Insight Publications

INSIGHT

Trial Exam Paper

2007

MATHEMATICAL METHODS

Written examination 1

Worked solutions

This book presents:

- correct answers
- worked solutions, giving you a series of points to show you how to work through the questions
- mark allocation details.

Question 1

Let $f(x) = 2x - 5$ and $g(x) = \cos x$. Write down the rule of $f(g(x))$.

Solution

$f(g(x)) = 2\cos x - 5$

1 mark

Mark allocation

- 1 mark for the correct answer

Question 2

For the function $f:(2,\infty) \to R,\ f(x) = 2\log_e(x-1)$,

2a. find the rule for the inverse function f^{-1}.

Solution

Interchange x and y to give

$x = 2\log_e(y-1)$

$e^{\frac{x}{2}} = y - 1$

$y = 1 + e^{\frac{x}{2}}$

2 marks

Mark allocation

- 1 mark for method
- 1 mark for the correct answer

2b. find the domain of the inverse function f^{-1}.

Solution

$dom\ f^{-1} = ran\ f$

$= R^+$

1 mark

Mark allocation

- 1 mark for the correct answer

Question 3

For the function $f:[-\pi,\pi]\rightarrow R, f(x)=-2\sin(3(x-\frac{\pi}{4}))$

3a. write down the amplitude and period of the function.

Solution

Amplitude is 2

Period is $\frac{2\pi}{3}$

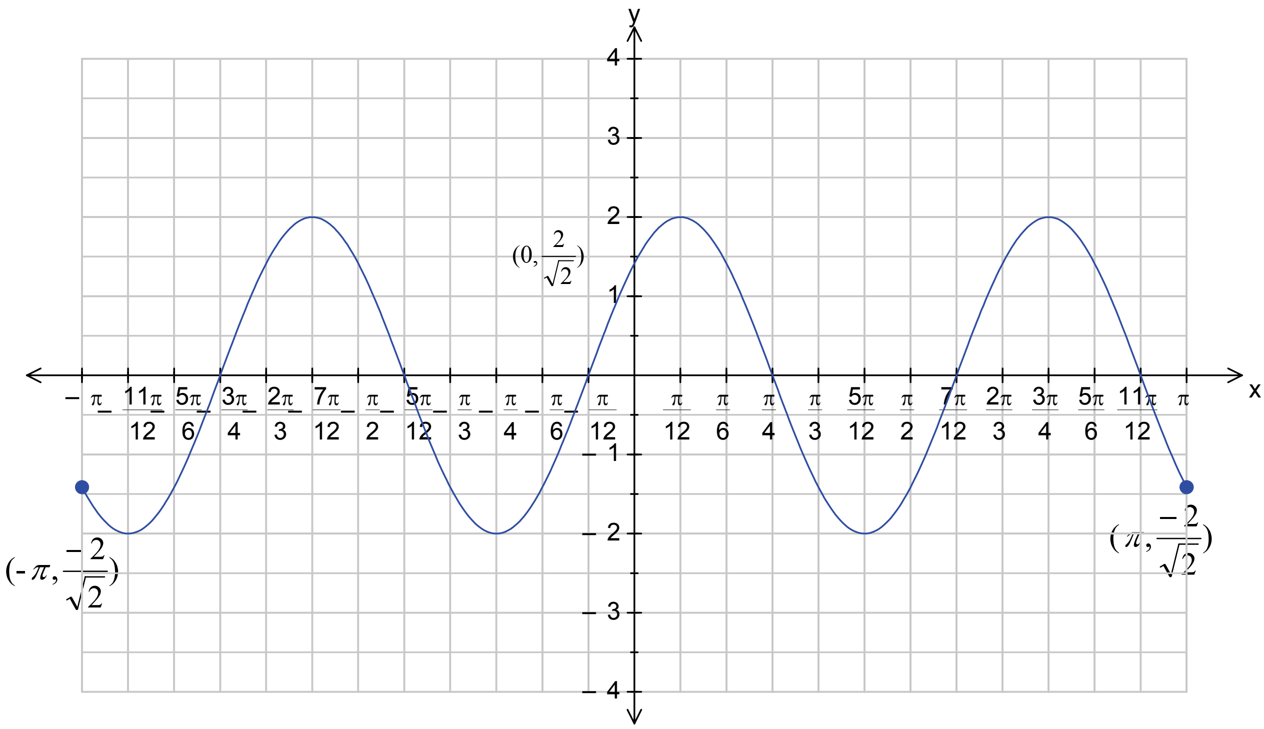
2 marks

Mark allocation

- 1 mark for each of amplitude and period

3b. on the set of axes below, sketch the graph of the function *f*. Label the axis intercepts with their coordinates. Label the end-points of the graph with their coordinates.

Solution



3 marks

Mark allocation

- 1 mark for shape – the graph must be a smooth, regular sine curve shape, must have 3 cycles and must have the correct amplitude
- 1 mark for correct intercepts – the intercepts must be correctly labelled with their coordinates; the *x*-intercepts are at $(\frac{\pi}{4},0)$, $(\frac{7\pi}{12},0)$, $(\frac{11\pi}{12},0)$, $(\frac{-\pi}{12},0)$, $(\frac{-5\pi}{12},0)$ and $(\frac{-3\pi}{4},0)$ and the *y*-intercept is at $(0,\frac{2}{\sqrt{2}})$
- 1 mark for correctly placing and labelling the graph end-points

Question 4

4a. Let $f(x) = \log_e(\sin(x))$. Find $f'(x)$.

Solution

Use the chain rule:

$$f'(x) = \frac{1}{\sin x}\cos x = \frac{\cos x}{\sin x}$$

1 mark

Mark allocation

- 1 mark for the correct answer

4b. Let $y = x^2\cos(x)$. Evaluate $\frac{dy}{dx}$ when $x = \frac{\pi}{3}$.

Solution

Using the product rule gives $\frac{dy}{dx} = 2x\cos x - x^2 \sin x$

At $x = \frac{\pi}{3}$,

$$x = \frac{\pi}{3} \Rightarrow \frac{dy}{dx} = \frac{2\pi}{3}\frac{1}{2} - \frac{\pi^2\sqrt{3}}{9\times 2}$$

$$= \frac{\pi}{3} - \frac{\sqrt{3}\pi^2}{18}$$

2 marks

Mark allocation

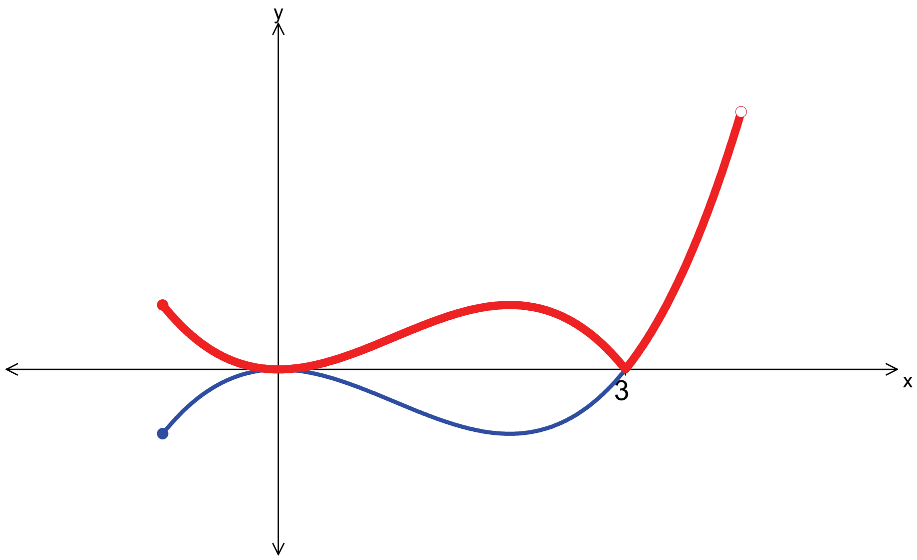
- 1 mark for evidence of using the product rule
- 1 mark for the correct answer

Question 5

The graph of $f:[-1,4]\to R$ where $f(x)=x^3-3x^2$ is shown below.

5a. Let $g(x)=|f(x)|$. On the same set of axes, sketch the graph of g.

Solution



2 marks

Mark allocation

- 1 mark for graph drawn as shown: in the interval from x = 3 to x = 4, it must be obvious that the original graph has been drawn over
- 1 mark for correct end-points

5b. State the domain of the derivative function g'.

Solution

$(-1,3)\cup(3,4)$ –note that the graph is not differentiable at the end-points or at the cusp

1 mark

Mark allocation

- 1 mark for the correct answer

Question 6

Solve the equation $\sin(2x)-\sqrt{3}\cos(2x)=0$ for $x\in[0,2\pi]$, giving exact values in terms of π.

Solution

$$\sin 2x=\sqrt{3}\cos 2x$$

$$\tan 2x=\sqrt{3}$$

$$2x=\frac{\pi}{3},\frac{4\pi}{3},\frac{7\pi}{3},\frac{10\pi}{3}$$

$$x=\frac{\pi}{6},\frac{2\pi}{3},\frac{7\pi}{6},\frac{5\pi}{3}$$

3 marks

Mark allocation

- 1 mark for getting equation in terms of tan
- 1 mark for obtaining the first quadrant angle of $\frac{\pi}{3}$
- 1 mark for getting all answers correct

Question 7

The probability density function of a continuous random variable X is given by

$$f(x)=\begin{cases}\dfrac{x}{k} & 2\le x\le 6\\ 0 & \text{otherwise}\end{cases}$$

7a. Show that $k=16$.

Solution

$$\int_2^6 \frac{x}{k}\,dx=1$$

$$\frac{1}{k}\left[\frac{x^2}{2}\right]_2^6=1$$

$$\frac{32}{2k}=1$$

$$\Rightarrow k=16$$

2 marks

Mark allocation

- 1 mark for setting the integral equal to 1
- 1 mark for solving to get k = 16

Question 7 – continued

7b. Find $\Pr(X > 4)$

Solution

$$\Pr(X > 4) = \frac{1}{16}\int_4^6 x\,dx$$
$$= \frac{1}{32}\left[x^2\right]_4^6$$
$$= \frac{1}{32}(36 - 16)$$
$$= \frac{5}{8}$$

2 marks

Mark allocation

- 1 mark for setting up an integral from 4 to 6
- 1 mark for the correct answer

7c. Find the median of X

Solution

$$\int_2^m \frac{x}{16}\,dx = 0.5$$
$$\frac{1}{32}\left[x^2\right]_2^m = 0.5$$
$$m^2 - 4 = 16$$
$$m^2 = 20$$
$$m = 2\sqrt{5} \quad (\text{since } m > 0)$$

2 marks

Mark allocation

- 1 mark for setting up the integral as equal to 0.5
- 1 mark for the correct answer

Question 8

The random variable X has the following probability distribution:

x	-1	0	1	2
$Pr(X = x)$	$a + b$	$2a - b$	$3a$	0.4

8a. Find the value of a.

Solution

The sum of the probabilities has to equal 1 so

$a + b + 2a - b + 3a + 0.4 = 1$

$6a + 0.4 = 1$

$6a = 0.6$

$a = 0.1$

1 mark

Mark allocation

- 1 mark for the correct answer

8b. If $E(X) = 0.95$, find the value of b.

Solution

$E(X) = \sum x \Pr(X) = -1 \times (a + b) + 3a + 2 \times 0.4 = 0.95$

$2a - b = 0.15$

$b = 0.05$

2 marks

Mark allocation

- 1 mark for equating the sum of the products to 0.95
- 1 mark for the correct answer

Question 9

The random variable X is normally distributed with mean 50 and standard deviation 5. The random variable Z is normally distributed with mean 0 and standard deviation 1.

9a. If $\Pr(X < 56) = \Pr(Z < a)$, find the value of a.

Solution

$$\Pr(X < 56) = \Pr(Z < \frac{56-50}{5})$$
$$= \Pr(Z < \frac{6}{5})$$
$$\Rightarrow a = \frac{6}{5}$$

2 marks

Mark allocation

- 1 mark for converting to a z-score
- 1 mark for the correct answer

9b. If $\Pr(50 < X < b) = 0.5 - \Pr(Z > 2)$, find the value of b.

Solution

Using symmetry, $0.5 - \Pr(Z > 2) \equiv \Pr(0 < Z < 2)$

Converting to standard normal distribution gives

$$\frac{b-50}{5} = 2$$
$$b - 50 = 10$$
$$b = 60$$

2 marks

Mark allocation

- 1 mark for recognising that $0.5 - \Pr(Z > 2) \equiv \Pr(0 < Z < 2)$
- 1 mark for the correct answer

Question 10

A hemispherical bowl is being filled with water at a constant rate of $150\pi \ cm^3/\min$. When the depth of the water in the bowl is h cm, the volume, V cm^3, of the water is given by $V = \pi h^2(30 - \frac{2}{3}h)$. Find the rate at which the depth of the water is increasing when the depth is 5 cm.

Solution

$$\frac{dV}{dt} = 150\pi$$

$$V = \pi h^2(30 - \frac{2}{3}h)$$

$$\Rightarrow \frac{dV}{dh} = 60\pi h - 2\pi h^2$$

Using the chain rule:

$$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$$

$$= 150\pi \times \frac{1}{60\pi h - 2\pi h^2}$$

$$= \frac{150}{60h - 2h^2}$$

At $h = 5$, $\frac{dh}{dt} = \frac{150}{250} = \frac{3}{5}$ cm/min

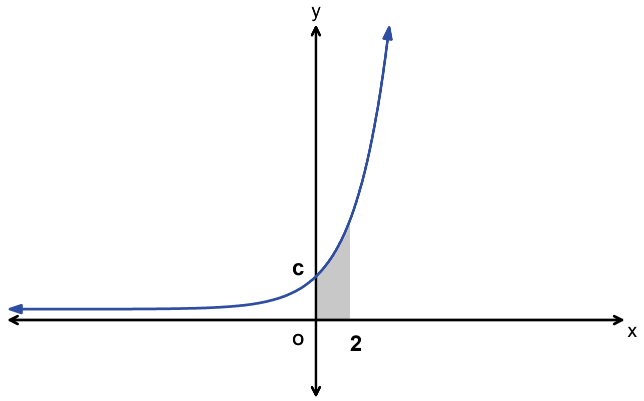
4 marks

Mark allocation

- 1 mark for recognising $\frac{dV}{dt}$
- 1 mark for determining $\frac{dV}{dh}$
- 1 mark for using the chain rule
- 1 mark for the correct answer

Question 11

Part of the graph of the function $f : R \to R,\ f(x) = ae^{2x} + b$ is shown below. If the shaded area is $3e^4 + 1$ square units, find one set of possible values for *a*, *b* and *c,* where *c* is the *y*-intercept of the graph $y = f(x)$.



Solution

At $x = 0,\ y = a + b \quad \Rightarrow a + b = c$

$$\int_0^2 ae^{2x} + b\ dx = 3e^4 + 1$$

$$\Rightarrow \left[\frac{a}{2}e^{2x} + bx\right]_0^2 = 3e^4 + 1$$

$$\Rightarrow \frac{a}{2}e^4 + 2b - \frac{a}{2} = 3e^4 + 1$$

equating coefficients on each side gives $\frac{a}{2} = 3$ so $a = 6$ and

$2b - \frac{a}{2} = 1$

$2b = 4$

$b = 2$

and

$a + b = c$ so $c = 8$

(Note other values for *a, b, c* are possible using the equation $b = -\frac{e^4 - 1}{4}a + \frac{3}{2}e^4 + \frac{1}{2}$.
However, only one set of values was required, and the simplest set is that obtained by equating coefficients, so there is no need to pursue other solutions.)

5 marks

Mark allocation

- 1 mark for setting up the integral from 0 to 2 as equal to $3e^4 + 1$
- 1 mark for antidifferentiating
- 1 mark for getting $a = 6$
- 1 mark getting $b = 2$
- 1 mark for getting $c = 8$

END OF SOLUTION BOOK