Insight Publications

INSIGHT

Trial Exam Paper

2007

MATHEMATICAL METHODS

Written examination 2

Worked solutions

This book presents:

- correct answers
- worked solutions, giving you a series of points to show you how to work through the questions
- mark allocation details.

SECTION 1

Question 1

The graph with the equation $y = x^3$ is translated 3 units down and then reflected in the x-axis. The resulting graph has the equation

A. $y = x^3 + 3$

B. $y = -x^3 - 3$

C. $\boldsymbol{y = -x^3 + 3}$

D. $y = x^3 - 3$

E. $y = -(x-3)^3$

Answer is C

Worked solution

- Translating 3 units down produces $y = x^3 - 3$
- Reflecting in the x-axis gives $y = -(x^3 - 3) = -x^3 + 3$

Question 2

The range of the function $f : [-1,4) \to R,\ f(x) = x^2 - 2$ is

A. $[-1,\ 14)$

B. $(-1, 14]$

C. **[–2, 14)**

D. $(-2, 14)$

E. R

Answer is C

Worked solution

- The turning point is at $(0,-2)$ which is the lowest point in the range. So, the range is $[-2,14)$.

SECTION 1 – continued

Question 3

The range of the function $f : R \to R,\ f(x) = -2\cos(x - \pi) + 3$ is

A. $[-2, 3]$

B. $\mathbf{[1, 5]}$

C. $[-5, -1]$

D. $[0, 3]$

E. $[-\pi, 3]$

Answer is B

Worked solution

- The centre point of the graph is 3 and the amplitude is 2 so the graph moves up and down 2 units from 3.

Question 4

The largest set of real values of *t* for which $|t - 2| > 4t$ is

A. $t > \frac{2}{5}$

B. $t > -\frac{2}{3}$

C. $t < \frac{-2}{3}$ *or* $t > \frac{2}{5}$

D. $\boldsymbol{t < \frac{2}{5}}$

E. $\frac{2}{5} < t < 2$

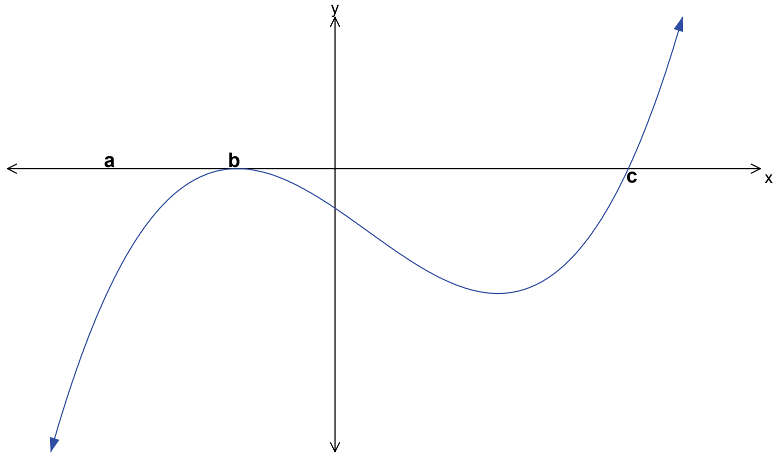
Answer is D

Worked solution

- Look at the problem graphically and find where the graph of $y_1 = |x - 2|$ is above the graph of $y_2 = 4x$. Intersection occurs at $x = \frac{2}{5}$, and the graph of y_1 is above the graph of y_2 for values of x less than $\frac{2}{5}$.

Question 5

Part of the graph of the function *f* is shown below.



The total area bounded by the curve of $y = f(x)$ and the x-axis over the interval (a, c) is given by

A. $\int_a^c f(x)dx$

B. $\boldsymbol{\int_c^a f(x)dx}$

C. $-\int_c^a f(x)dx$

D. $\int_a^b f(x)dx - \int_b^c f(x)dx$

E. $\int_a^b f(x)dx + \int_b^c f(x)dx$

Answer is B

Worked solution

- The entire area is below the x-axis so there is no need to split the area up. Be careful – the integral from a to c would give a negative value, and an area cannot be negative, hence the terminals are swapped over.

Question 6

The function *f* has the rule $f(x) = \log_e|2x-1|$. The maximal domain of *f* is

A. $(-1, \infty)$

B. $(\frac{1}{2}, \infty)$

C. $(1, \infty)$

D. $\boldsymbol{R} \setminus \left\{\frac{\mathbf{1}}{\mathbf{2}}\right\}$

E. R^{+}

Answer is D

Worked solution

- Look at the graph of this function – note that there is an asymptote at $x = \frac{1}{2}$.

Question 7

A bag contains three red and six blue balls. Three balls are drawn one at a time from the bag without replacement. The probability that at least one is red is

A. $\frac{15}{28}$

B. $\frac{5}{21}$

C. $\mathbf{\frac{16}{21}}$

D. $\frac{5}{28}$

E. $\frac{83}{84}$

Answer is C

Worked solution

- $$\begin{aligned}\Pr(\textit{at least one red}) &= 1 - \Pr(\textit{no red}) \\ &= 1 - \frac{6}{9} \times \frac{5}{8} \times \frac{4}{7} = \frac{16}{21}\end{aligned}$$

Question 8

The function $f:[a,\infty)\to R$, with the rule $y=x^{\frac{2}{3}}$, will have an inverse function if

A. $a<1$

B. $a>-1$

C. $\boldsymbol{a>0}$

D. $a<-1$

E. $a<0$

Answer is C

Worked solution

- To have an inverse function, the original function must be one-to-one, and the domain for $a>0$ gives a one-to-one function.

Question 9

The graph of the function $y=-e^{(x+2)}-5$ is obtained from the graph of $y=e^{x}$ by

A. a reflection in the x-axis, a translation by 2 units up and a translation 5 units down.

B. a reflection in the x-axis, a translation by 5 units right and a translation 2 units left.

C. a reflection in the y-axis, a translation by 5 units right and a translation 2 units down.

D. a reflection in the x-axis and a dilation by a factor of 5 from the x-axis.

E. **a reflection in the x-axis, a translation by 2 units left and a translation 5 units down.**

Answer is E

Worked solutions

Reflection in the x-axis produces $y=-e^{x}$, a translation 2 units left gives $y=-e^{x+2}$, and a translation 5 units down results in $y=-e^{(x+2)}-5$.

Question 10

If $y = 2a^{\frac{x}{3}} - b$ then x is equal to

A. $\mathbf{3\log_a(\frac{y+b}{2})}$

B. $3\log_a(\frac{y-b}{2})$

C. $3\log_a \frac{y}{2} + b$

D. $\frac{1}{3}\log_a(\frac{y+b}{2})$

E. $\frac{1}{3}\log_a(\frac{2y+b}{2})$

Answer is A

Worked solution

$$y + b = 2a^{\frac{x}{3}}$$

$$\frac{y+b}{2} = a^{\frac{x}{3}}$$

$$\log_a(\frac{y+b}{2}) = \frac{x}{3}$$

$$so\ x = 3\log_a(\frac{y+b}{3})$$

Question 11

A fair coin is tossed 20 times. The probability, correct to four decimal places, of getting fewer than 12 heads is

A. 0.1201

B. 0.8684

C. **0.7483**

D. 0.1316

E. 0.2517

Answer is C

Worked solution

Use a binomial distribution with $n = 20, p = 0.5$ to find $\Pr(X < 12)$ or use the calculator function binomcdf (20, 0.5, 11).

Question 12

The function $f:(5,\infty)\to R$ has the rule $f(x)=\dfrac{2x+1}{x-5}$. The rule for the inverse function is

A. $\boldsymbol{f^{-1}(x)=\dfrac{11}{x-2}+5}$

B. $f^{-1}(x)=\dfrac{11}{x+2}-5$

C. $f^{-1}(x)=\dfrac{11}{x+5}-2$

D. $f^{-1}(x)=\dfrac{1}{x+5}-2$

E. $f^{-1}(x)=5-\dfrac{11}{x+2}$

Answer is A

Worked solution

Re-express y to get $y=\dfrac{2x+1}{x-5}=2+\dfrac{11}{x-5}$. Interchange x and y to get

$$x=2+\frac{11}{y-5}$$

$$x-2=\frac{11}{y-5}$$

$$y-5=\frac{11}{x-2}$$

$$y=5+\frac{11}{x-2}$$

Question 13

If $4^x=e^{kx}$ for all $x\in R$, then k equals

A. 4

B. $\boldsymbol{\log_e 4}$

C. $\log_e 4x$

D. $\dfrac{4}{e}$

E. $4x$

Answer is B

Worked solution

$4=e^k$ therefore $k=\log_e 4$.

Question 14

A function $f : R \to R$ is such that:

$f'(x) = 0$ at $x = -2$ and $x = 3$
$f'(x) > 0$ for $x < -2$ and $-2 < x < 3$
$f'(x) < 0$ for $x > 3$

Which one of the following is true?

A. The graph has a local maximum turning point at $x = -2$.

B. **The graph has a stationary point of inflection at $x = -2$.**

C. The graph has a local minimum at $x = -2$.

D. The graph has a local minimum at $x = 3$.

E. The graph has a stationary point of inflection at $x = 3$.

Answer is B

Worked solution

A sketch of the details shows a stationary point of inflection at $x = -2$ and a maximum turning point at $x = 3$.

Question 15

If $f'(x) = 4x - g'(x)$, $f(0) = 3$ and $g(0) = -2$, then $f(x)$ is given by

A. $f(x) = 4x - g(x) + 1$

B. $f(x) = 4x - g(x) - 5$

C. $\boldsymbol{f(x) = 2x^2 - g(x) + 1}$

D. $f(x) = 2x^2 - g(x) - 5$

E. $f(x) = 4x - g'(x)$

Answer is C

Worked solution

Antidifferentiate to get $f(x) = 2x^2 + g(x) + c$

$f(0) = 3 \Rightarrow f(0) = -g(0) + c = 3$

$+2 + c = 3$

$c = 1$

$\Rightarrow f(x) = 2x^2 - g(x) + 1$

Question 16

The derivative of $\dfrac{\cos(3x)}{2x-e^{4x}}$ with respect to x is

A. $\dfrac{-3\sin(3x)}{2-4e^{4x}}$

B. $\dfrac{\cos(3x)(2-4e^{4x})-3(2x-e^{4x})\sin(3x)}{4x^2-4xe^{4x}+e^{8x}}$

C. $\dfrac{-3(2x-e^{4x})\sin(3x)-\cos(3x)(2-4e^{4x})}{4x^2+e^{8x}}$

D. $\boldsymbol{\dfrac{-3(2x-e^{4x})\sin(3x)-\cos(3x)(2-4e^{4x})}{4x^2-4xe^{4x}+e^{8x}}}$

E. $\dfrac{(2x-e^{4x})\cos(3x)-\sin(3x)(2-4e^{4x})}{4x^2-4xe^{4x}+e^{8x}}$

Answer is D

Worked solution

Use the quotient rule to differentiate:

$$y=\frac{\cos(3x)}{2x-e^{4x}}=\frac{u}{v};\quad \frac{dy}{dx}=\frac{vu'-uv'}{v^2}$$

$$\frac{dy}{dx}=\frac{(2x-e^{4x})\times-3\sin(3x)-\cos(3x)(2-4e^{4x})}{(2x-e^{4x})^2}$$

$$=\frac{-3(2x-e^{4x})\sin(3x)-\cos(3x)(2-4e^{4x})}{(4x^2-4xe^{4x}+e^{8x})}$$

Question 17

If $y=\left|\log_e(x)\right|$, then the rate of change of y with respect to x at $x=k,\ 0<k<1$, is

A. $\dfrac{1}{k}$

B. $\boldsymbol{-\dfrac{1}{k}}$

C. $-\log_e k$

D. $\log_e k$

E. e^k

Answer is B

Worked solution

For the region $0<x<1$ the graph of $y=\left|\log_e(x)\right|=-\log_e x$ and so $\dfrac{dy}{dx}=\dfrac{-1}{x}$, and for $x=k,\ 0<k<1$, $\dfrac{dy}{dx}=\dfrac{-1}{k}$.

Question 18

If $f : R \to R$ be a differentiable function, then for all $x \in R$, the derivative of $f(e^{4x})$ with respect to x is equal to

A. $4e^{4x}f'(x)$

B. $e^{4x}f'(x)$

C. $f'(e^{4x})$

D. $\mathbf{4e^{4x}f'(e^{4x})}$

E. $e^{4x}f'(e^{4x})$

Answer is D

Worked solution

Use the chain rule. The derivative of $f(g(x)) = f'(g(x))g'(x)$. Differentiating e^{4x} gives $4e^{4x}$ and differentiating the function *f* gives f'.

Question 19

An antiderivative of $e^{3x} + 3x + 3$ is

A. $9e^{3x} + 6x + 3$

B. $\frac{1}{3}e^{3x} + 3x^2 + 3x$

C. $3e^{3x} + 3x^2 + 3$

D. $\mathbf{\frac{1}{3}e^{3x} + \frac{3}{2}x^2 + 3x}$

E. $3e^{3x} + 3$

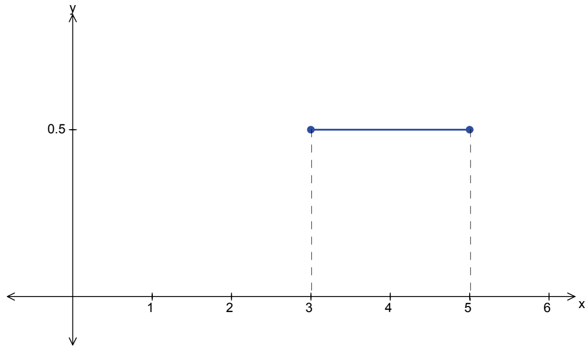
Answer is D

Question 20

A continuous random variable *X* has a probability density function given by

$$f(x) = \frac{1}{2} \text{ for } 3 < x < 5$$

The graph of *f* is shown below.



The mean of *X* is

A. 0.25

B. 0.5

C. 3

D. **4**

E. 5

Answer is D

Worked solution

Because of the symmetry of the distribution, the median and the mean are the same.

Question 21

A probability density function is given by

$$f(x) = \frac{1}{8}(2 + 2x) \qquad \text{for } 0 < x < k$$

The value of k is

A. 1

B. **2**

C. 4

D. 8

E. 12

Answer is B

Worked solution

To be a probability density function the area under the curve must equal 1. The shape under the curve is a trapezium, so the area is:

$$\begin{aligned}\text{area} &= \frac{1}{2}(a+b)h \\ &= \frac{1}{2}\left(\frac{2}{8} + \frac{(2+2k)}{8}\right)k \\ &= \frac{(2+k)k}{8} = 1 \\ \Rightarrow 0 &= k^2 + 2k - 8 \\ (k+4)(k-2) &= 0 \\ k &= 2, \text{since } k > 0\end{aligned}$$

Question 22

The masses of ‘55 gram’ chocolate bars are normally distributed with a mean of 55.1 g and a standard deviation of 0.15 g. It would be expected that about two-thirds of the chocolate bars produced would be between

A. 54.65 g and 55.55 g

B. 55.10 g and 55.40 g

C. **54.95 g and 55.25 g**

D. 54.80 g and 55.40 g

E. 51.10 g and 55.55 g

Answer is C

Worked solution

About two-thirds of the population lie within one standard deviation of the mean, and $55.1 \pm 0.15 = 54.95\text{g}$ to 55.25g.

END OF SECTION 1

SECTION 2

Question 1

Speak-Easy is a mobile phone service provider, with more than two million customers, and has two mobile phone plans, the Quickchat Plan and the Gasbagger Plan.

70% of customers have the Quickchat plan and 30% of customers have the Gasbagger Plan.

1a. If 12 Speak-Easy customers are selected at random

1a i. what is the probability, correct to four decimal places, that exactly nine customers are on the Quickchat plan?

Worked solution

Use a calculator program or inbuilt facility such as binompdf(12, 0.7, 9) to calculate

$$X \sim Bi(n = 12, p = 0.7)$$

$$\Pr(X = 9) = 0.2397$$

Mark allocation

- 1 mark for recognising and stating the correct distribution and parameters
- 1 mark for getting the correct answer

1a. ii. how many customers would be expected to be on the Quickchat plan?

Worked solution

Expected value $E(X) = np = 8.4$

Mark allocation

- 1 mark for getting the correct answer.

Tips

- *Do not round the answer off – the answer is 8.4 not 8.*

2 + 1 = 3 marks

1b. The sales manager for Speak-Easy has calculated that the monthly time in hours used by Quickchat customers is a random variable with the probability density function given by

$$f(t) = \begin{cases} 0.04e^{-0.04t} & \text{for } t > 0 \\ 0 & \text{for } t \le 0 \end{cases}$$

1b. i. What percentage, correct to the nearest per cent, of Quickchat customers use more than 20 hours in a month?

Worked solution

$$\begin{aligned}\Pr(t > 20) &= \int_{20}^{\infty} 0.04e^{-.04t}\,dt \\ &= 1 - \int_{0}^{20} 0.04e^{-.04t}\,dt \\ &= 1 - 0.55067 \\ &= 0.44933 \\ &= 45\%\end{aligned}$$

This question can be handled with a graphics calculator, using either a program or the integral facility. Because the question is worth 2 marks, a working step needs to be given.

Mark allocation

- 1 mark for setting up an integral
- 1 mark for getting the correct answer

1b. ii. What is the probability, correct to three decimal places, that a randomly chosen Quickchat customer uses more than 30 hours per month, given that the customer uses more than 20 hours per month?

Worked solution

$$\begin{aligned}\Pr(t > 30 | t > 20) &= \frac{\Pr(t > 30 \cap t > 20)}{\Pr(t > 20)} \\ &= \frac{\Pr(t > 30)}{\Pr(t > 20)} \\ &= \frac{0.30119}{0.44933} \\ &= 0.670\end{aligned}$$

Mark allocation

- 1 mark for recognising conditional probability and for having a fraction involving 0.44933 on the denominator
- 1 mark for getting the correct answer

1b. iii. Find the mean number of hours used in a month by a randomly selected Quickchat customer.

Worked solution

$$\text{mean } = \int_0^\infty t\, 0.04e^{-0.04t}\, dt = 25 \text{ hours}$$

Tip

- *This question can be handled with a graphics calculator, using either a program or the integral facility. Because the question is worth 2 marks, a working step needs to be given.*

Mark allocation

- 1 mark for stating the integral
- 1 mark for getting the correct answer

1b. iv. Find the median number of hours, correct to 2 decimal places, used in a month by a randomly selected Quickchat customer.

Worked solution

The median m is such that $\int_0^m f(t)\, dt = 0.5$

The best way to tackle this question is to use a graphics calculator.

Set up equations for y_1 and y_2 as follows:

$$y_1 = f\,\text{int}(f(x), x, 0, x)$$

$$y_2 = 0.5$$

and find the point of intersection.

This gives $m = 17.33$ hours.

Alternatively, the integral can be used as follows:

$$\int_0^m 0.04e^{-0.04t}\, dt = \frac{1}{2}$$

$$-\left[e^{-0.04t}\right]_0^m = \frac{1}{2}$$

$$e^{-0.04m} = \frac{1}{2}$$

$$m = 17.33 \text{ hours}$$

Mark allocation

- 1 mark for evidence of some relevant method that involves finding an integral = 0.5
- 1 mark for getting the correct answer.

2 + 2 + 2 + 2 = 8 marks

1c. To try to increase profits, Speak-easy is conducting a marketing campaign to persuade customers to switch to the more profitable Gasbagger mobile phone plan. The marketing manager estimates that each month 30% of customers on the Quickchat plan will switch to the Gasbagger plan, and 5% of the customers on the Gasbagger plan will switch to the Quickchat plan. No existing customers of Speak-Easy are estimated to stop using Speak-Easy.

According to the estimates, after two months, what proportion of the present customers will be on the Quickchat plan (correct to two decimal places)?

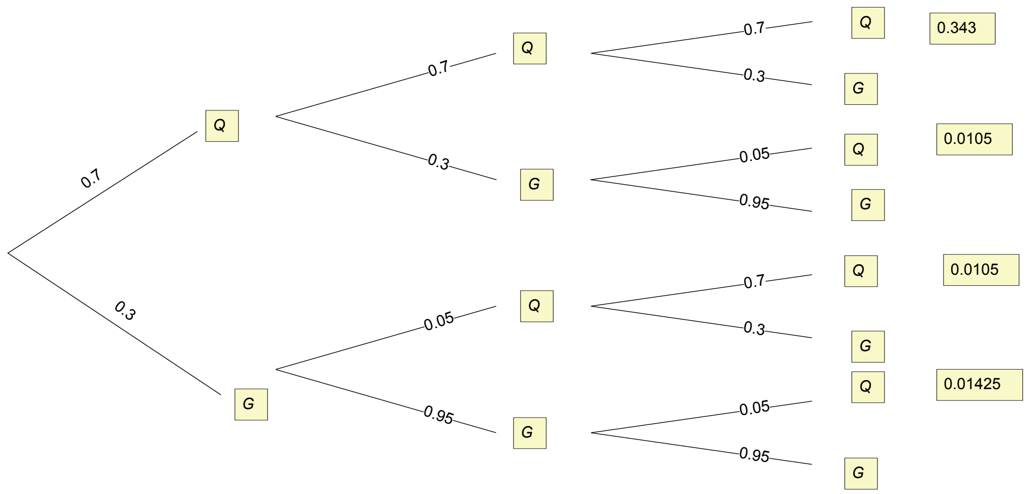
Worked solution

This question can be tackled using either matrices (not really part of a mathematical methods course) or a tree diagram.

Using matrices:

$$\begin{bmatrix} 0.7 & 0.05 \\ 0.3 & 0.95 \end{bmatrix}^2 \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} = \begin{bmatrix} 0.3783 \\ 0.6217 \end{bmatrix}$$

Using a tree diagram, with G indicating the Gasbagger plan and Q indicating the Quickchat plan, we get:



The total probability of Q as a final outcome $= 0.343 + 0.0105 + 0.0105 + 0.01425 = 0.378 =$ 37.83%

3 marks

Mark allocation

- 1 mark for drawing tree diagram with eight outcomes
- 1 mark for determining four probabilities associated with switching to Quickchat
- 1 mark for getting the correct answer of 37.83%

Total 14 marks

Question 2

Consider the function $f : R \to R, f(x) = x(x-2)^3 + 2$.

2a. If $f'(x) = (ax+b)(x-2)^2$, where a and b are constants, use calculus to find the values of a and b.

Worked solution

Using the product rule gives

$$\begin{aligned} f'(x) &= 3x(x-2)^2 + (x-2)^3 \\ &= (x-2)^2(3x + x - 2) \\ &= (x-2)^2(4x-2) \end{aligned}$$

so $a = 4, b = -2$

2 marks

Mark allocation

- 1 mark for evidence of use of the product rule
- 1 mark for getting the correct answer

2b. The coordinates of the stationary points of the graph of $y = f(x)$ are $(u,2)$ and $(v,\frac{5}{16})$.

Find the values of u and v.

Worked solution

To find stationary points, let $f'(x) = 0 \Rightarrow (x-2)^2(4x-2) = 0$

$$\Rightarrow x = 2,\ x = \frac{1}{2}$$

at $x = 2,\ y = 2$ and at $x = \frac{1}{2},\ y = \frac{5}{16}$

so $u = 2$ and $v = \frac{1}{2}$

2 marks

Mark allocation

- 1 mark for getting u correct
- 1 mark for getting v correct

2c. Find the values of x for which $f'(x) \le 0$.

Solution

Looking at the graph of $f(x)$ gives $x \le \frac{1}{2}$ or $x = 2$

2 marks

Mark allocation

- 1 mark for each one of the two intervals that is given correctly

2d. Find the real values of k for which both solutions to the equation $f(x+k) = 2$ are positive.

Solution

Look at a graph of the function – both solutions are positive when the graph is moved any distance to the right. To move right, $k < 0$.

1 mark

Mark allocation

- 1 mark for answer $k < 0$.

The function *f(x)* can also be written as $f(x) = x^4 - 6x^3 + 12x^2 - 8x + 2$.

2e. Use calculus to find the area bounded by the graph of $y = f(x) - 2$ and the x-axis.

Worked solution

Be careful here as the area cannot be negative – it is best to use absolute value signs around the calculation.

$$\text{Area} = \left| \int_0^2 x^4 - 6x^3 + 12x^2 - 8x \; dx \right|$$

$$= \left| \left[\frac{x^5}{5} - \frac{6x^4}{4} + \frac{12x^3}{3} - \frac{8x^2}{2} \right]_0^2 \right|$$

$$= \frac{16}{10} = 1.6 \text{ square units}$$

3 marks

Mark allocation

- 1 mark for setting up the integral
- 1 mark for getting the correct anti-derivative
- 1 mark for getting the correct answer

2f. i. Describe a sequence of transformations which maps the graph of $y = f(x)$ on to the graph of $y = f(2x) - 2$.

Worked solution

Dilation by a factor of $\frac{1}{2}$ in x-direction (or from the y-axis), then translation of 2 units down.

Mark allocation

- 1 mark for dilation specified correctly
- 1 mark for translation specified correctly

2f. ii. Find the x-axis intercepts of the graph of $y = f(2x) - 2$.

Worked solution

The two intercepts of $(0,0)$ $(1,0)$ can be found by looking at the x-intercepts of $y = f(x) - 2$ and halving.

Mark allocation

- 1 mark for getting coordinates for both intercepts correct

2f. iii. Use the answers to 2e., 2fi. and 2fii. above to write down the area of the region bounded by the graph of $y = f(2x) - 2$ and the x-axis, correct to two decimal places.

Worked solution

The area should be calculated as:

$$\text{area} = \frac{1}{2} \times \text{ area found in question 2e.}$$

To get any marks it is necessary to have this statement or other evidence of halving the answer to question 2e. If the area is calculated by using an integral, then zero marks are awarded.

$$= \frac{1}{2} \times 1.6 = 0.80 \text{ square units}$$

Mark allocation

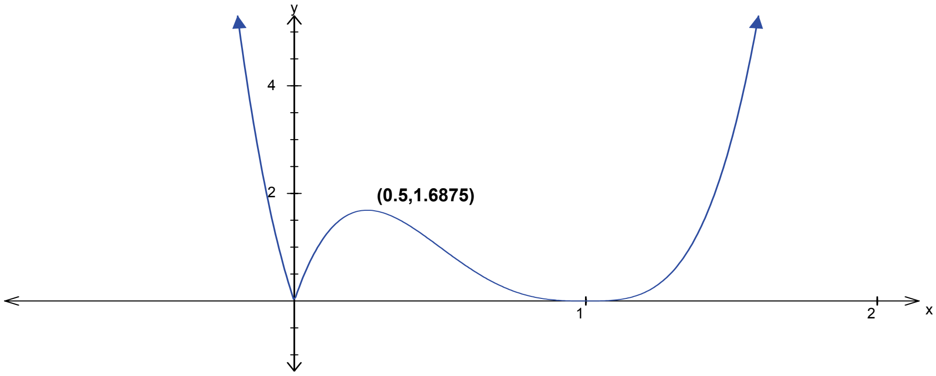
- 1 mark for getting the correct answer

2 + 1 + 1 = 4 marks

2g. Find the real values of p for which the equation $|f(2x)-2| = p$ has exactly four solutions.

Worked solution

Draw the graph of $y = |f(2x)-2|$ as follows:



A horizontal line drawn through the graph will intersect the graph at 4 points in the interval from $y = 0$ to the y coordinate of the turning point.

So, $p \in (0, 1.6875)$.

2 marks

Mark allocation

- 1 mark for finding the turning point
- 1 mark for getting the correct interval

Total 16 marks

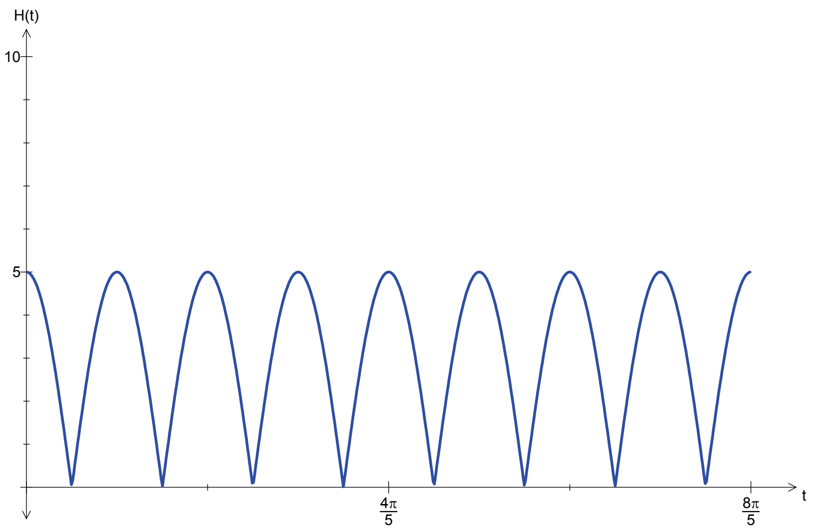
Question 3

A ball is dropped from a height of 5 metres and bounces continuously in a regular motion for the next $\frac{8\pi}{5}$ seconds. The height of the ball above the ground is given by

$$H(t) = |5\cos(5t)|$$

where H is the height in metres and t is the time in seconds after the ball is dropped.

The graph of the motion as modelled by the equation $H(t) = |5\cos(5t)|$ for $t \in [0, \frac{8\pi}{5}]$ is shown below.



3a. Find when the ball first hits the ground.

Worked solution

Look at the graph and divide the first section of the graph from $[0, \frac{4\pi}{5}]$ by 8 to get $\frac{\pi}{10}$. So the ball hits the ground at $\frac{\pi}{10}$ seconds.

1 mark

Mark allocation

- 1 mark for getting the correct answer

3b. Find when the ball first rebounds to maximum height.

Worked solution

The ball reaches maximum height $\frac{\pi}{10}$ seconds further on from the first bounce point –so this occurs at $\frac{\pi}{10}+\frac{\pi}{10}=\frac{\pi}{5}$ seconds.

1 mark

Mark allocation

- 1 mark for getting the correct answer

Realistically, the ball cannot rebound to its previous height. Instead, the rebound height on each bounce will diminish with each bounce. It is found that the height of a ball released from a height of 5 metres can be more accurately modelled by the equation

$$H(t)=\left|5e^{-0.8t}\cos(5t)\right|$$

3c. Find the first time, correct to 2 decimal places, when the ball is exactly 3 metres above the ground.

Worked solution

Use a graphics calculator to find the point of intersection between the lines $y_1=\left|5e^{-0.8x}\cos(5x)\right|$ and $y_2=3$.

Intersection occurs when t = 0.16 seconds.

1 mark

Mark allocation

- 1 mark for getting the correct answer

Tip

- *You should be careful to have the correct window setting and to ensure the mode is in radians.*

3d. How many times during the motion as described by the model is the ball exactly at a height of 2 metres above the ground?

Worked solution

Using a graphics calculator, find the intersections of the graph $y_1=\left|5e^{-0.8x}\cos(5x)\right|$ with the line $y_2=2$ and find the number of times an intersection occurs – be careful to have the correct domain sketched.

It can be seen that the graph of y_1 intersects the line y_2 at 3 points.

1 mark

Mark allocation

- 1 mark for getting the correct answer

3e. i. Find an expression for $\frac{dH}{dt}$ for values of t such that $\frac{\pi}{10} < t < \frac{3\pi}{10}$.

Worked solution

For the interval $t \in [\frac{\pi}{10}, \frac{3\pi}{10}]$ the graph behaves like $H(t) = -5e^{-0.8t}\cos(5t)$ so $\frac{dH}{dt}$ can be found using the product rule and chain rule to give

$$\frac{dH}{dt} = 4e^{-0.8t}\cos(5t) + 25e^{-0.8t}\sin(5t)$$

Mark allocation

- 1 mark for knowing that *H(t)* can be written as $H(t) = -5e^{-0.8t}\cos(5t)$
- 1 mark for evidence of use of the chain and product rules
- 1 mark for getting the correct formula for $\frac{dH}{dt}$

3e. ii. Find $\frac{dH}{dt}$ for $t = \frac{\pi}{5}$. Give your answer correct to two decimal places.

Worked solution

Using a calculator, let $y_1 = \left|5e^{-0.8x}\cos(5x)\right|$ and use the calculator menu to find $\frac{dy}{dx}$ at $x = \frac{\pi}{5}$.

This gives $\frac{dy}{dx} = -2.42$.

Mark allocation

- 1 mark for getting the correct answer

3e. iii. Using your answer to ei., write down an equation the solution of which gives the value of t when the ball has rebounded fully from each bounce. Find this value of t for the first rebound, correct to two decimal places. Also find, according to the model, the height of the first full rebound, correct to two decimal places.

Worked solution

The maximum rebound height occurs when $\frac{dH}{dt} = 0$, so let

$$4e^{-0.8t}\cos(5t) + 25e^{-0.8t}\sin(5t) = 0$$

This gives

$$4e^{-0.8t}\cos(5t) = -25e^{-0.8t}\sin(5t)$$

$$\frac{-4}{25} = \tan(5t)$$

$$\pi - \tan^{-1}(\frac{4}{25}) = 5t$$

$$\frac{1}{5}(\pi - \tan^{-1}(\frac{4}{25})) = t$$

$$t = 0.60,\ H(t) = 3.06$$

Mark allocation

- 1 mark for stating that $4e^{-0.8t}\cos(5t)+25e^{-0.8t}\sin(5t)=0$
- 1 mark for finding $t=0.60$
- 1 mark for finding $H(t)=3.06$

Tip

- *The solution can be found from the graphics calculator. There is no requirement to find it algebraically, it is just necessary to state that* $4e^{-0.8t}\cos(5t)+25e^{-0.8t}\sin(5t)=0$.

3 + 1 + 3 = 7 marks

3f. The company that manufactures the balls can modify the elastic component of the balls and produce balls that bounce back to different rebound heights according to the model

$$H(t)=\left|5e^{-at}\cos(5t)\right| \text{ for } a>0$$

Find the exact value of *t* that gives the time when a ball rebounds fully after the first bounce.

Worked solution

The graph in this interval behaves like $H(t)=-5e^{-at}\cos(5t)$.

Letting $\frac{dH}{dt}=0$ gives:

$$5ae^{-at}\cos(5t)=-25e^{-at}\sin(5t)$$

$$\frac{-5a}{25}=\tan(5t)$$

$$\pi-\tan^{-1}(\frac{5a}{25})=5t$$

$$\frac{1}{5}(\pi-\tan^{-1}(\frac{5a}{25}))=t$$

$$t=\frac{1}{5}(\pi-\tan^{-1}(\frac{a}{5}))$$

3 marks

Mark allocation

- 1 mark for obtaining the correct derivative and setting it equal to 0
- 1 mark for obtaining an equation involving tan
- 1 mark for getting the correct answer

3g. A ball is considered flat if the first full rebound height is less than 2 metres. Find, correct to 3 decimal places, the smallest value of *a* for a ball to be considered flat.

Worked solution

When $t = \frac{1}{5}(\pi - \tan^{-1}(\frac{a}{5}))$, this gives $H(t) = -5e^{-a(\frac{\pi}{5} - \frac{\tan^{-1}(\frac{a}{5})}{5})}\cos(\pi - \tan^{-1}(\frac{a}{5}))$

(There is no need to try to simplify this equation.)

We want $H(t) < 2$, so set $y_1 = -5e^{-x(\frac{\pi}{5} - \frac{\tan^{-1}(\frac{x}{5})}{5})}\cos(\pi - \tan^{-1}(\frac{x}{5}))$ and $y_2 = 2$, and find when $y_1 < y_2$ by finding the point of intersection with a graphics calculator.

Intersection occurs at x = 1.532, so the smallest value of *a* is 1.532.

3 marks

Mark allocation

- 1 mark for finding the correct expression for *H(t)*
- 1 mark for setting $H(t) < 2$
- 1 mark for getting the correct answer

Total 17 marks

Question 4

Two street lights *A* and *B*, of power 40 units and 320 units respectively, are placed 20 metres apart. The intensity of light illumination, *I* units, along the straight road between the two lights is given by

$$I(x) = \frac{40}{x^2} + \frac{320}{(20-x)^2},$$

where *x* is the distance in metres from light *A*.

4a. What is the intensity of light illumination at a point 5 metres from light *B*? Express the answer correct to 2 decimal places.

Worked solution

5 metres from *B* means 15 metres from *A*, so $x = 15$.

This gives $I = \frac{40}{15^2} + \frac{320}{5^2} = 12.98$

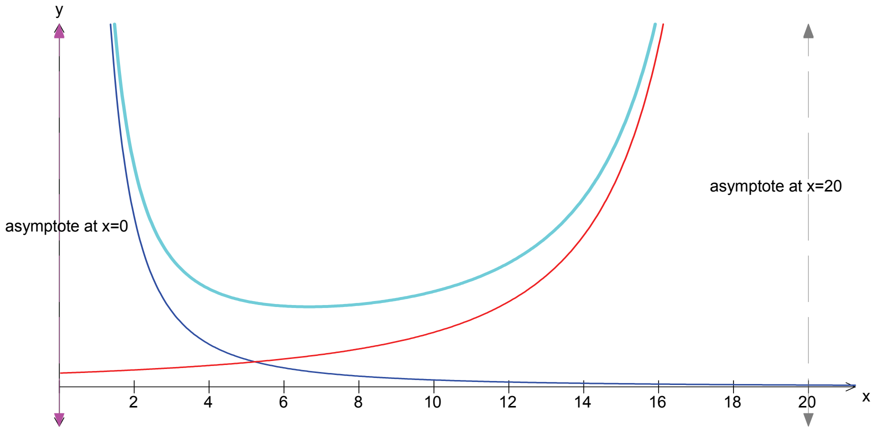
1 mark

Mark allocation

- 1 mark for getting the correct answer

4b. The graphs of $y_1 = \dfrac{40}{x^2}$ and $y_2 = \dfrac{320}{(20-x)^2}$ are shown below. On the same set of axes, sketch the graph of I for $0 < x < 20$.

Worked solution



3 marks

Mark allocation

- 1 mark for getting the correct shape of the graph
- 1 mark for labelling asymptotes at $x = 0$ and $x = 20$
- 1 mark for showing the new graph above both existing graphs

4c. Diana stands on the straight road between lights A and B. Where should she stand to ensure that the light illumination intensity is more than 18 units? Express your answer correct to two decimal places.

Worked solution

Use a graphics calculator to find the point of intersection between the graph of

$y_1 = \dfrac{40}{x^2} + \dfrac{320}{(20-x)^2}$ and the line $y_2 = 18$.

The required region is $x \in [0, 1.53] \cup [15.76, 20]$. Diana can stand anywhere along the straight road within the required region.

2 marks

Mark allocation

- 1 mark for each one of the 2 intervals that is given correctly

4d. Use calculus to find the exact point on the straight road between the two lights that has the least light illumination intensity.

Worked solution

$$y = \frac{40}{x^2} + \frac{320}{(20-x)^2}$$

$$= 40x^{-2} + 320(20-x)^{-2}$$

$$\frac{dy}{dx} = -80x^{-3} + 640(20-x)^{-3}$$

$$= \frac{-80}{x^3} + \frac{640}{(20-x)^3}$$

Least illumination intensity occurs when $\frac{dy}{dx} = 0$.

Setting $\frac{dy}{dx} = 0$ gives:

$$\frac{80}{x^3} = \frac{640}{(20-x)^3}$$

$$640x^3 = 80(20-x)^3$$

$$8x^3 = (20-x)^3$$

$$2x = (20-x)$$

$$3x = 20$$

$$x = \frac{20}{3} \text{ metres from } A$$

5 marks

Mark allocation

- 1 mark for attempting to find $\frac{dy}{dx}$
- 1 mark for correctly finding $\frac{dy}{dx}$
- 1 mark for setting $\frac{dy}{dx} = 0$
- 1 mark for solving the resulting equation
- 1 mark for getting the correct answer

Total: 11 marks

END OF WORKED SOLUTIONS