 + 3x + 9,\ x \in (0,5]$ is

A. $(0,5]$
B. $(9,24)$
C. $[9,24]$
D. $(0,24]$
E. $[0,24]$

Question 2

A parabola of the form $y = ax^2 + bx + c$ passes through the points $(-1,0)$ $(0,2)$ $(1,2)$.

A matrix equation that can be used to solve the resulting system of simultaneous linear equations to find *a, b* and *c* is

A. $\begin{bmatrix} a \\ b \\ c \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$

B. $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$

D. $\begin{bmatrix} 0 & 0 & 1 \\ 4 & 2 & 1 \\ 4 & 2 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

E. $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Question 3

The number of solutions to the equation $(x^2 - a)(x^3 - b^3)(x + c) = 0$ where $a, b, c \in R^+$ is

A. 6
B. 5
C. 4
D. 3
E. 2

Question 4

Given that $\log_2 c + \log_2 5 = 3\log_2 3$, then c is equal to

A. 22
B. 1.8
C. $2^{5.4}$
D. 5.4
E. 4

Question 5

The curve with equation $h(x) = x^3 - bx^2 - 9x + 7$ has a stationary point when $x = -1$.
The value of b is

A. -3
B. 3
C. $-\frac{1}{2}$
D. 2
E. 6

Question 6

The maximal domain, *D*, of the function $f : D \to R$ with the rule $f(x) = \cos(\sqrt{(2x-3)})$ is

A. $R \setminus \left\{\frac{3}{2}\right\}$
B. $R \setminus \{3\}$
C. R
D. $(\frac{3}{2}, \infty)$
E. $[\frac{3}{2}, \infty)$

Question 7

The period and amplitude of the function $f(x) = 1 - 2\sin(\frac{\pi}{4} - 2x)$ are respectively

A. $\frac{\pi}{2}$, 2

B. $\frac{\pi}{4}, -2$

C. π, 2

D. 2π, 2

E. $\pi, -2$

Question 8

$\left|p^2 - 5p\right| = 6$ for

A. $p = -1$ only

B. $p = 6$ only

C. $p = -1$ or $p = 6$ only

D. $p = 2$ or $p = 3$ only

E. $p = -1$ or $p = 6$ or $p = 2$ or $p = 3$

Question 9

The transformation $T : R^2 \to R^2$, which maps the curve with the equation $y = x^2 - 4$

to the curve with the equation $y = \left(\frac{x}{2}\right)^2 - 1$, could have the rule

A. $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

B. $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \end{bmatrix}$

C. $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

D. $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

E. $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \end{bmatrix}$

Question 10

Let $f: R \to R$ be a differentiable function.

Then for all $x \in R$ and k constant, the derivative of $e^{f(kx)}$ is equal to

A. $e^{f(kx)}$

B. $ke^{f(kx)}$

C. $kf(kx)e^{f(kx)}$

D. $f'(kx)e^{f(kx)}$

E. $kf'(kx)e^{f(kx)}$

Question 11

The average rate of change of the function with the rule $f(x) = \sqrt{x^2 + 2x}$ between $x = 0$ and $x = 4$ is

A. $2\sqrt{6}$

B. $\frac{\sqrt{6}}{2}$

C. $\frac{2}{\sqrt{6}}$

D. $\sqrt{6}$

E. 4

Question 12

Which one of the following is **not** true about the function $f: R \to R,\ f(x) = (x-4)^{\frac{2}{3}}$?

A. The graph of *f* is continuous everywhere.

B. $f(x) \geq 0$ for all values of *x*.

C. $f'(x) > 0$ for $x > 4$

D. $f'(x) < 0$ for $x < 4$

E. The graph of *f* is differentiable everywhere.

Question 13

The average value of the function $y = x^2$ over the interval $[0, 2]$ is

A. 4

B. 2

C. $\frac{4}{3}$

D. $\frac{8}{3}$

E. 1

Question 14

The function f satisfies the functional equation $f(x-y)=\frac{f(x)}{f(y)}$ where x and y are any non-zero real numbers.

A possible rule for the function is

A. $f(x)=\log_e(x)$

B. $f(x)=e^x$

C. $f(x)=2x$

D. $f(x)=\sin(x)$

E. $f(x)=\frac{1}{x}$

Question 15

If $f(x)=\frac{(x-a)^2}{g(x)}$ then the derivative of $f(x)$ is

A. $\frac{2(x-a)}{g'(x)}$

B. $\frac{2(x-a)}{g(x)}$

C. $\frac{2(x-a)g'(x)}{(g(x))^2}$

D. $\frac{(x-a)[2g(x)-(x-a)g'(x)]}{(g(x))^2}$

E. $\frac{(x-a)[2g(x)-(x-a)g'(x)]}{(g'(x))^2}$

Question 16

$\int(\cos(3x-1)+12x^2)\,dx$ is equal to

A. $\frac{1}{3}\sin(3x-1)+24x+c$

B. $-3\sin(3x-1)+24x+c$

C. $\frac{1}{3}\sin(3x)+4x^3+c$

D. $\frac{1}{3}\sin(3x-1)+4x^3+c$

E. $3\sin(3x-1)+4x^3+c$

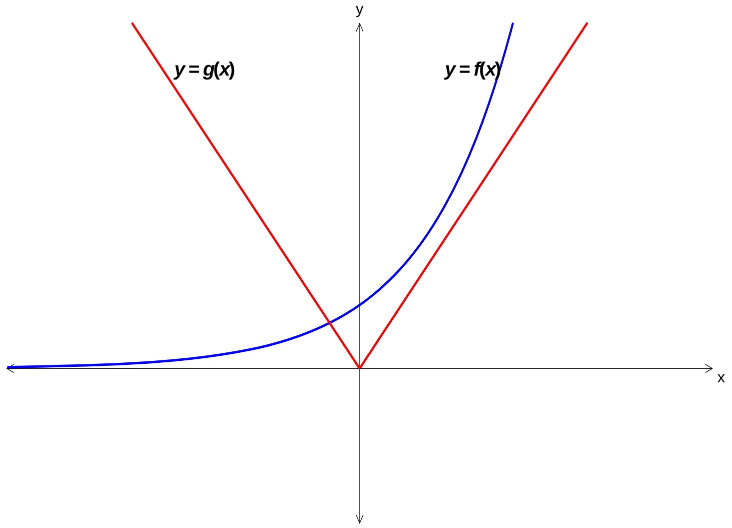
Question 17

If $\int_0^{\pi} f(x)dx = 2$ then $\int_0^{\pi} (2f(x) - \sin x)dx$ is equal to

A. 2
B. 3
C. 4
D. -2
E. 0

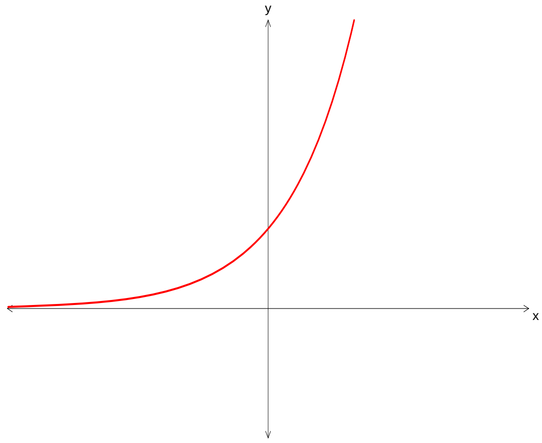
Question 18

The graphs of $y = f(x)$ and $y = g(x)$ are as shown.



The graph of $y = f(g(x))$ is best represented by

A.



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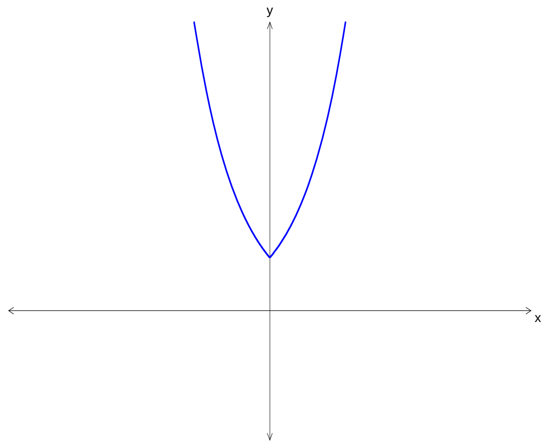
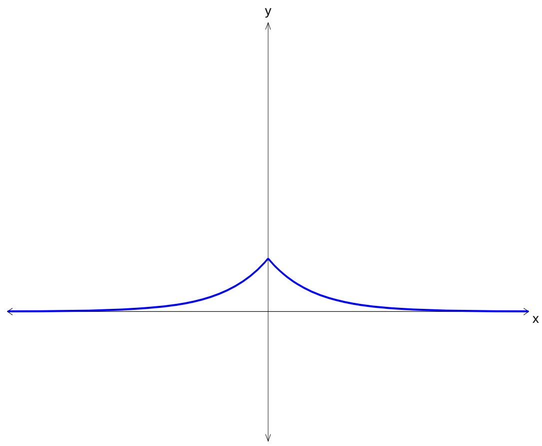
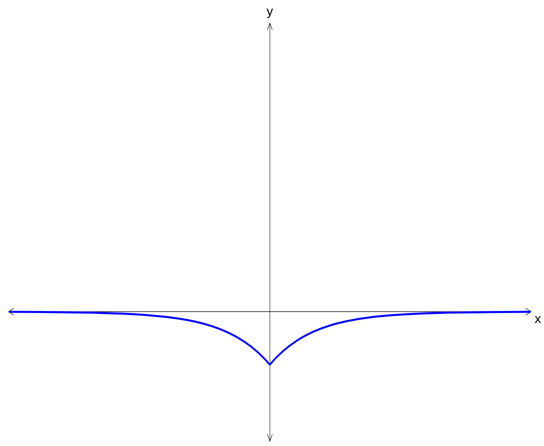


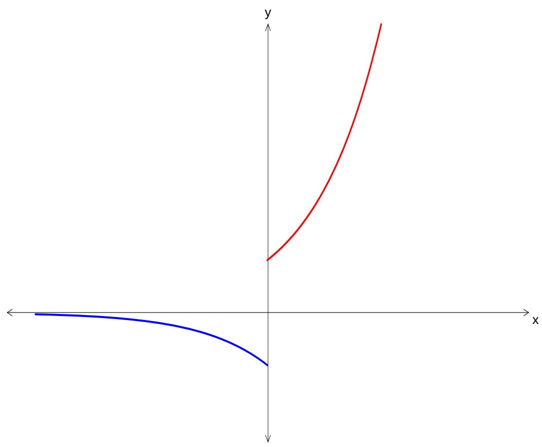
C.



D.



E.



SECTION 1 – continued
TURN OVER

Question 19

A bag contains 4 red, 3 white and 2 blue marbles. James selects a ball from the bag (and does not return it). After him, Kevin selects another ball from the bag. What is the probability that James does not select a blue ball and Kevin does not select a white ball?

A. $\frac{7}{12}$

B. $\frac{14}{27}$

C. $\frac{1}{12}$

D. $\frac{19}{36}$

E. $\frac{41}{72}$

Question 20

The speeds of vehicles travelling along a particular section of Citylink freeway are normally distributed with a mean of 95 km/h and a standard deviation of σ. 15% of drivers are found to be exceeding the 100 km/h speed limit.

The value of σ is closest to

A. 0.150

B. 1.036

C. 4.824

D. 5.182

E. 0.207

Question 21

Ben has constructed a spinner that will randomly display an integer between 0 and 4 with the following probabilities.

Number	*x*	0	1	2	3	4
Probability	*Pr(X=x)*	0.2	0.3	0.15	0.25	0.1

Ben spins the spinner 5 times. The probability of obtaining at least 3 odd numbers is

A. 0.55

B. 0.55^3

C. $(0.55)^3 + (0.55)^4 + (0.55)^5$

D. $(0.45)^2(0.55)^3 + (0.45)(0.55)^4 + (0.55)^5$

E. $10(0.45)^2(0.55)^3 + 5(0.45)(0.55)^4 + (0.55)^5$

Question 22

The continuous random variable X has a probability density function given by

$$f(x)=\begin{cases}3x^2 & \text{if } 0<x<1\\ 0 & \text{otherwise}\end{cases}$$

The value of a such that $\Pr(X>a)=0.875$ is

A. 0.875
B. 0.540
C. 0.204
D. 0.956
E. 0.500

END OF SECTION 1

SECTION 2

Instructions for Section 2

Answer **all** questions in the spaces provided.
A decimal answer will not be accepted if an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working must be shown.
Where an instruction to **use calculus** is stated for a question, you must show an appropriate derivative or antiderivative.
Unless otherwise stated diagrams are not drawn to scale.

Question 1

The "peanut" spider, a rare South American spider, weaves a peanut-shaped web—hence the name.

An araneologist (person who studies spiders) observes the web-making process.

Initially the spider weaves a strand that has the shape that can be described by
$y = \frac{1}{35}x(x-7)(x^2-8x+25)$ for $x \in [0,9]$ with all measurements in cm.

a. Show that $x^2 - 8x + 25 > 0$ for $x \in R$.

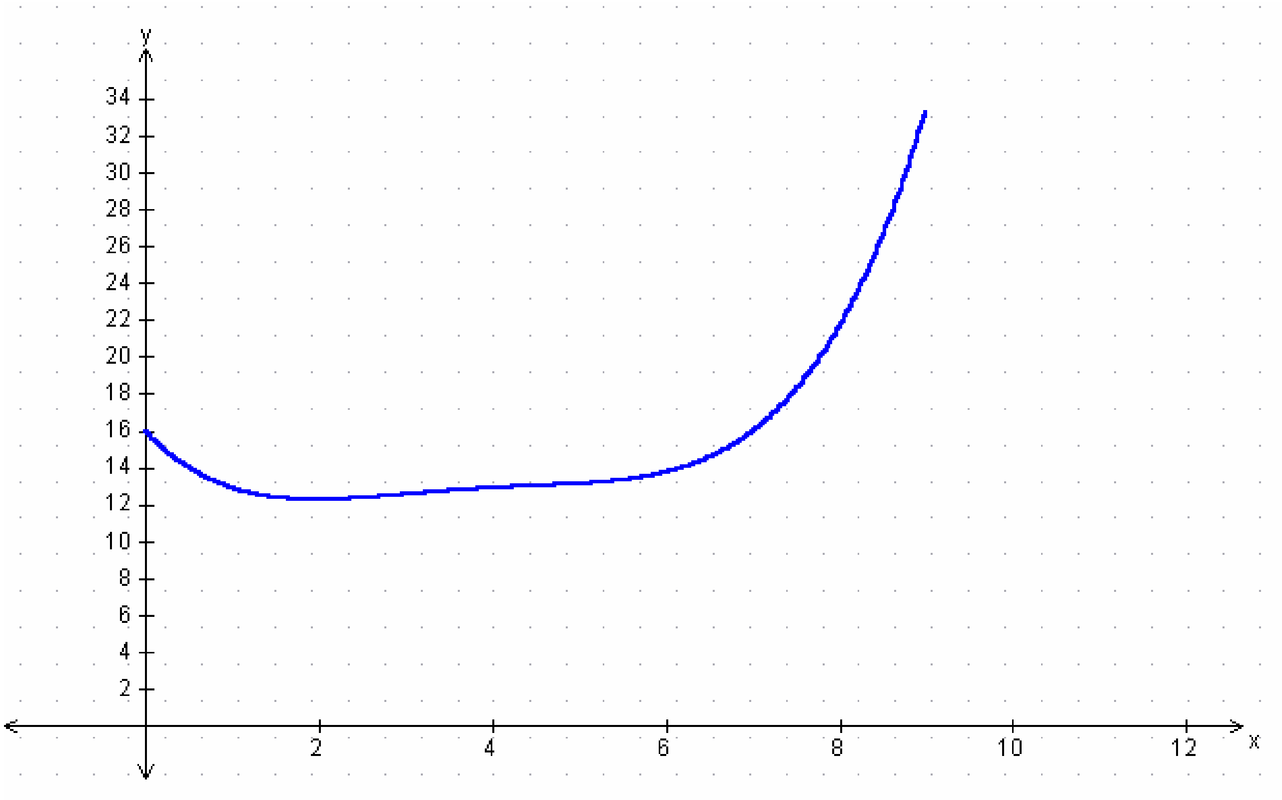
2 marks

b. Find the x-intercepts of the graph of $y = \frac{1}{35}x(x-7)(x^2-8x+25)$.

1 mark

SECTION 2 – continued

The shape below describes the bottom boundary line of the web which is the graph of *y* translated up 16 cm above the ground as shown in the graph below.



c. State the equation of the bottom boundary of the web, y_b, as shown in the graph above.

1 mark

The spider begins to weave the top boundary of the web. The symmetry of the web becomes apparent. The top boundary of the web is a reflection of the bottom boundary in the line $y = 18$. Its equation is given by $y_T = 20 - \frac{1}{35}x(x-7)(x^2 - 8x + 25)$.

d. State the transformations (in correct order) involved in producing y_T from *y*.

2 marks

e. The two webs are attached to trees at their endpoints. State the coordinates of the endpoints (correct to 2 decimal places).

2 marks

f. The two boundary webs intersect at one point in the domain. Find the point of intersection (correct to 2 decimal places).

1 mark

The spider then begins to weave vertical portions on the web joining the top boundary web with the bottom boundary web.

g. One vertical web is placed at $x = 5$ cm. Find the minimum length of this web (correct to 2 decimal places).

2 marks

h. Three vertical webs of length 5 cm are required. Where should these be positioned? Answer correct to 3 decimal places.

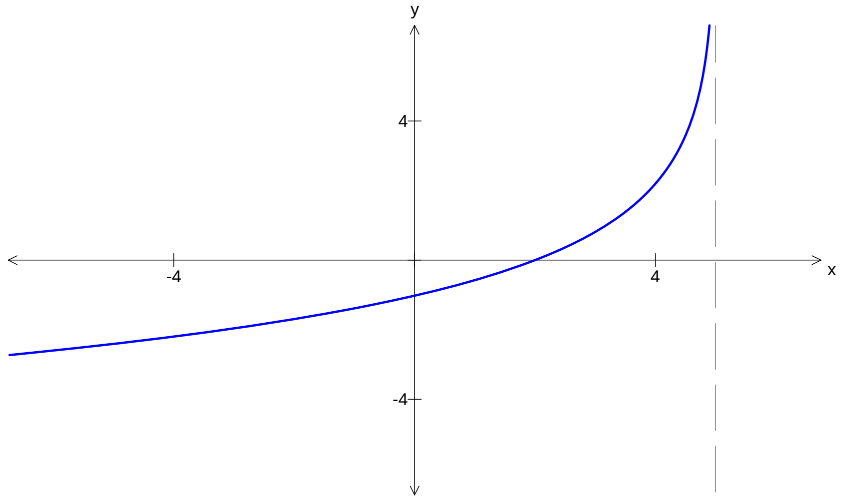
2 marks

i. Find the maximum length of a vertical web (correct to 2 decimal places).

2 marks
Total 15 marks

Question 2

Part of the graph of the function $f: R \to R,\ f(x) = -2\log_e\left(\frac{5-x}{3}\right)$ is shown below.



a. State the equation of the asymptote.

1 mark

b. Find the equation of the inverse function f^{-1}.

2 marks

c. Sketch and label the inverse function f^{-1} on the axes above. Label any asymptotes with their equation. Label axes intercepts with coordinates.

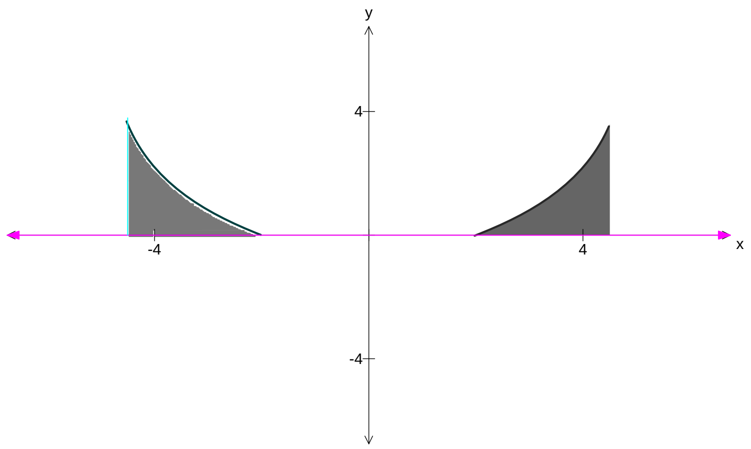
2 marks

A skateboard ramp is made in the shape as shown below. The ramp consists of a horizontal section between two curved surfaces. The curved surfaces are described by the equations

$$g:[2, 4.5] \rightarrow R,\ g(x) = f(x)$$

and $$h:[-4.5, -2] \rightarrow R,\ h(x) = f(-x)\ .$$

All measurements are in metres.



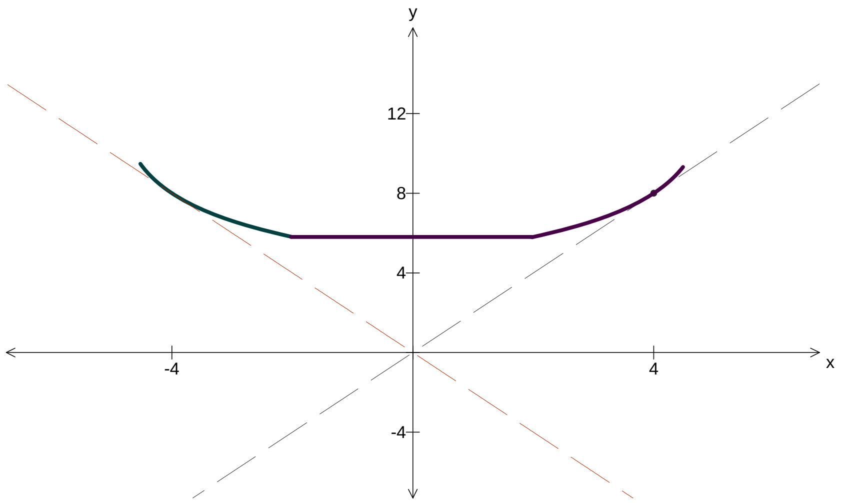
d. How many metres above the ground is the ramp when $x = 4.5$? Give your answer as an exact value.

1 mark

e. Use calculus to find the area of the shaded regions correct to two decimal places.

3 marks

The ramp is to be used for a public exhibition by a group of international skateboarders. For the public display, the ramp is to be lifted and secured above the ground by a pair of diagonal supporting beams as shown in the diagram below. The equations of the supporting beams are described by the equations of the tangents to the ramp at the points $x = 4$ and $x = -4$.



f. If the beams must pass through the origin, find how high the horizontal section of the ramp is lifted above the ground. Answer in exact form.

2 marks

g. For safety reasons the horizontal section of the ramp cannot be lifted more than 4 metres above the ground—find where on the curve, correct to 3 decimal places, the tangent line supporting beams should be placed so that they still go through the origin.

3 marks

Total 14 marks

SECTION 2 – continued

TURN OVER

Question 3

A rare species of flower is grown in a hothouse. The temperature inside the hothouse is monitored and can be observed to go through two phases—an elevated temperature phase and a constant phase. These phases are cyclical and repeat regularly.

The temperature inside the hothouse is observed for 35 minutes and can be modelled by a continuous function of time described by

$$T(t) = \begin{cases} 20\sqrt{\sin\frac{t}{2} + \cos\frac{t}{2}} + 30 & \textit{for } t \in [0, \frac{3\pi}{2}) \cup (\frac{7\pi}{2}, \frac{11\pi}{2}) \cup (a, b) \\ m & \text{otherwise} \end{cases}$$

where T is the temperature inside the hothouse in ^0C and t is the time in minutes.

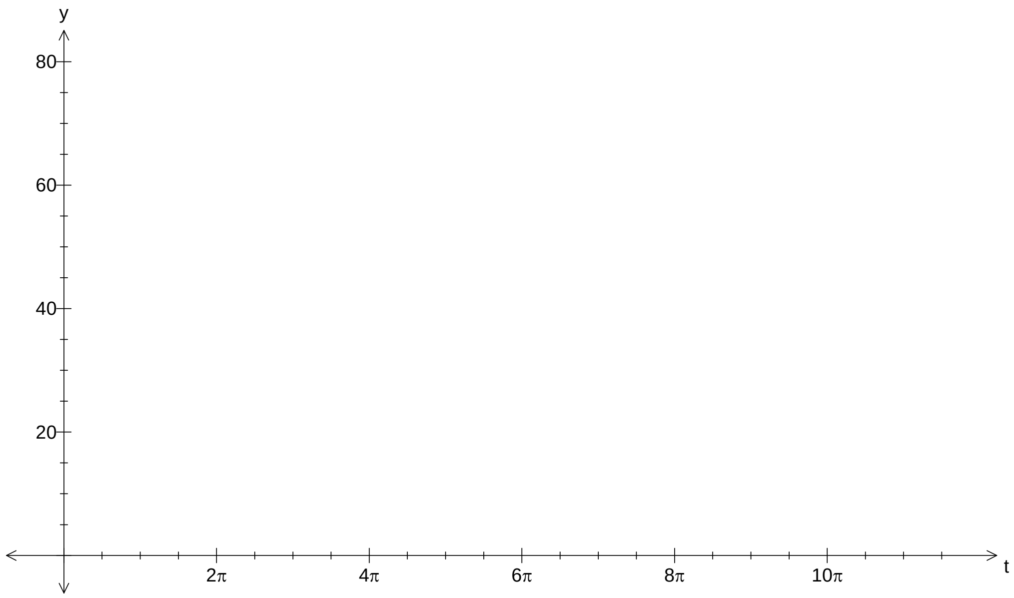
a. What is the initial temperature?

1 mark

b. State the values of *a*, *b* and *m*.

3 marks

c. Sketch the graph of T for $0 < t < 35$ (Coordinates of points are not needed)



3 marks

d. **i.** Find an expression for $\frac{dT}{dt}$ in the elevated phase.

1 mark

ii. Hence write down an equation, the solution of which is the first value of t when the temperature is a maximum. Find this value of t in exact form.

2 marks

iii. Find the maximum temperature correct to 2 decimal places.

1 mark

e. To ensure the flowers flourish and an adequate quantity is produced, the temperature must remain above a particular temperature R for a continuous period of exactly 3 minutes. Find the value of R for this hothouse. Give your answer correct to 3 decimal places.

2 marks
Total 13 marks

Question 4

Type A butterflies are known to inhabit a remote Queensland island. On the Island there are 2 separate colonies of type A butterflies—the South Colony and the North Colony with 40% initially living in South colony and 60% initially living in the North Colony. Research has shown that each year 10% of the butterflies in the South Colony will move to the North Colony and 15% of the butterflies in the North Colony will move to the South Colony.

a. Determine the percentage of butterflies living in the North Colony at the end of the first year.

1 mark

b. Determine the percentage, correct to 2 decimal places, of butterflies living in the North Colony at the end of the fifth year.

2 marks

c. Find the percentage of butterflies that will eventually be living in the North Colony in the long term.

2 marks

The length, X_A centimetres, of the wings of the type A butterfly has been found to have a probability density function

$$f(x) = \begin{cases} 0.05e^{-0.05x} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$$

d. **i.** Find the mean length of the wings of the type A butterfly.

2 marks

ii. What percentage, correct to 2 decimal places, of type A Butterflies have wings of length more than 30 centimetres?

2 marks

e. Four type A butterflies are captured. What is the probability, correct to 2 decimal places, that exactly two of the four type A butterflies have a wing of length more than 30 centimetres?

2 marks

Type B butterflies also inhabit the island. The two butterflies are nearly identical in shape, colour and size. The length X_B centimetres of the wings of a type B butterfly is given by the probability density function defined by

$$g(x) = \begin{cases} \frac{1}{2} - |a(x-20)|, & 18 < x < 22 \\ 0 & \text{otherwise} \end{cases}$$

f. Find the value of a, if $a > 0$

1 mark

g. If $c = 20$ calculate the probability, correct to 3 decimal places, that a type A butterfly is misclassified, and the probability, correct to 3 decimal places, that a type B butterfly is misclassified.

2 marks

h. Find the value of *c*, correct to 3 decimal places, for which the two probabilities of misclassification are equal.

2 marks
Total 16 marks

END OF QUESTION AND ANSWER BOOK