Year 2007

VCE

Mathematical Methods CAS Solutions Trial Examination 2

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SECTION 1

ANSWERS

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2	A	B	C	**D**	E
3	**A**	B	C	D	E
4	A	B	C	**D**	E
5	A	B	C	**D**	E
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9	A	**B**	C	D	E
10	A	B	C	D	**E**
11	A	B	**C**	D	E
12	A	B	**C**	D	E
13	A	B	**C**	D	E
14	A	**B**	C	D	E
15	A	B	**C**	D	E
16	**A**	B	C	D	E
17	**A**	B	C	D	E
18	A	**B**	C	D	E
19	A	B	C	**D**	E
20	**A**	B	C	D	E
21	A	B	C	**D**	E
22	A	**B**	C	D	E

SECTION 1

Question 1 **Answer E**

$y = x^3$ into $y = (4x-6)^3 + 1$ or $y - 1 = (4x-6)^3$

$y = y' - 1$ and $x = 4x' - 6$ become

$x' = \frac{x}{4} + \frac{3}{2} = 0.25x + 1.5$ and $y' = y + 1$ in matrix form

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0.25 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1.5 \\ 1 \end{bmatrix}$$

Question 2 **Answer D**

The equations can be written as

$(1)\quad -3x + (m+1)y = 5$

$(2)\quad -(m+3)x + 8y = (2m+4)$

$(m+3) \times (1) - 3(2)$ becomes

$((m+3)(m+1) - 24)y = 5(m+3) - 3(2m+4)$

$(m^2 + 4m - 21)y = -m + 3$

$$y = \frac{-(m-3)}{(m-3)(m+7)}$$

There is a unique solution if $m \in R \setminus \{3, -7\}$

If $m = 3$ the equations become

$(1)\quad -3x + 4y = 5$

$(2)\quad -6x + 8y = 10$ which are the same line and have an infinite number of solutions

If $m = -7$ the equations become

$(1)\quad -3x - 6y = 5$

$(2)\quad 4x + 8y = -10$ which are parallel lines, with no intersection points, and hence there is no solution when $m = -7$

Question 3 **Answer A**

The function $f:(-\infty,2)\to R$ has the rule $y=f(x)=\dfrac{9}{(x-2)^2}-4$ interchanging x and y

the inverse is f^{-1} $x=\dfrac{9}{(y-2)^2}-4$ transposing to make y the subject

$$\frac{9}{(y-2)^2}=x+4 \qquad (y-2)^2=\frac{9}{x+4}$$

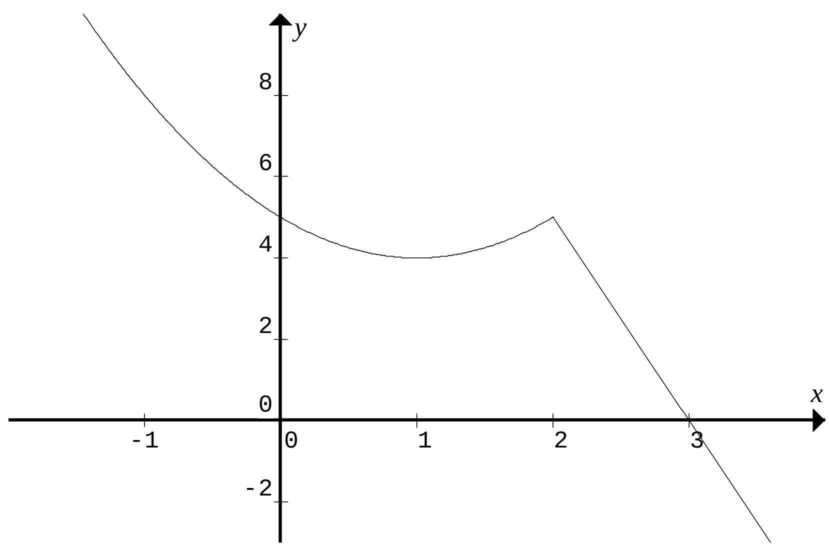
$y-2=\pm\dfrac{3}{\sqrt{x+4}}$ we need to take the negative, since the ran $f^{-1}=$ dom $f=(-\infty,2)$

$$y=f^{-1}(x)=2-\frac{3}{\sqrt{x+4}}$$

Question 4 **Answer D**

$$f(x)=\begin{cases}x^2-2x+5 & x\le 2\\ 15-5x & x>2\end{cases}$$

at $x=2$ $f(2)=5$ the hybrid graph joins up at $x=2$, so the function is continous for $x\in R$ but the function is not differentiable at $x=2$.



Question 5 **Answer D**

$P(t)=\dfrac{500}{0.03+e^{-0.1t}}$, the average rate is $\dfrac{P(2)-P(0)}{2-0}$

$$\frac{P(2)-P(0)}{2-0}=\frac{589.115-485.437}{2}=\frac{103.678}{2}=51.8$$

Question 6 **Answer C**

If $f(x) = \log_e(x)$ then for $x > 0$ and $y > 0$

$f(xy) = \log_e(xy) = \log_e(x) + \log_e(y) = f(x) + f(y)$

Question 7 **Answer E**

$|5 - 3x| > 2$ means that

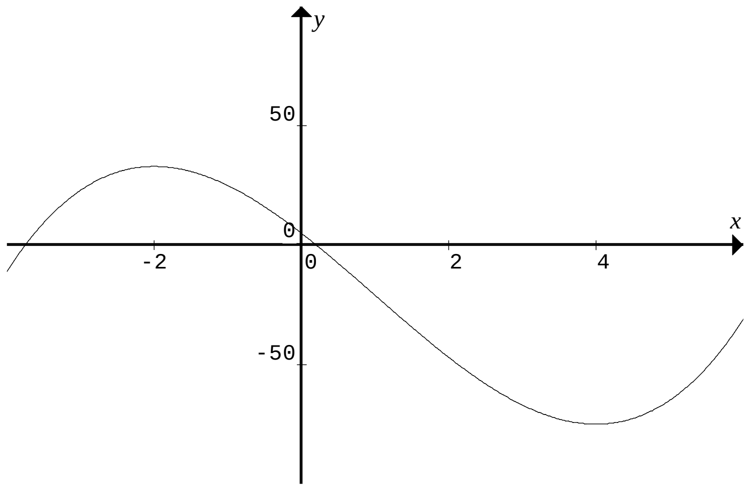
$5 - 3x > 2$ or $5 - 3x < -2$

$3 > 3x$ $\quad$ $7 < 3x$

$x < 1$ $\quad$ $x > \frac{7}{3}$

Question 8 **Answer E**

The function $f(x) = x^3 - 3x^2 - 24x + 5$ will have an inverse only if it is a one-one function, it has turning points at $x = -2$ and $x = 4$, the only possible value of a is $a \le -2$



Question 9 **Answer B**

The graph of $y = \frac{1}{x+2} - 3$ has a domain of $R \setminus \{-2\}$ and a range of $R \setminus \{-3\}$

so that $a = 2$ $c = -3$

Question 10 **Answer E**

$f(x) = h(x)\cos(4x)$ now differentiating using the product rule

$f'(x) = h'(x)\cos(4x) - 4h(x)\sin(4x)$ equating to

$f'(x) = -4e^{-4x}\left(\sin(4x) + \cos(4x)\right)$ gives $h'(x) = -4e^{-4x}$ and $h(x) = e^{-4x}$

Question 11 **Answer C**

$\int_b^a f(x)dx = A$ then $\int_a^b \left(\alpha f(x) + \beta\right)dx = \alpha\int_a^b f(x)dx + \left[\beta x\right]_a^b = -\alpha\int_b^a f(x)dx + \left[\beta(b-a)\right]$

$\int_a^b \left(\alpha f(x) + \beta\right)dx = \beta(b-a) - \alpha A$

Question 12 **Answer C**

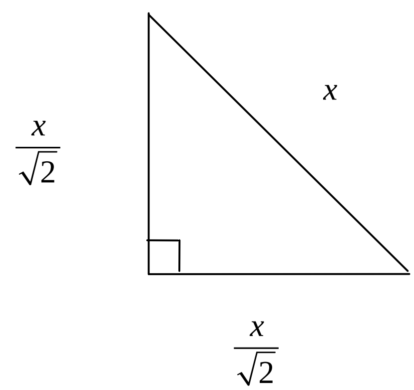
Let the isosceles triangle, have its hypotenuese length of x, by Pythagorus, the other two equal sides are $\frac{x}{\sqrt{2}}$, the area of the triangle is

$A = \frac{1}{2}\left(\frac{x}{\sqrt{2}}\right)^2 = \frac{x^2}{4}$ now given that $\frac{dx}{dt} = \sqrt{2}$

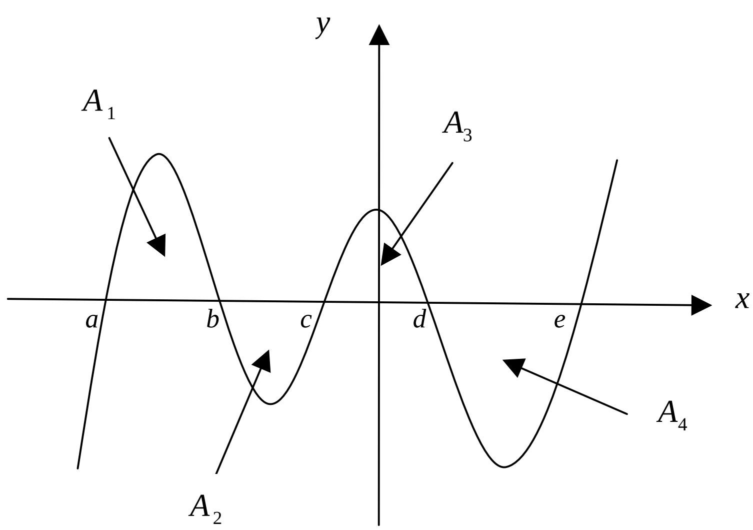
We need to find $\frac{dA}{dt}$ now $\frac{dA}{dx} = \frac{x}{2}$

by the chain rule

$\frac{dA}{dt} = \frac{dA}{dx}\frac{dx}{dt} = \frac{\sqrt{2}\,x}{2}$ when $x = 4\sqrt{2}$ $\frac{dA}{dt} = 4$



Question 13 **Answer C**



Now $A_1=\int_a^b f(x)\,dx \quad A_2=\int_b^c f(x)\,dx \quad A_3=\int_c^d f(x)\,dx \quad A_4=\int_d^e f(x)\,dx$

but $A_2<0$ and $A_4<0$ since they are below the *x*-axis, the area is

$$A=A_1-A_2+A_3-A_4=\int_a^b f(x)\,dx-\int_b^c f(x)\,dx+\int_c^d f(x)\,dx-\int_d^e f(x)\,dx$$

Question 14 **Answer B**

$y=\log_e(ax+b)$ The graph crosses the *x*-axis when $y=0$ since $\log_e(1)=0$

$ax+b=1$ so that $x=\dfrac{1-b}{a}$ and has a

maximal domain when $ax+b>0$ $\quad x>-\dfrac{b}{a}$ or $\left(-\dfrac{b}{a},\infty\right)$.

Question 15 **Answer C**

The gradient is positive when the graph of the function is sloping upwards, that is when $x<b$ and $0<x<e$

Question 16 **Answer A**

$$\lim_{\delta x\to 0}\sum_{i=1}^{n}3\sin(3x)\delta x=\int_0^3 3\sin(3x)\,dx$$

Question 17 **Answer A**

$f(x)=\int 4e^{-4x}\,d(4x)=16\int e^{-4x}\,dx=-4e^{-4x}+c$ to find *c* use $f(0)=0$

$0=-4+c$ so that $c=4$ $\quad f(x)=-4e^{-4x}+4=4\left(1-e^{-4x}\right)$

Question 18 **Answer B**

Given that $\Pr(Z<c)=a$

$\Pr(|Z|<c)=\Pr(-c<Z<c)$

$\Pr(|Z|<c)=2\Pr(0<Z<c)$

$\Pr(|Z|<c)=2(a-0.5)$

$\Pr(|Z|<c)=2a-1$

Question 19 **Answer D**

The mean value is $\frac{1}{b-a}\int_a^b f(x)\,dx$

$$\frac{1}{5-0}\int_0^5 5\sin\left(\frac{\pi x}{5}\right)dx = \frac{5}{\pi}\left[-\cos\left(\frac{\pi x}{5}\right)\right]_0^5$$

$$= -\frac{5}{\pi}\left(\cos(\pi) - \cos(0)\right)$$

$$= \frac{10}{\pi}$$

Question 20 **Answer A**

Since it is a discrete random variable, the probabilities add to one, so that $a+b=1$

$E\left(X^2\right) = \sum x^2 \Pr(X = x) = a + 4b$ but $a = 1-b$ so that

$E\left(X^2\right) = 1 + 3b$

Question 21 **Answer D**

X is the number of people listening to an Ipod $X \overset{d}{=} Bi\left(n = 10,\, p = \frac{8}{20} = 0.4\right)$

$\Pr(X = 4) = {}^{10}C_4\, 0.4^4\, 0.6^6$

Question 22 **Answer B**

$\Pr(\text{all different}) = 6 \times \frac{100}{250} \times \frac{90}{249} \times \frac{60}{248} \approx 0.2099$ Note that the 6 needs to be included as there are six different orderings, ABC ACB BAC BCA CAB CBA

END OF SECTION 1 SUGGESTED ANSWERS

SECTION 2

Question 1

a. $y = 9 - 4x^2$ at $x = a$ $y = 9 - 4a^2$ $P\left(a, 9 - 4a^2\right)$

$\frac{dy}{dx} = -8x$ $m_T = \left.\frac{dy}{dx}\right|_{x=a} = -8a$ M1

the equation of the tangent is $y - \left(9 - 4a^2\right) = -8a(x - a)$

$y = -8ax + 9 + 4a^2$ A1

b.i. at C $x = 0$ $y = c$

$c = 9 + 4a^2$ A1

ii. at B $x = b$ $y = 0$

$0 = -8ab + 9 + 4a^2$

$8ab = 9 + 4a^2$

$b = \frac{9 + 4a^2}{8a}$ A1

c. i. the area of the triangle BOC $A = \frac{1}{2}bc = \frac{\left(9 + 4a^2\right)^2}{16a}$ A1

ii. $\frac{dA}{da} = \frac{3\left(4a^2 - 3\right)\left(4a^2 + 9\right)}{16a^2} = 0$

for minimum area $\frac{dA}{da} = 0$ so that $a^2 = \frac{3}{4}$ but $0 < a < \frac{3}{2}$ A1

$a = \frac{\sqrt{3}}{2}$ A1

iii. $A_{\min}\left(\frac{\sqrt{3}}{2}\right)=6\sqrt{3}$ A1

using a sign test if

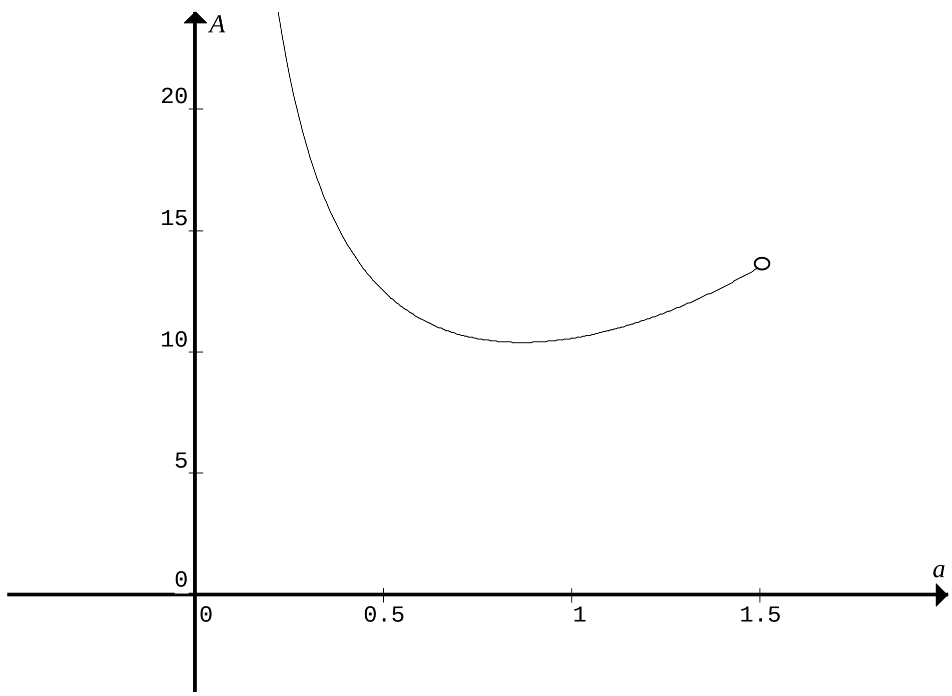
$a=0.8<\frac{\sqrt{3}}{2}$ $\frac{dA}{da}=-1.49$

M1

$a=0.9>\frac{\sqrt{3}}{2}$ $\frac{dA}{da}=0.68$

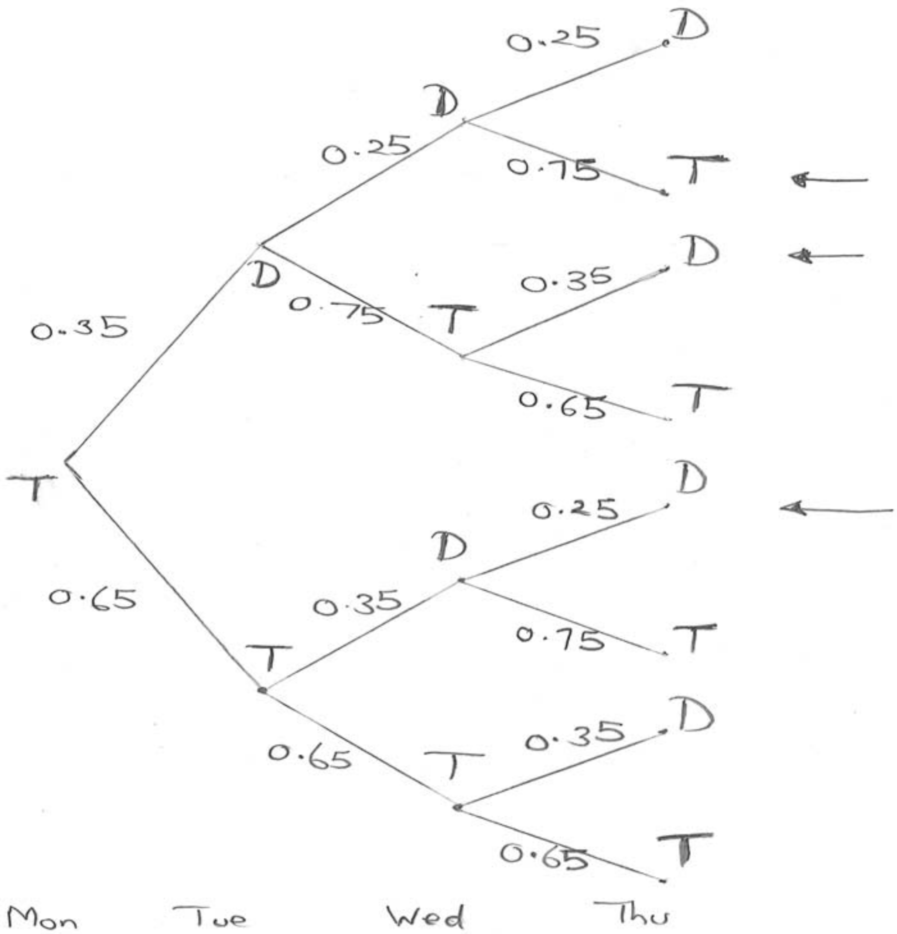
$\frac{dA}{da}$ goes from negative to zero to positive, it is a minimum.

iv. Graph restricted domain $(0,1.5)$ $a=0$ is a vertical asymptote A1



Question 2

a.



DDT or DTD or TDD using a tree diagram M1

$0.35 \times 0.25 \times 0.75 + 0.35 \times 0.75 \times 0.35 + 0.65 \times 0.35 \times 0.25$

$= 0.2144$ A1

b. $\dfrac{0.35}{0.35 + 0.75} = 0.318$

or alternatively $\begin{bmatrix} 0.65 & 0.75 \\ 0.35 & 0.25 \end{bmatrix}^{100} = \begin{bmatrix} 0.682 & 0.682 \\ 0.318 & 0.318 \end{bmatrix}$

in the long run, the probability that he takes the train to work 0.318 A1

c. D is the driving time $D \stackrel{d}{=} N\left(\mu = 35, \sigma^2 = ?^2\right)$

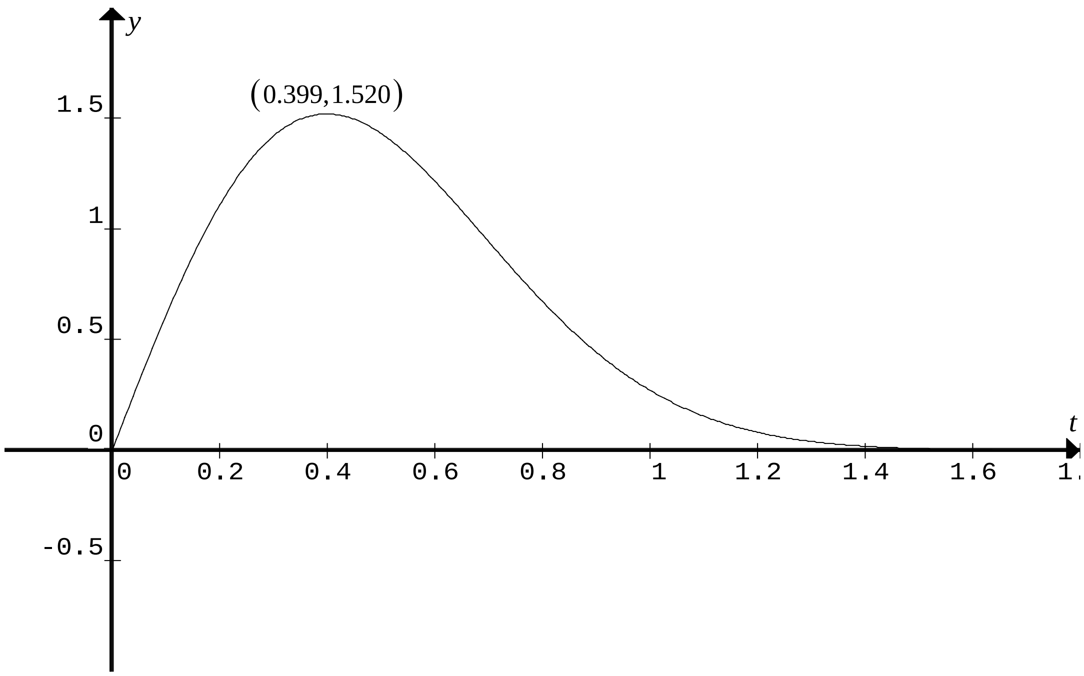
$\Pr(D < 30) = 0.30$ now $\Pr(Z < -0.524) = 0.30$

$\dfrac{30-35}{\sigma} = -\dfrac{5}{\sigma} = -0.5244$ M1

$\sigma = 9.5$ minutes A1

d. the maximum is $(0.399, 1.520)$, t axis a horizontal asymptote A1

graph correct shape, domain $[0, \infty)$, as $t \to \infty$ $y \to 0$ A1



e. 20 minutes $= \dfrac{1}{3}$ hour and 30 minutes $= \dfrac{1}{2}$ hour

$$\Pr\left(\frac{1}{3} < T < \frac{1}{2}\right) = \int_{\frac{1}{3}}^{\frac{1}{2}} 2\pi t e^{-\pi t^2}\, dt$$

$$\Pr\left(\frac{1}{3} < T < \frac{1}{2}\right) = -\left[e^{-\pi t^2}\right]_{\frac{1}{3}}^{\frac{1}{2}}$$ M1

$$\Pr\left(\frac{1}{3} < T < \frac{1}{2}\right) = e^{-\frac{\pi}{9}} - e^{-\frac{\pi}{4}}$$ A1

f. $\Pr\left(\frac{1}{3} < T < \frac{1}{2}\right) = e^{-\frac{\pi}{9}} - e^{-\frac{\pi}{4}} = 0.249$ A1

$Y \overset{d}{=} Bi\left(n = 3, p = 0.249\right)$ M1

$\Pr\left(Y \geq 1\right) = 1 - \Pr\left(Y = 0\right) = 1 - \left(1 - 0.249\right)^3$

$\Pr\left(Y \geq 1\right) = 0.577$ A1

g. $E\left(T\right) = \int_0^\infty 2\pi t^2 e^{-\pi t^2}\, dt = 0.5$ hour A1

the mean train travel time is 30 minutes A1

h. $\int_0^m 2\pi t e^{-\pi t^2}\, dt = 0.5$ A1

$-\left[e^{-\pi t^2}\right]_0^m = -e^{-\pi m^2} + 1 = 0.5$ M1

$e^{-\pi m^2} = 0.5$

$e^{\pi m^2} = 2$

$m = 0.47$ hours

$m = 28$ minutes A1

Question 3

a.i. $f(0)=90=s$ A1

ii. $f(100)=(100)^4 q+(100)^2 r+s=0$

$10^8 q+10^4 r=-90 \quad (1)$ A1

b.i. $y=qx^4+rx^2+s$

$\frac{dy}{dx}=4qx^3+2rx$

at $x=50$ $\frac{dy}{dx}=0$ M1

$0=4q(50)^3+2r \times 50=0$

$r=-5,000q \quad (2)$ A1

ii. in the equation $q\,x^4-5000q\,x^2+90=0$ let $u=x^2$ as a quadratic in u

$q\,u^2-5000q\,u+90=0$

for more than one solution, the discriminant must be positive M1

$\Delta=(-5000q)^2-4\,q \times 90$

$\Delta=25,000,000q^2-360q$ M1

$\Delta=360q\left(\frac{625000q}{9}-1\right)$

$q<0$ and $q>\frac{9}{625000}$ A2

iii. substitute (2) into (1)

$10^8 q-5000 \times 10^4 q=-90$ M1

$50,000,000\,q=-90$

$q=-\frac{9}{5,000,000} \qquad r=\frac{9}{1,000}$ A1

c. $y=-qx^4-rx^2-s$ reflection in the x-axis

$y=\frac{9x^4}{5,000,000}-\frac{9x^2}{1,000}-90 \qquad$ for $\quad 0\le x\le 100$ A1

d. now $y(50)=101.25$

the maximum width is 202.5 cm A1

e. $A=4\int_0^{100}\left(-\frac{9x^4}{5,000,000}+\frac{9x^2}{1,000}+90\right)dx$ A1

$A=33,600\ \text{cm}^2$ A1

f. $y=-a\sqrt{-bx-x^2}$ for $-100\le x\le -50$ A1

g. $y=a\sqrt{bx-x^2}$ when $x=100$ $y=0$

$0=a\sqrt{100b-100^2}$

so that $b=100$ A1

$y=a\sqrt{bx-x^2}$ when $x=50$ $y=101.25=\frac{405}{4}$

$\frac{405}{4}=a\sqrt{100 \text{ x } 50-50^2}$

$50a=\frac{405}{4}$

and $a=\frac{81}{40}$ A1

now $\frac{dy}{dx}=\frac{a}{2}(b-2x)\left(bx-x^2\right)^{-\frac{1}{2}}=\frac{a(b-2x)}{2\sqrt{bx-x^2}}$

when $x=50$ $b=100$ $\Rightarrow$ $\frac{dy}{dx}=0$

the join is smooth gradients are equal at $x=50$ A1

Question 4

a. The maximum number of hours of daylight is $\frac{1}{2}(24+5)=14.5$ hours

and occurs when $\cos\left(\frac{\pi(t-22)}{183}\right)=1$ so that $\frac{\pi(t-22)}{183}=0$ or $t=22$

on the 22nd of January. A1

b. The minimum number of hours of daylight is $\frac{1}{2}(24-5)=9.5$ hours

and occurs when $\cos\left(\frac{\pi(t-22)}{183}\right)=-1$ so that M1

$\frac{\pi(t-22)}{183}=\pi$ or $t=183+22=205$

on the 205th day of the year. A1

c.i. $2\cos\left(\frac{\pi(x-22)}{183}\right)+1=0$

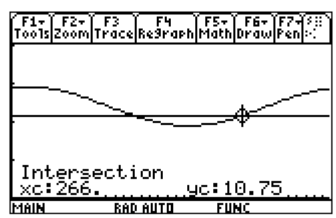
$\cos\left(\frac{\pi(x-22)}{183}\right)=-\frac{1}{2}$

$\frac{\pi(x-22)}{183}=2n\pi\pm\cos^{-1}\left(-\frac{1}{2}\right)=2n\pi\pm\frac{2\pi}{3}$ M1

$x-22=366n\pm122$

$x=366n-100$ or $x=366n+144$ where $n\in Z$ A1

ii. first solve $h(t)=10.75$ 10 hours 45 minutes



$\frac{1}{2}\left(24+5\cos\left(\frac{\pi(t-22)}{183}\right)\right)=10.75$

$\cos\left(\frac{\pi(t-22)}{183}\right)=-\frac{1}{2}$

$t=366n-100$ when $n=1$ $t=266$

$t=366n+144$ when $n=0$ $t=144$ M1

$266-144=122$ and $366-122=244$ days

so daylight of at least 10 hours and 45 minutes occurs for 244 days A1

(these values could have been obtained graphically)

d. $h(t)=12+\frac{5}{2}\cos\left(\frac{\pi(t-22)}{183}\right)$

$\frac{dh}{dt}=-\frac{5\pi}{366}\sin\left(\frac{\pi(t-22)}{183}\right)$ hours/day A1

e. $\frac{dh}{dt}$ has a maximum value $\frac{5\pi}{366}$ hours/day A1

and occurs when $\sin\left(\frac{\pi(t-22)}{183}\right)=-1$ that is when

$\frac{\pi(t-22)}{183}=\frac{3\pi}{2}$ or $t=22+\frac{3\text{x}183}{2}=296.5$

during the 296th day A1

f. $\int_1^{31}\left(12+\frac{5}{2}\cos\left(\frac{\pi(t-22)}{183}\right)\right)dt$ A1

$=433.781$ hours

$=433$ hours and 47 minutes A1

g. $a=12$ and $b=-4.5$ A1

END OF SECTION 2 SUGGESTED ANSWERS