The Mathematical Association of Victoria

MATHEMATICAL METHODS and MATHEMATICAL METHODS (CAS)

Trial written examination 1

2007

Reading time: 15 minutes
Writing time: 1 hour

Student's Name: ....

QUESTION AND ANSWER BOOK

Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
8	8	40

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

These questions have been written and published to assist students in their preparations for the 2007 Mathematical Methods and Mathematical Methods (CAS) Examination 1. The questions and associated answers and solutions do not necessarily reflect the views of the Victorian Curriculum and Assessment Authority. The Association gratefully acknowledges the permission of the Authority to reproduce the formula sheet.

Published by The Mathematical Association of Victoria
"Cliveden", 61 Blyth Street, Brunswick, 3056
Phone: (03) 9380 2399 Fax: (03) 9389 0399
E-mail: office@mav.vic.edu.au Website: http://www.mav.vic.edu.au

Working space

Instructions

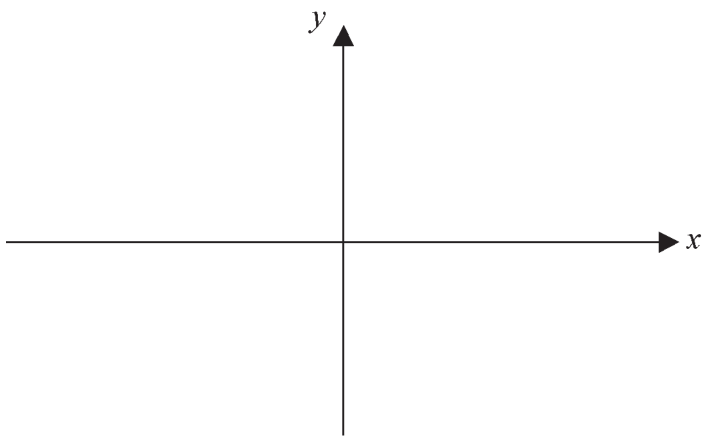
Answer **all** questions in the spaces provided.
A decimal approximation will not be accepted if an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working must be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

a. $\frac{x+2}{x-3}$ can be expressed in the form $A + \frac{B}{x-3}$, where A and B are real constants. Show that $A = 1$ and $B = 5$.

1 mark

b. Hence, sketch the graph of $y = g(x)$ given $g : R \setminus \{3\} \to R$, where $g(x) = A + \frac{B}{x-3}$ on the set of axes below, clearly showing the coordinates of any intercepts with the coordinate axes and the equations of the asymptotes.



Consider the function with rule $f(x) = \sqrt{\frac{x+2}{x-3}}$.

3 marks

c. State the maximal domain of $f(x)$.

1 mark

1 + 3 + 1 = 5 marks

Question 2

Consider the function $f:(-\infty,\ a)\rightarrow R$ where $f(x)=\log_e|2x-1|$, where a is a real constant.

a. Find the largest value of a so that f^{-1} exists.

1 mark

b. State the rule for f^{-1}, for this value of a.

2 marks

1 + 2 = 3 marks

Question 3

For what values of k, where k is a real constant, does the equation $4^x - 5(2^x) = k$, have two distinct solutions?

4 marks

Question 4

Consider the functions with rules $f(x) = (1 - x)^{\frac{1}{3}}$ and $g(x) = \log_e(2x)$.

a. Find $f(g(x))$.

1 mark

b. Find the equation of the tangent to $f(g(x))$ at $x = \dfrac{1}{2}$.

4 marks

1 + 4 = 5 marks

Question 5

Find the area bounded by the x-axis and the curve with equation $y = x(x + 1)^2$.

3 marks

TURN OVER

Question 6

Consider the function $s:[0,9]\rightarrow R$, where $s(t)=4\sin\left(\frac{\pi}{6}t\right)-2$.

a. For the function, write down

i. the range and

1 mark

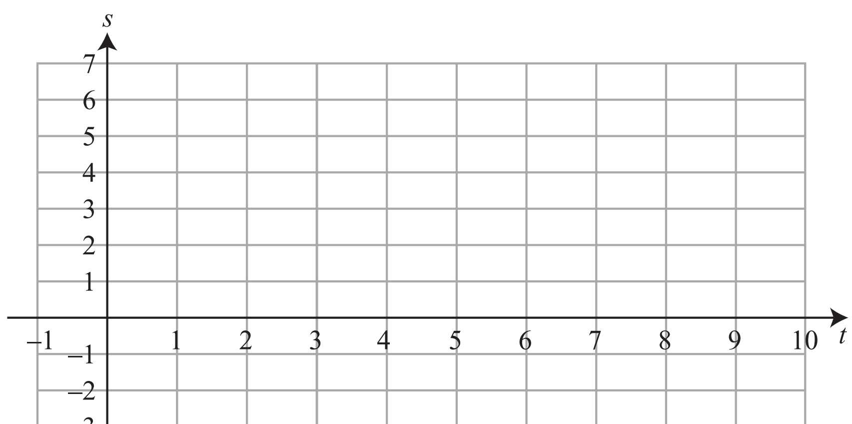
ii. the period.

1 mark

b. Solve the equation $s(t) = 0$, for $t\in[0, 9]$.

2 marks

c. Sketch the graph of the function s on the set of axes below. Label the axes intercepts and endpoints with their coordinates.



3 marks

1 + 1 + 2 + 3 = 7 marks

Question 7

A shop in the historic town of Brugge sells hand-made Belgian chocolates. The mass, in grams, of a box of twenty chocolates is a normally distributed random variable, X, with a mean of 510 grams and a standard deviation of 40 grams.

a. If $\Pr(k < X < 590) \approx 0.95$, find the value of k.

1 mark

b. Let Z be the standard normal random variable. To answer the following, use the result that, correct to two decimal places, $\Pr(Z < -0.5) \approx 0.31$. Give your answers correct to two decimal places.

i. John randomly selects a box of these chocolates from the shelf. What is the probability that its mass is less than 530 grams?

2 marks

ii. The shopkeeper sold Marie a box of chocolates with a mass **greater than** the mean value of 510 grams. What is the probability that its mass is less than 530 grams?

2 marks

1 + 2 + 2 = 5 marks

Question 8

A continuous random variable, X, has a probability density function given by

$$f(x)=\begin{cases} a(x+1) & \text{for } -1\le x\le 3 \\ 0 & \text{otherwise} \end{cases},$$

where a is a real constant.

a. Show that $a=\frac{1}{8}$.

2 marks

b. Evaluate $\Pr(X<0)$.

1 mark

c. Find the mean value of X.

2 marks

d. Find the value(s) of *m* for which $\int_{-1}^{m}\left(\frac{1}{8}(x+1)\right)dx = \frac{1}{2}$. Hence state the median value of *X*.

3marks

2 + 1 + 2 + 3 = 8 marks

END OF QUESTION AND ANSWER BOOK

Mathematical Methods Exam 1: Solutions

Question 1

a. $\frac{x+2}{x-3}=1+\frac{5}{x-3}$

$\frac{x+2}{x-3}=\frac{(x-3)+5}{x-3}$

$=\frac{(x-3)}{x-3}+\frac{5}{x-3}$ **1M**

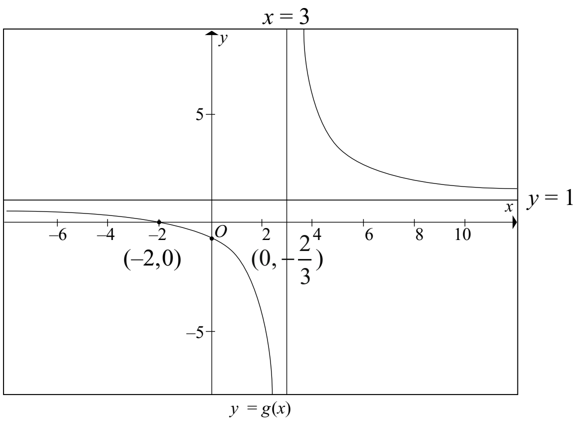
$=1+\frac{5}{x-3}$

Alternatively, use the long division algorithm.

$$\begin{array}{r} 1 \\ x-3\overline{)\,x+2} \\ \underline{x-3} \\ 5 \end{array}$$ **1M**

$\frac{x+2}{x-3}=1+\frac{5}{x-3}$

b.



Shape **1A**

Asymptotes $y = 1$ and $x = 3$ **1A**

Intercepts $\left(0,\ -\frac{2}{3}\right)$ and (–2, 0) **1A**

c. $(-\infty,-2]\cup(3,\infty)$ **1A**

Question 2

a. $a=\frac{1}{2}$ **1A**

b. Let $y=\log_e|2x-1|$, where $x<\frac{1}{2}$

For the inverse swap x and y

$x=\log_e|2y-1|$, where $y<\frac{1}{2}$ **1A**

$x=\begin{cases}\log_e(2y-1), & y>\frac{1}{2}\\ \log_e(1-2y), & y<\frac{1}{2}\end{cases}$

$e^x = 1 - 2y$

$y=\frac{1-e^x}{2}$

$f^{-1}(x)=\frac{1-e^x}{2}$ **1A**

Question 3

$4^x - 5(2^x) = k$

Let $a = 2^x$, $a > 0$

$a^2 - 5a - k = 0$ **1M**

$a=\frac{5\pm\sqrt{25+4k}}{2}$

$0 < \Delta < 25$ as $a > 0$

$0 < 25 + 4k < 25$ **1M** for discriminant
1A for restriction

$-\frac{25}{4}<k<0$ **1A**

Question 4

a. $f(g(x)) = (1 - \log_e(2x))^{\frac{1}{3}}$ **1A**

b. By the chain rule,

$f'(g(x)) \times g'(x) = \frac{1}{3}(1 - \log_e(2x))^{\frac{-2}{3}} \times \frac{-1}{x}$ **1M**

Substitute $x = \frac{1}{2}$ into the derivative to find m.

$m = \frac{1}{3}(1 - \log_e(1))^{\frac{-2}{3}} \times -2$

$= \frac{-2}{3}$ **1M**

$f\left(g\left(\frac{1}{2}\right)\right) = (1 - \log_e(1))^{\frac{1}{3}} = 1$ **1M**

The equation of the tangent is

$y - 1 = \frac{-2}{3}(x - \frac{1}{2})$

$y = -\frac{2}{3}x + \frac{4}{3}$ **1A**

Question 5

Area $= -\int_{-1}^{0}\left(x(x+1)^2\right)dx = \int_{0}^{-1}\left(x(x+1)^2\right)dx$ **1A**

$= \int_{0}^{-1}\left(x^3 + 2x^2 + x\right)dx$

$= \left[\frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2}\right]_0^{-1}$ **1M**

$= \left((\frac{1}{4} - \frac{2}{3} + \frac{1}{2}) - 0\right)$

$= \frac{3 - 8 + 6}{12}$

$= \frac{1}{12}$ units^2 **1A**

Question 6

a. i. Range: [–4 – 2, 4 – 2] = [–6, 2] **1A**

ii. Period $= \frac{2\pi}{\pi/6} = \frac{2\pi}{1} \times \frac{6}{\pi} = 12$ **1A**

b. Solve $4\sin\left(\frac{\pi}{6}t\right) - 2 = 0.$

$\sin\left(\frac{\pi}{6}t\right) = \frac{1}{2}$ **1M**

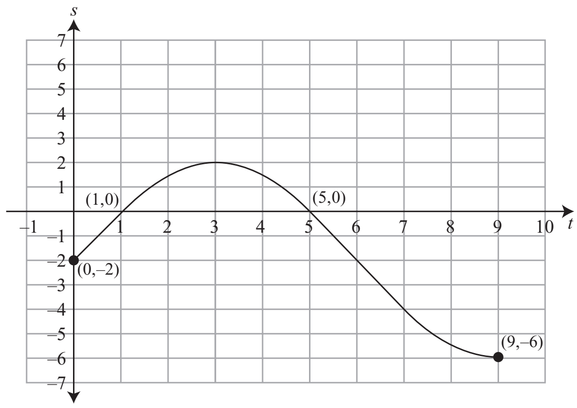
$\frac{\pi}{6}t = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, ...$

$t = \frac{\pi}{6} \times \frac{6}{\pi}, \frac{5\pi}{6} \times \frac{6}{\pi}, \frac{13\pi}{6} \times \frac{6}{\pi}, ...$

Since $t \in [0, 9]$,

$t = 1$ *or* $t = 5$ **1A**

c.



Correct shape: **1A**
Coordinates of x-axis intercepts labelled: **1A**
Endpoints labelled: **1A**

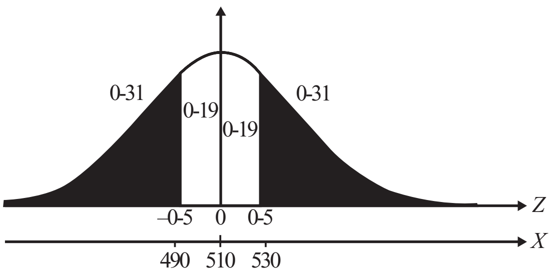
Question 7

a. $\Pr(\mu - 2\sigma < X < \mu + 2\sigma) \approx 0.95$

Therefore $\Pr(430 < X < 590) \approx 0.95$

$k = 430$ **1A**

b.



$\Pr(X < 530) = 1 - \Pr(Z < -0.5)$ **1M**

$= 1 - 0.31$

$= 0.69$ **1A**

c. $\Pr\left(X < 530 | X > 510\right) = \dfrac{\Pr\left(X > 510 \cap X < 530\right)}{\Pr\left(X > 510\right)}$

$= \dfrac{\Pr\left(510 < X < 530\right)}{\Pr\left(X > 510\right)}$ **1M**

$= \dfrac{0.19}{0.5} = 0.19 \times 2$

$= 0.38$ **1A**

Question 8

a. For f to be a probability density function,

$\int_{-\infty}^{\infty} f(x) = 1$. Therefore,

$0 + a\int_{-1}^{3}(x+1)dx = 1$

$a\left[\dfrac{x^2}{2} + x\right]_{-1}^{3} = 1$ **1M**

$a\left[\left(\dfrac{9}{2} + 3\right) - \left(\dfrac{1}{2} - 1\right)\right] = 1$

$8a = 1$

$a = \dfrac{1}{8}$, *as required* **1M**

b.

$\Pr(X < 0) = \dfrac{1}{8}\int_{-1}^{0}(x+1)dx$

$= \dfrac{1}{8}\left[\dfrac{x^2}{2} + x\right]_{-1}^{0}$

$= \dfrac{1}{8}\left[0 - \left(\dfrac{1}{2} - 1\right)\right]$

$= \dfrac{1}{16}$ **1A**

c.

$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$

$= 0 + \dfrac{1}{8}\int_{-1}^{3}(x^2 + x)dx$

$= \dfrac{1}{8}\left[\dfrac{x^3}{3} + \dfrac{x^2}{2}\right]_{-1}^{3}$ **1M**

$= \dfrac{1}{8}\left[\left(9 + \dfrac{9}{2}\right) - \left(-\dfrac{1}{3} + \dfrac{1}{2}\right)\right]$

$= \dfrac{1}{8}\left[\dfrac{26}{2} + \dfrac{1}{3}\right]$

$= \dfrac{1}{8} \times \dfrac{40}{3}$

$= \dfrac{5}{3}$ **1A**

d.

$$\frac{1}{8}\int_{-1}^{m}\left(x+1\right)dx=\frac{1}{2}$$

$$\int_{-1}^{m}\left(x+1\right)dx=4$$

$$\left[\frac{x^2}{2}+x\right]_{-1}^{m}=4$$ **1M**

$$\left[\left(\frac{m^2}{2}+m\right)-\left(\frac{1}{2}-1\right)\right]=4$$

$$\frac{m^2}{2}+m+\frac{1}{2}=4$$

$$m^2+2m-7=0$$

Use quadratic formula or complete the square

$$m=\frac{-2\pm\sqrt{4+28}}{2}$$

$$=\frac{-2\pm\sqrt{32}}{2}$$

$$=\frac{-2\pm4\sqrt{2}}{2}$$

$$=-1\pm2\sqrt{2}$$ **1M**

Note that $-1-2\sqrt{2}$ is outside the domain because $m>-1$.

Median value is $-1+2\sqrt{2}$ **1A**

MATHEMATICAL METHODS AND MATHEMATICAL METHODS (CAS)

Written examinations 1 and 2

FORMULA SHEET

Directions to students

Detach this formula sheet during reading time.
This formula sheet is provided for your reference.

Version 2 – March 2006

Mathematical Methods and Mathematical Methods CAS Formulas

Mensuration

area of a trapezium:	$\frac{1}{2}(a+b)h$	volume of a pyramid:	$\frac{1}{3}Ah$
curved surface area of a cylinder:	$2\pi rh$	volume of a sphere:	$\frac{4}{3}\pi r^3$
volume of a cylinder:	$\pi r^2 h$	area of a triangle:	$\frac{1}{2}bc\sin A$
volume of a cone:	$\frac{1}{3}\pi r^2 h$		

Calculus

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c, n \neq -1$$

$$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$$

$$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$$

$$\int \frac{1}{x}\,dx = \log_e|x| + c$$

$$\frac{d}{dx}\left(\sin(ax)\right) = a\cos(ax)$$

$$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + c$$

$$\frac{d}{dx}\left(\cos(ax)\right) = -a\sin(ax)$$

$$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + c$$

$$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\sec^2(ax)$$

product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

approximation: $f(x+h) \approx f(x) + hf'(x)$

Probability

$\Pr(A) = 1 - \Pr(A')$

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$

mean: $\mu = \mathrm{E}(X)$

variance: $\mathrm{var}(X) = \sigma^2 = \mathrm{E}((X-\mu)^2) = \mathrm{E}(X^2) - \mu^2$

probability distribution		mean	variance
discrete	$\Pr(X = x) = p(x)$	$\mu = \sum x\,p(x)$	$\sigma^2 = \sum (x-\mu)^2\,p(x)$
continuous	$\Pr(a < X < b) = \int_a^b f(x)\,dx$	$\mu = \int_{-\infty}^{\infty} x\,f(x)\,dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$

END OF FORMULA SHEET

Version 2 – March 2006