Test 3

Section A: Technology free. 41 marks
Section B: CAS technology assumed. 39 marks
Suggested time: 90 minutes

Section A: Short answer and extended response questions. Technology free.

Specific instructions to students

- Answer **all** questions in the spaces provided.
- A decimal approximation will not be accepted if an **exact** answer is required to a question.
- In questions where more than one mark is available, appropriate working **must** be shown.

QUESTION 1 **3 marks**

Find matrix A, given
$2A - 3\begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$.

Matrix A has a 2×2 degree.

$$2A - \begin{bmatrix} 3 & 12 \\ -6 & 9 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$$

$$2A = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 12 \\ -6 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 13 \\ -5 & 14 \end{bmatrix}$$

$$A = \frac{1}{2}\begin{bmatrix} 8 & 13 \\ -5 & 14 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & \frac{13}{2} \\ -\frac{5}{2} & 7 \end{bmatrix}$$

QUESTION 2 **3 marks**

Let $A = \begin{bmatrix} a & 1 \\ b & 3 \end{bmatrix}$, $B = \begin{bmatrix} a^2 + 1 & b \\ 2a + b & -3 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & 2 \\ h & k \end{bmatrix}$.
If $A + B = C$, find a, b, h and k.

LHS: $A + B = \begin{bmatrix} a^2 + a + 1 & b + 1 \\ 2a + 2b & 0 \end{bmatrix}$

Equate components

$a^2 + a + 1 = 3$

$a^2 + a - 2 = 0$

$(a + 2)(a - 1) = 0$

$a = -2, 1$

$b + 1 = 2$

$b = 1$

$2a + 2b = h$

when $a = -2$ and $b = 1$, $h = -2$

when $a = 1$ and $b = 1$, $h = 4$

$k = 0$

Two sets of solutions: $a = -2$, $b = 1$, $h = -2$, $k = 0$
and $a = 1$, $b = 1$, $h = 4$, $k = 0$

QUESTION 3 **2 marks**

If $A = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, find AB. Write the answer in the form $a\begin{bmatrix} b & c \\ d & e \end{bmatrix}$.

$$AB = \begin{bmatrix} \frac{\sqrt{3}}{2} \times 0 + -\frac{1}{2} \times 1 & \frac{\sqrt{3}}{2} \times -1 + -\frac{1}{2} \times 0 \\ \frac{1}{2} \times 0 + \frac{\sqrt{3}}{2} \times 1 & \frac{1}{2} \times -1 + \frac{\sqrt{3}}{2} \times 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$$

QUESTION 4 **Total 4 marks**

a Find the determinant of $A = \begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix}$. **1 mark**

The determinant is: $\Delta = (3 \times 2 - 5 \times 4)$
$= -14$

b Find A^{-1}, the inverse of A. **1 mark**

$$A^{-1} = -\frac{1}{14}\begin{bmatrix} 2 & -5 \\ -4 & 3 \end{bmatrix} \text{ or } \begin{bmatrix} -\frac{1}{7} & \frac{5}{14} \\ \frac{2}{7} & -\frac{3}{14} \end{bmatrix}$$

c Find X if $AX = B$, where $B = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. **2 marks**

The degree of X is (2×1).

$$X = A^{-1} B$$

$$= -\frac{1}{14}\begin{bmatrix} 2 & -5 \\ -4 & 3 \end{bmatrix} \times \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= -\frac{1}{14}\begin{bmatrix} 2 \times -2 + -5 \times 1 \\ -4 \times -2 + 3 \times 1 \end{bmatrix}$$

$$= -\frac{1}{14}\begin{bmatrix} -9 \\ 11 \end{bmatrix} \text{ or } \begin{bmatrix} \frac{9}{14} \\ -\frac{11}{14} \end{bmatrix}$$

QUESTION 5 — Total 3 marks

a Set up a matrix equation to solve the pair of simultaneous equations $2x - y = 7$ and $3y - 5x = -19$. **1 mark**

$$\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -19 \end{bmatrix}$$

b Solve the matrix equation to find the solution set. **2 marks**

The inverse of $\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$ is $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$.

Solution: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 7 \\ -19 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

QUESTION 6 — Total 3 marks

a Write the following transformations in matrix form: A dilation of 3 from the x axis followed by a reflection in the line $y = x$ followed by a translation of 4 in the x direction and -3 in the y direction. **2 marks**

$$\begin{bmatrix} 4 \\ -3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \text{ or } \begin{bmatrix} 3y + 4 \\ x - 3 \end{bmatrix}$$

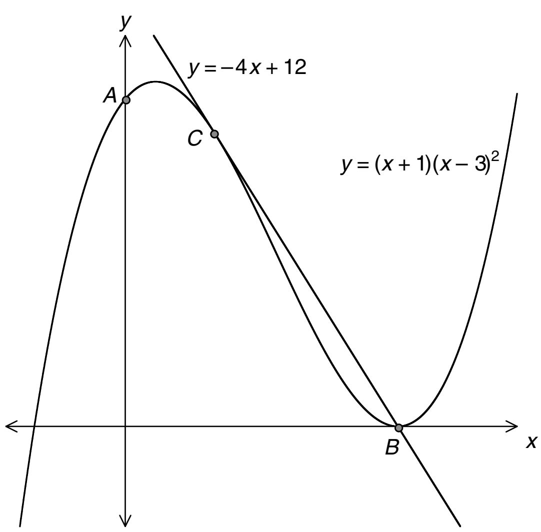
b Hence, find the image of the point $(3, -2)$ under these transformations. **1 mark**

$\begin{bmatrix} 4 \\ -3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ or $\begin{bmatrix} 3 \times -2 + 4 \\ 3 - 3 \end{bmatrix}$

$= \begin{bmatrix} -2 \\ 0 \end{bmatrix}$ $\qquad = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$

QUESTION 7 — Total 7 marks

The graph of $y = (x + 1)(x - 3)^2$ is shown.



a State the coordinates of the points A and B. **2 marks**

Point A: y intercept

$(x = 0)$, $y = 1 \times (-3)^2 = 9$

Point B: x intercept

$(y = 0)$, $(x + 1)(x - 3)^2 = 0$

$x = -1, 3$

Since $B > 0$, $B = 3$

b Hence, find the average rate of change of y with respect to x from A to B. **2 marks**

Average rate of change:

$$\frac{0 - 9}{3 - 0} = \frac{-9}{3} = -3$$

The tangent to the curve at C, where $x = 1$, also passes through the point B.

c Show that the equation to the tangent at C is $y = -4x + 12$. **2 marks**

Coordinates of C:

$y = (1 + 1)(1 - 3)^2 = 8$

Gradient of line joining B and C:

$$\frac{8 - 0}{1 - 3} = -4$$

Equation of line passing through B and C:

$y = -4x + c$

$(3, 0)$: $0 = -4 \times 3 + c$

$c = 12$

$y = -4x + 12$

d Hence, state the instantaneous rate of change of y with respect to x of the curve at C. **1 mark**

The instantaneous rate of change of y with respect to x at C is the gradient of the tangent to the curve at $C = -4$.

QUESTION 8 — Total 4 marks

The temperature (T°C) of a cup of coffee at time t minutes after the coffee is made is modelled by the formula: $T(t) = \frac{1050}{4t + 15} + 20, 0 \leq t \leq 20$.

a Find the temperature of the coffee when it is first made. **1 mark**

When it is first made $t = 0$, $T = \frac{1050}{15} + 20$

$= 90$°C

b Find the temperature of the coffee 5 minutes after it was made. **1 mark**

$t = 5$, $T = \frac{1050}{35} + 20$

$= 50$°C

c Find the average rate of change of the temperature of the coffee from when it was first made until 5 minutes later. **2 marks**

Average rate of change: $\frac{50 - 90}{5 - 0}$

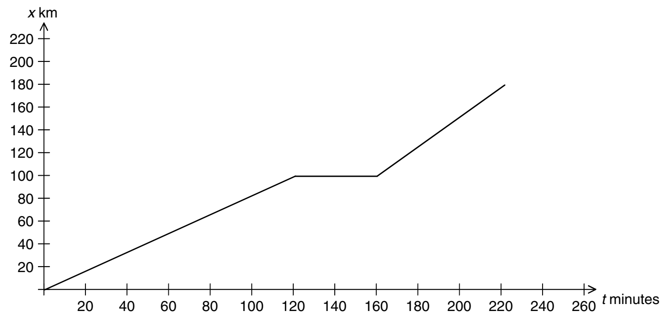
$= \frac{-40}{5}$

$= -8°\text{C/minute}$.

QUESTION 9 — Total 6 marks

A family going on a holiday drives 100 km in 120 minutes. They stop for 40 minutes before resuming their journey, then travel a further 80 km in 60 minutes.

a On the axes provided, draw a displacement-time graph representing their journey. Label the axes appropriately. **3 marks**



b Find the average speed for the journey in km/minute, to the nearest whole number. **2 marks**

Average speed: $\frac{\text{distance travelled}}{\text{time taken}}$

$= \frac{100 + 80}{120 + 40 + 60}$

$= \frac{180}{220}$

$= \frac{9}{11}$ km/minute

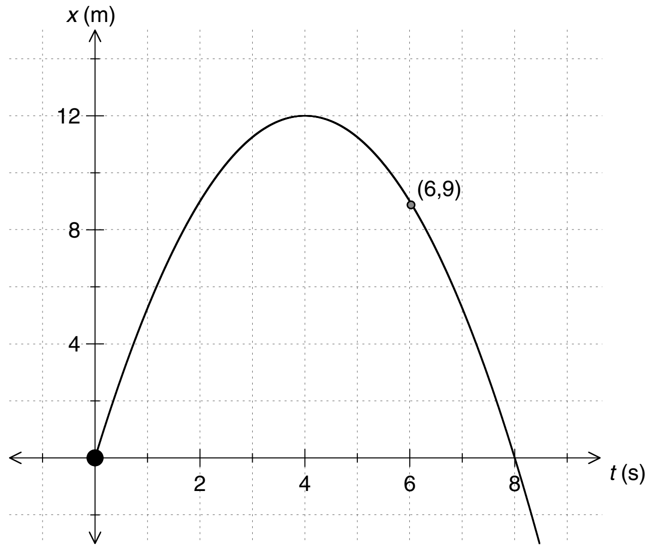
c If the return journey took 3 hours and there were no stops, what was the average speed for the return journey, in km/hour? **1 mark**

Average speed: $\frac{180}{3}$

$= 90$ km/hr

QUESTION 10 — Total 6 marks

The diagram shows the displacement-time graph of a particle moving along a horizontal straight line. The equation of the graph is $x(t) = ax(x - 8)$.



a Show that the value of a is $-\frac{3}{4}$. **1 mark**

$(6, 9): 9 = 6a(6 - 8)$

$a = \frac{9}{6} \times -\frac{1}{2}$

$= -\frac{3}{4}$

b At what time is the velocity zero? What is the displacement at this time? **2 marks**

From the graph, velocity is zero when $t = 4$ s; displacement is 12 m.

c What is the maximum displacement of the particle for $0 \leq t \leq 8$? **1 mark**

Maximum displacement is 12 m.

d Over which values of t is the velocity positive? **2 marks**

The velocity is positive when $0 < t < 4$.

Section B: Multiple-choice questions. CAS technology assumed.

Specific instructions to students

- A correct answer scores 1, and an incorrect answer scores 0.
- Marks are not deducted for incorrect answers.
- No marks are given if more than one answer is given.
- Choose the alternative which most correctly answers the question and mark your choice on the multiple-choice answer section at the bottom of each page, as shown in the example below.

1 A B C D E   USE PENCIL ONLY

- Use pencil only.

QUESTION 11

If P = $\begin{bmatrix} -2 & 1 & 3 \end{bmatrix}$ and Q = $\begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}$, then the product, PQ is:

A $\begin{bmatrix} 2 & 0 & 12 \end{bmatrix}$

B $\begin{bmatrix} 2 \\ 0 \\ 12 \end{bmatrix}$

C $\begin{bmatrix} 2 & -1 & -3 \\ 0 & 0 & 0 \\ -8 & 4 & 12 \end{bmatrix}$

D $\begin{bmatrix} 14 \end{bmatrix}$

E not defined

QUESTION 12

Let A = $\begin{bmatrix} 4 & -3 \\ 5 & 1 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The dimension of AB is:

A 2×2

B 2×3

C 3×2

D 2×1

E not defined

QUESTION 13

The matrix equation to solve the simultaneous equations

$x + 2y + z = 1$

$2x - 3y + z = -1$

$-x + y = 3$

is:

A $\begin{bmatrix} 1 & 2 & 1 \\ 2 & -3 & 1 \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x & y & z \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 \end{bmatrix}$

B $\begin{bmatrix} 1 & 2 & 1 \\ 2 & -3 & 1 \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$

C $\begin{bmatrix} x & y & z \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 2 & -3 & 1 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 \end{bmatrix}$

D $\begin{bmatrix} x & y & z \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 2 & -3 & 1 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$

E $\begin{bmatrix} 1 & 2 & -1 \\ 2 & -3 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 \end{bmatrix}$

QUESTION 14

The average rate of change of y with respect to x for $y = f(x)$ from $x = 3$ to $x = 3 + h$ can be found by evaluating:

A $\frac{f(3)}{3}$

B $\frac{f(x+h) - f(x)}{h}$

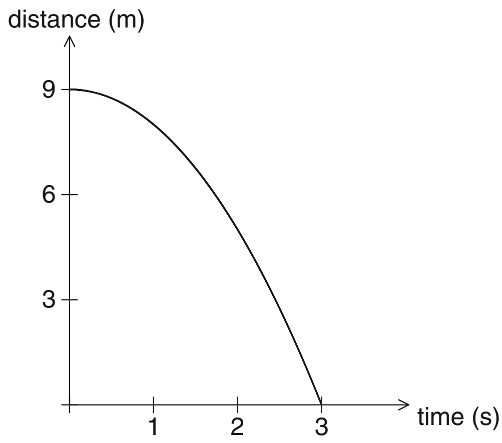
C $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

D $\lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$

E $\frac{f(3+h) - f(3)}{h}$

QUESTION 15

The average speed from $t = 0$ to $t = 3$ for the displacement-time graph shown is:



A $\frac{1}{3}$ m/s

B $-\frac{1}{3}$ m/s

C -6 m/s

D -3 m/s

E 3 m/s

ONE ANSWER PER LINE   USE PENCIL ONLY

11	A	B	C	**D**	E
12	A	**B**	C	D	E
13	A	**B**	C	D	E
14	A	B	C	D	**E**
15	A	B	C	**D**	E

Section B: Extended response questions. CAS technology assumed.

Specific instructions to students

- Answer **all** questions in the spaces provided.
- In questions where more than one mark is available, appropriate working **must** be shown.

QUESTION 16 **Total 8 marks**

The hyperbola $y = 1 + \frac{1}{k + x}$ crosses the x axis at the point A and the y axis at the point B.

a Express the coordinates of A and of B in terms of k. **2 marks**

y intercept ($x = 0$) $\quad y = 1 + \frac{1}{k}$

Using CAS: x intercept: solve $1 + \frac{1}{k + x} = 0$ for x

$x = -k - 1$

Coordinates are: $A = (-k - 1, 0)$ and $B = \left(0, 1 + \frac{1}{k}\right)$.

b Write an expression to find the gradient of the line joining A and B. Write the expression for the gradient in simplest form. **2 marks**

$$m = \frac{\left(1 + \frac{1}{k}\right) - 0}{0 - (-k - 1)}$$

Using CAS: $m = \frac{1}{k}$

c Write the equation of the line joining A and B. **1 mark**

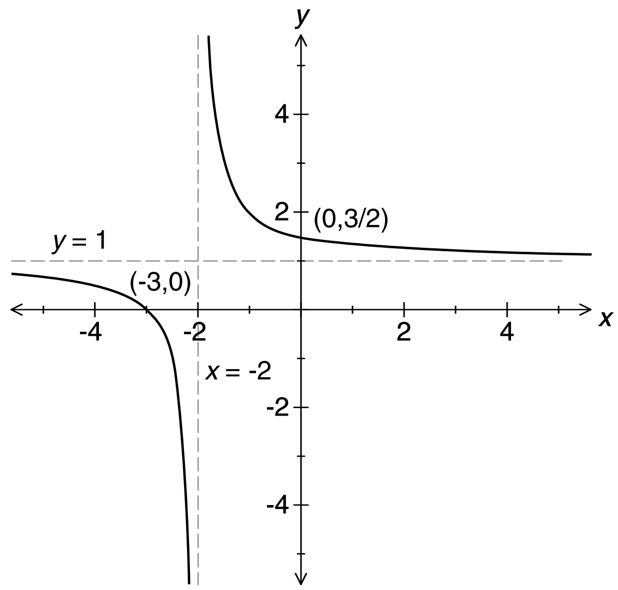
$y = \frac{1}{k}x + 1 + \frac{1}{k}$

d On the axes provided, sketch the graph of the hyperbola when $k = 2$. Label any asymptotes and x and y intercepts. **3 marks**

$k = 2$: $y = 1 + \frac{1}{x + 2}$. This hyperbola has a vertical asymptote at $x = -2$ and a horizontal asymptote at $y = 1$.

y intercept is $1 + \frac{1}{2} = \frac{3}{2}$

x intercept is $-2 - 1 = -3$



QUESTION 17 **Total 9 marks**

The general equation of a quartic polynomial that passes through the origin is given by $y = ax^4 + bx^3 + cx^2 + dx$. A particular quartic polynomial that passes through the origin also passes through the points $(-2, 15)$, $(-1, 2)$, $(1, -3)$ and $(2, -7)$.

a Write four simultaneous equations using a, b, c and d. **2 marks**

$16a - 8b + 4c - 2d = 15$

$a - b + c - d = 2$

$a + b + c + d = -3$

$16a + 8b + 4c + 2d = -7$

b Rewrite these equations as a matrix equation. **1 mark**

$$\begin{bmatrix} 16 & -8 & 4 & 2 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 16 & 8 & 4 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 15 \\ 2 \\ -3 \\ -7 \end{bmatrix}$$

c Find the values of a, b, c and d. Hence, write the equation of the particular quartic polynomial. **2 marks**

CAS: Either use SOLVE with equations or solve the matrix equation.

$a = \frac{1}{2}$, $b = -1$, $c = -1$, $d = -\frac{3}{2}$

Quartic equation: $y = \frac{1}{2}x^4 - x^3 - x^2 - \frac{3}{2}x$

d Find the coordinates of the minimum point to two decimal places. **2 marks**

From the graph: (2.13, −7.10).

e State the range of $f: [0, 3] \rightarrow R$, $f(x) = \frac{1}{2}x^4 - x^3 - x^2 - \frac{3}{2}x$ to two decimal places. **2 marks**

From the graph, the range is [−7.10, 0].

QUESTION 18 **Total 10 marks**

The combined water storage levels of all dams for a certain Australian city for 2007 can be modelled by the function

$$P(t) = \begin{cases} 40.527 - 2.0743t & 0 \le t \le 6 \\ -0.0194t^3 + 0.0369t^2 + 6.0897t - 5.2976 & 6 < t < 13, \end{cases}$$

where $P(t)$ is the percentage of total capacity of water stored in all dams t months after 1 January 2007. ($t = 0$ is 1 January 2007, $t = 1$ is 1 February 2007.)

a What percentage of water is stored in all dams when $t = 0$ and $t = 9$? Give answers correct to one decimal place. **2 marks**

$t = 0$:

$P(0) = 40.527 - 2.0743 \times 0$

$= 40.5\%$

$t = 9$:

$P(9) = -0.0194 \times 9^3 + 0.0369 \times 9^2 + 6.0897 \times 9 - 5.2976$

$= 38.4\%$

b Find the average rate of change of the percentage of stored water from $t = 0$ to $t = 6$, and from $t = 7$ to $t = 12$. Give a brief description of the significance of these results. **5 marks**

Average rate of change:

$t = 0$ to $t = 6$:

Over the domain $0 \le t \le 6$ the gradient of the linear function gives the average rate of change, which is a constant value of -2.0743. To one decimal place: -2.1% per month.

$t = 7$ to $t = 12$:

$$\frac{P(12) - P(7)}{12 - 7} = \frac{39.569 - 32.484}{5}$$

$= 1.4\%$ per month

Over the first 6 months the storage level dropped at a constant rate of 2.1% of full capacity per month. During the 5 months from the start of July to December, the storage level rose by an average of 1.4% of full capacity per month.

c The following table gives the percentage storage levels for several months of the second half of 2007.

t	7	10	12
P	31.9	39.9	39.7

It was thought that a quadratic polynomial of the form $P(t) = at^2 + bt + c$ could give a more accurate model over these months.

i Find the values of a, b and c, correct to three decimal places. **2 marks**

Use CAS to solve the simultaneous equations:

$49a + 7b + c = 31.9$

$100a + 10b + c = 39.9$

$144a + 12b + c = 39.7$

Or solve $f(7) = 31.9$ and $f(10) = 39.9$ and $f(12) = 39.7$ for a, b and c.

$a = -0.553$, $b = 12.073$, $c = -25.5$

The quadratic function is:

$P(t) = -0.553t^2 + 12.073t - 25.5$

ii Using this quadratic function, find the value for t, to the nearest whole number, when the water storage was a maximum. **1 mark**

CAS: Find the maximum value from the table of values or the graph. $\therefore t = 11$

QUESTION 19 **Total 7 marks**

$\text{P} = \begin{bmatrix} a & 0 \\ b & 1 \end{bmatrix}$ and $\text{P}^{-1} = \begin{bmatrix} u & 0 \\ v & 1 \end{bmatrix}$

a By equating $\text{P} \times \text{P}^{-1} = \text{I}$, find u and v in terms of a and b. **4 marks**

$$\text{P} \times \text{P}^{-1} = \begin{bmatrix} a & 0 \\ b & 1 \end{bmatrix} \times \begin{bmatrix} u & 0 \\ v & 1 \end{bmatrix}$$

$$\text{LHS: } = \begin{bmatrix} a \times u + 0 \times v & a \times 0 + 0 \times 1 \\ b \times u + 1 \times v & b \times 0 + 1 \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} au & 0 \\ bu + v & 1 \end{bmatrix}$$

Equate components:

$au = 1 \Rightarrow u = \frac{1}{a}$

$bu + v = 0$

$v = -bu$

$v = -\frac{b}{a}$

b If $a = 3$ and $b = 4$, write P^{-1} in the form $\frac{1}{3}\begin{bmatrix} m & n \\ q & r \end{bmatrix}$. State the values of m, n, q and r. **3 marks**

$$\text{P}^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{4}{3} & 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1 & 0 \\ -4 & 3 \end{bmatrix}$$

Thus $m = 1$, $n = 0$, $q = -4$ and $r = 3$.