MATHEMATICAL METHODS

Units 3 & 4 – Written examination 2

2007 Trial Examination

SOLUTIONS

SECTION 1: Multiple-choice questions (1 mark each)

Question 1

Answer: D

Explanation:

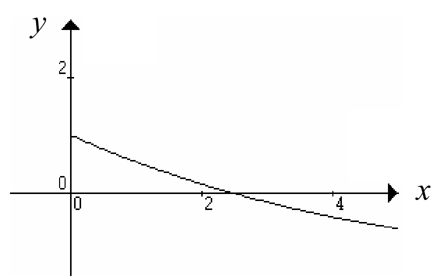
End points: when $x = -1$ $f(x) = -4$
when $x = 5$ $f(x) = -2$
$\therefore$ Range = $(-2, 4]$

Question 2

Answer: B

Explanation:

$f(g(x)) = f(\sqrt{x}) = cos\sqrt{x}$



From the graph of $f(g(x)) = f(\sqrt{x}) = cos\sqrt{x}$
Dom = $[0, \infty)$

Question 3

Answer: C

Explanation:

$f(x) = 3e^{x-1} + 2$

$x = 3e^{y-1} + 2$

$3e^{y-1} = x - 2$ Domain $= [5, \infty)$

$e^{y-1} = \dfrac{x-2}{3}$

$y = \log_e\left(\dfrac{x-2}{3}\right) + 1$

Question 4

Answer: A

Explanation:

$y = -log_2(3 - x) + 2$

- Negative sign in front of the log implies the graph is reflected in the *x*-axis
- $3 - x = -(x - 3)$ implies a horizontal translation of 3 units to the right
- + 2 implies a vertical translation of 2 units up

Question 5

Answer: B

Explanation:

$n = b \;\; \therefore$ period $= \dfrac{2\pi}{b}$

amplitude = a

range = $[-a + c, a + c]$

Question 6

Answer: C

Explanation:

$$2x+\frac{\pi}{6}=\frac{\pi}{6},\ 2\pi-\frac{\pi}{6},\ 2\pi+\frac{\pi}{6}$$
$$2x+\frac{\pi}{6}=\frac{\pi}{6},\frac{11\pi}{6},\frac{13\pi}{6}$$
$$2x=0,\ \frac{10\pi}{6},\frac{12\pi}{6}$$
$$x=0,\frac{5\pi}{6},\pi$$
$$\therefore sum\ of\ solutions=0+\frac{5\pi}{6}+\pi$$
$$=\frac{11\pi}{6}$$

Question 7

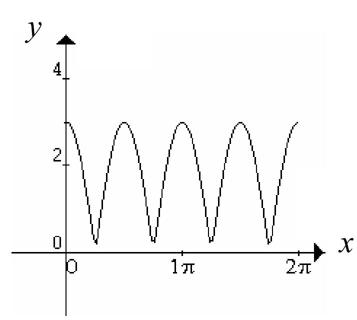
Answer: C

Explanation:

$y=|3\cos 2x|$

Since the range of $3\cos 2x$ is $[-3, 3]$
then the range of $y=|3\cos 2x|$ is $[0, 3]$

From the graph of $y=|3\cos 2x|$ below the range is $[0,3]$



Question 8

Answer: D

Explanation:

$2\log_e 3x - 1 = \log_e a$

$\log_e 9x^2 - \log_e e = \log_e a$

$$log_e \frac{9x^2}{3} = log_e a$$

$$\frac{9x^2}{3} = a$$

$$x^2 = \frac{ea}{9}$$

$$x = \frac{\sqrt{ae}}{3}$$

Question 9

Answer: D

Explanation :

$$y = \frac{cos(2x)}{3e^x - x}$$

$Let\ u = \cos 2x$ and $v = 3e^x - 1$

$$\frac{du}{dx} = \cos 2x \text{ and } \frac{dv}{dx} = 3e^x$$

$$\frac{dy}{dx} = \frac{-2\sin 2x(3e^x - x) - \cos 2x(3e^x - 1)}{9e^{2x} - 6xe^x + x^2}$$

Question 10

Answer: B

Explanation:

$V = \frac{4}{3}\pi\, r^3$

$\frac{dv}{dr} = 4\pi r^2$

$r = 7\,, \frac{dv}{dr} = 4\pi \times 7^2$

$\frac{dv}{dr} = 196\pi cm^3 / cm$

Question 11

Answer: A

Explanation:

$f(x) = 2x^3 + ax^2 + bx$

$f'(x) = 6x^2 + 2ax + b$

$x = -2, f'(x) = 0;$

$0 = 6(4) - 4a + b$

$4a - b = 24 \quad ....(1)$

Stationary point at (2,–4)

$-4 = 16 + 4a - 2b$

$12 = 4a - 3b$

$6 = 2a - b \quad ....(2)$

$b = 12 \quad and \quad a = 9$

Question 12

Answer: A

Explanation:

An approximate value for $\frac{1}{\sqrt{9.5}}$ is:

$$y = x^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = -\frac{1}{2}x^{-\frac{3}{2}}$$

$$x = 9, \frac{dy}{dx} = -\frac{1}{2}(9)^{-\frac{3}{2}}$$

$$\frac{dy}{dx} = -\frac{1}{54}$$

$$x = 9, \ \delta x = 0.5$$

$$\frac{dy}{dx} = \frac{\delta y}{\delta x}$$

$$\delta y = \frac{dy}{dx}\delta x$$

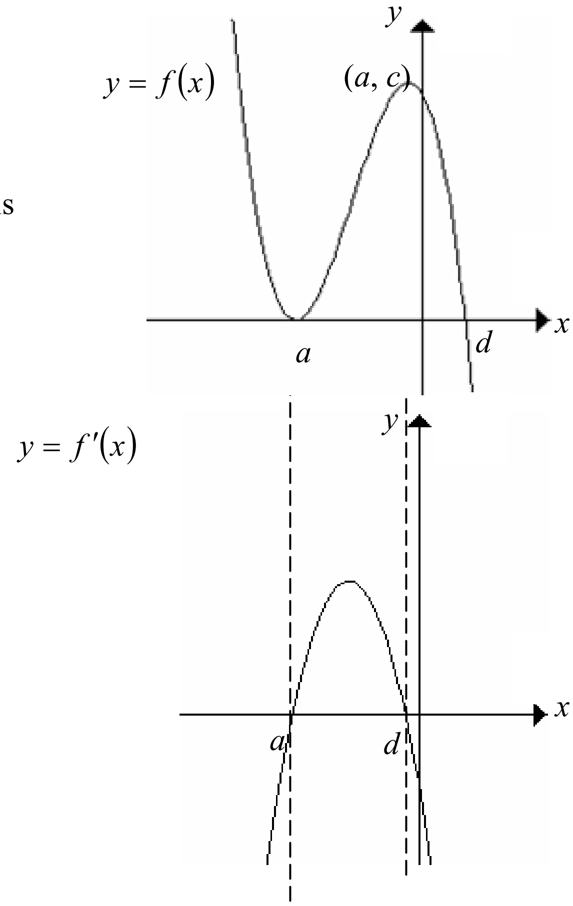
$$\delta y = -\frac{1}{54} \times 0.5$$

Question 13

Answer: C

Explanation:

From the gradient graph shown, $f'(x)$ is negative when $x < -3$ or $x > 1$.

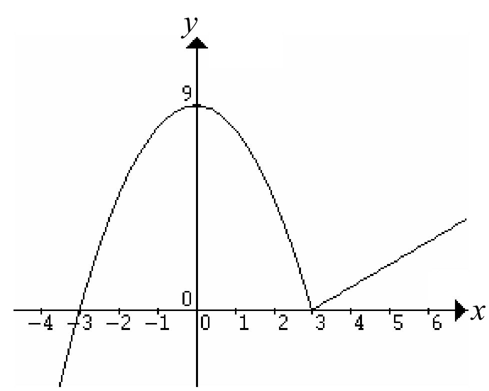


Question 14

Answer: E

Explanation:

Since a unique tangent cannot be drawn at the sharp point
$x = 3$ the function is not differentiable but is continuous at
$x = 3$
Continuous but not differentiable at $x = 3$



Question 15

Answer: E

Explanation:

$$\Pr(X < 2) = \int_0^2 \frac{1}{2}e^{-\frac{1}{2}}\ dx$$
$$= \left[-e^{-\frac{1}{2}x}\right]_0^2$$
$$= -\frac{1}{e} + 1$$
$$= 0.6321$$

Question 16

Answer: D

Explanation:

$n = 10,\ p = 0.2$

$$\Pr(X = x) = \binom{10}{x} 0.2^2\ \ 0.8^{10-x}$$

$$\Pr(X = 5) = \binom{10}{5} 0.2^5\ \ 0.8^5$$
$$= 0.0264$$

Question 17

Answer: A

Explanation:

$0.2 + 3k + 2a + 3a = 1$
$5a + 3k = 0.8$……………………..(1)

$E(X) = 2.6$
$2.6 = 0.2 + 6k + 6a + 12a$
$2.4 = 18a + 6k$………………..(2)
$8a = 0.8$
$a = 0.1$ and $k = 0.1$

Question 18

Answer: D

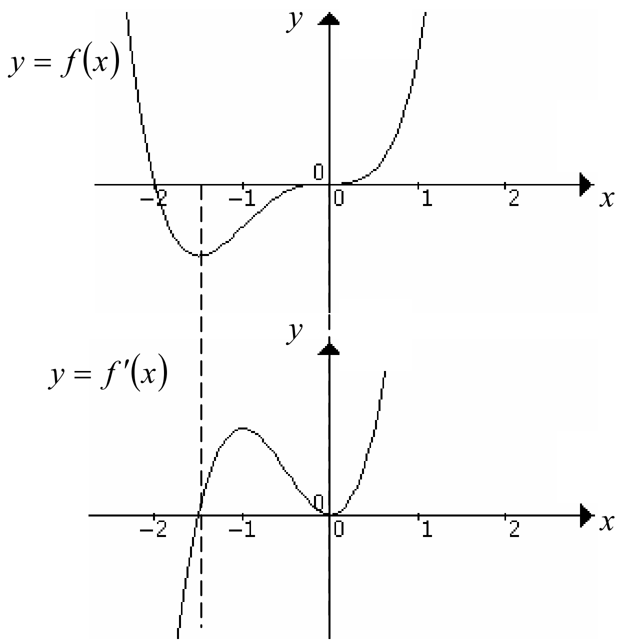
Explanation:

$$Pr(X \geq 20) = Pr\left(z \geq \frac{20-16}{4}\right)$$
$$= Pr(z \geq 1)$$

Question 19

Answer: B

Explanation:



Question 20

Answer: A

Explanation:

$A = 0.5(8.75 + 8 + 6.75 + 5 + 2.75)$
$= 15.625$ units squared

Question 21

Answer: C

Explanation:

$$A = \int_0^2 m\left(x^2 + 2\right)dx$$

$$2 = m\left[\frac{x^3}{3} + 2x\right]_0^2$$

$$2 = \left[\frac{8}{3} + 4\right]$$

$$2 = \frac{20}{3}m$$

$$20m = 6$$

$$m = \frac{3}{10}$$

Question 22

Answer: B

Explanation:

The area between $x = -3$ and $x = 1$ is below the x-axis therefore must take absolute value of the area or $\int_{-3}^{1} f(x)\,dx$ and add the area of $\int_{1}^{5} f(x)\,dx$.

$$-A = \int_{-3}^{1} f(x)\,dx + \int_{1}^{5} f(x)\,dx$$

SECTION 2 : Analysis questions

Question 1

a. $f(x) = \log_e(5-2x)$

x - intercept : $0 = \log_e(5-2x)$ M1

$5 - 2x = e^0$

$2x = 4$

$x = 2$

$(2,0)$ A1

y - intercept : $y = \log_e 5$

$(0, \log_e 5)$ A1

b. For f to be defined:

$5 - 2x > 0$

$-2x > -5$

$x < \frac{5}{2}$

$\therefore D = \left(-\infty, \frac{5}{2}\right)$ A1

c.

$f(x) = \log_e(5-2x)$

$f'(x) = \frac{-2}{5-2x}$ M1

From **part b** $5 - 2x > 0$ (ie always positive) A1

from $f'(x) = \frac{-2}{5-2x}$ the answer will always be negative since -2 will always be divided by a positive number.

d.

i.

$$y = \log_e(5-2x)$$

inverse: $x = \log_e(5-2y)$

$5-2y = e^x$ M1

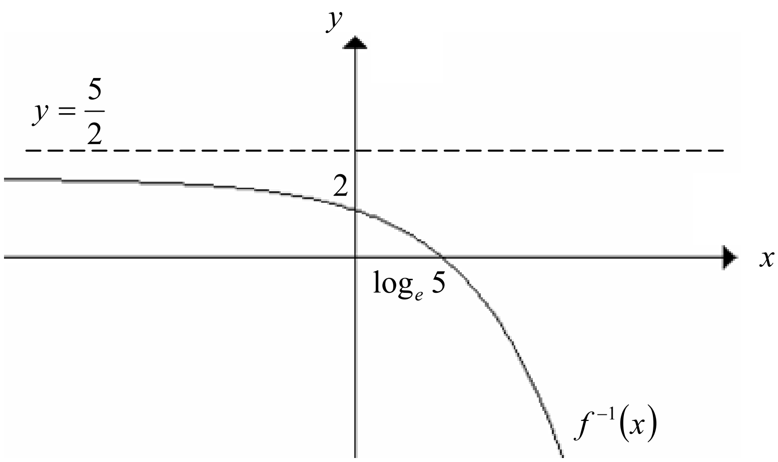
$2y = 5-e^x$

$y = \frac{1}{2}(5-e^x)$

$\therefore f^{-1}(x) = \frac{1}{2}(5-e^x)$ A1

ii. dom f^{-1} = ran f = R A1

e.



M1 correct shape of graph

M1 labelling axes intercepts

f.

$$A = \int_0^{\log_e 5} f(x)\,dx$$ M1

$$= \int_0^{\log_e 5} \frac{1}{2}\left(5 - e^x\right)dx$$

$$= \frac{1}{2}\left[5x - e^x\right]_0^{\log_e 5}$$

$$= \frac{5}{2}\log_e 5 - 2 \text{ square units}$$ A1

Total 12 marks

Question 2

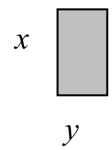
a.

Area window = 24 M1

$yx + yx = 24$

$2xy = 24$ A1

$y = \dfrac{12}{x}$



b.

$L = 6 + 2y$ M1

$L = 6 + 2\left(\dfrac{12}{x}\right)$

$L = 6 + \left(\dfrac{24}{x}\right)$ A1

c. $H = 4 + x$ A1

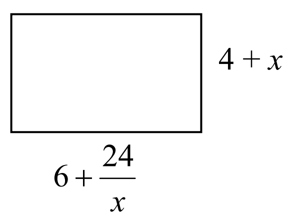
d. A = area shop front – Area window

$A = LH - 24$

$= \left(6 + \dfrac{24}{x}\right)(x + 4) - 24$ M1

$= \left(24 + 6x + \dfrac{96}{x} + 24\right) - 24$

$= 24 + 6x + \dfrac{96}{x}, \quad x > 0$ M1, A1



e.

$\dfrac{dA}{dx} = 6 - \dfrac{96}{x^2}$ M1

f.

For minimum value let $\dfrac{dA}{dx} = 0$ M1

$0 = 6 - \dfrac{96}{x^2}$

$0 = 6x^2 - 96$

$x^2 = 16$

$x = 4$ M1

minimum value of $x = 4$

minimum Area = $24 + 6x + \dfrac{96}{x}$ = 24 + 24 + $\dfrac{96}{x}$ = 72 m^2

A1

g.

To verify $x = 4$ is a minimum

If the stationary points of a function, $A = 24 + 6x + \frac{96}{x}$ are at $x = 4$ and $\frac{dA}{dx} = 6 - \frac{96}{x^2}$

For $x = 4$

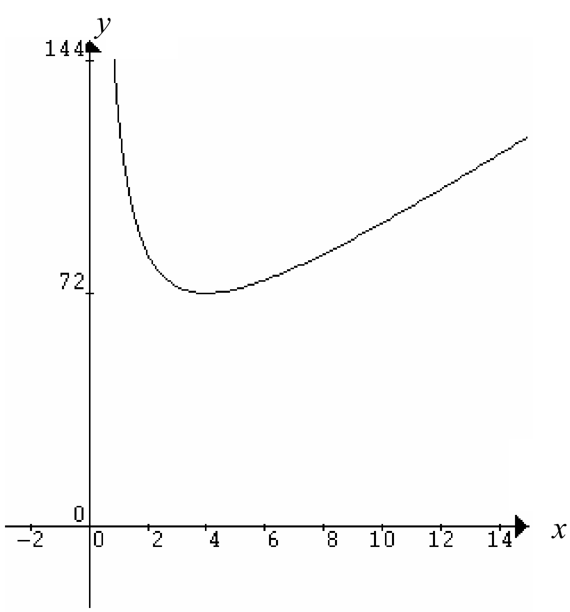
	$x < 4$	$x = 4$	$x > 4$
***x* value**	$x = 2$	$x = 4$	$x = 5$
$\frac{dy}{dx}$	$\frac{dy}{dx} = -18$	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} = 3.33$
	$\frac{dy}{dx} < 0$	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} > 0$
	\	—	/

M1

A1

Give co-ordinate of stationary point (4, 72)

h.



M1 correct shape graph
M1 labelling stationary point

i. $y = \frac{12}{x} = \frac{12}{4} = 3$ A1

Total 16 marks

Question 3

a.

i. Pr ($X = 5$) = 0.3 (from the table) A1

ii. Pr (less than half of their Music questions correctly)

= Pr ($X = 0$) + Pr ($X = 1$) + Pr ($X = 2$) M1

= 0.06 + 0.04 + 0.3

= 0.4 A1

b. $p + 3q + q + p + p + p = 1$ M1

$4p + 4q = 1$

$4p + p = 1$ since $p = 4q$

$5q = 1$

M1

$p = \frac{1}{5} = 0.2$

$\therefore$ Pr ($X = 3$) = p = 0.2 A1

c. $\therefore$ Pr (Total of 10 for Maths and Music) = 0.2 × 0.3

= 0.06 A1

d. Let Y = the number of times the celebrity team competes on the next 6 shows

Y is a Binomial distribution where $n = 6$ and $p = 0.85$

$$Pr(Y = y) = \binom{6}{6} 0.85^6 \left(0.1^0\right)$$ M1

$= 0.3771$

e.

$0.9^n > 0.5$ M1

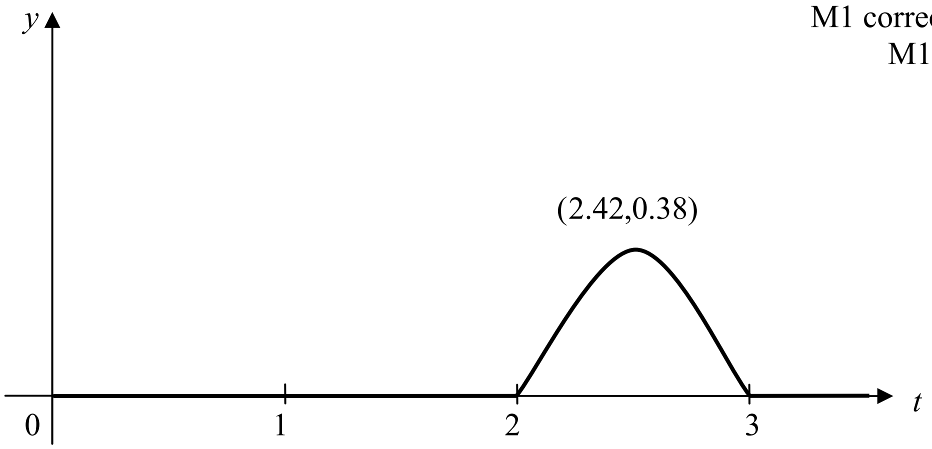
$\log_e 0.85^n \geq \log_e 0.5$

$n \leq \frac{\log_e 0.5}{\log_e 0.85}$

$n \leq 4.26$

$\therefore$ A probability of greater than 0.5 of playing on the next 4 shows. A1

f.



M1 correct shape of graph
M1 stationary point.

g.

$$\Pr(T < 150) = \int_2^{\frac{150}{60}} \left(t^3 - 9t^2 + 26t - 24\right) dt$$ M1

$$= \left[\frac{t^4}{4} - 3t^3 + 13t^2 - 24t\right]_2^{2.5}$$

$$= 0.141$$

A1

h.

$$E(T) = \int_2^3 t\, f(t)\, dt$$

$$= \int_2^3 t\left(t^3 - 9t^2 + 26t - 24\right) dt$$ M1

$$= \int_2^3 \left(t^4 - 9t^3 + 26t^2 - 24t\right) dt$$

$= 0.62hrs$

$= 37$ minutes

A1

Total 17 marks

Question 4

a. y- intercept : $y = 4 - \frac{1}{2}e^{\frac{x}{2}} - \frac{1}{2}e^{-\frac{x}{2}}$

$y = 4 - \frac{1}{2}e^{0} - \frac{1}{2}e^{0}$

$y = 3$ M1

$\Rightarrow b = 3$

b. x-intercept :

$0 = 4 - \frac{1}{2}e^{\frac{x}{2}} - \frac{1}{2}e^{-\frac{x}{2}}$

M1

$0 = 8 - e^{\frac{x}{2}} - e^{-\frac{x}{2}}$

M1

$Let\ e^{\frac{x}{2}} = y$

$0 = 8 - y - \frac{1}{y}$

$y^2 - 8y + 1 = 0$

$y = \frac{8 \pm \sqrt{60}}{2} = 4 \pm \sqrt{15}$

$sub\ \ y = e^{\frac{x}{2}}$ M1

$e^{\frac{x}{2}} = 4 \pm \sqrt{15}$

$\log_e e^{\frac{x}{2}} = \log_e\left(4 \pm \sqrt{15}\right)$

$\frac{x}{2} = \log_e\left(4 \pm \sqrt{15}\right)$

$x = 2\log_e\left(4 \pm \sqrt{15}\right)$

$\Rightarrow a = 2\log_e\left(4 + \sqrt{15}\right)$ A1

c.

i.

$A = 2[f(1) + f(2) + f(3)]$ A1

$= 2(2.872 + 2.457 + 1.648)$ M1

$= 13.95 \text{ m}^2$ A1

ii. cost = Area $\times 35$

= \$485.25 A1

d. $y = ax^2 + bx + c$

$c = 3$

$(0,3) \quad y = 0 + 3$

(0, 3) $y = ax^2 + bx + 3$

$(4, 0) \quad 0 = 16a - 4b + 3$ ....(1)

$(-4, 0) \quad 0 = 16a - 4b + 3$ ....(2)

(2) + (3) $\quad 32a + 6 = 0$

$a = \frac{-3}{16}$

$b = 0$

$y = -\frac{3}{16}x^2 + 3$

M1 for finding value of *a*
M1 for finding value of *b*
M1 for finding value of *a*

e.

$A = 2\int_0^4 \left(3 - \frac{3}{16}x^2\right)dx$ M1

$= 2\left[3x - \frac{x^3}{16}\right]_0^4$

$= 16m^2$

A1
Total 13 marks