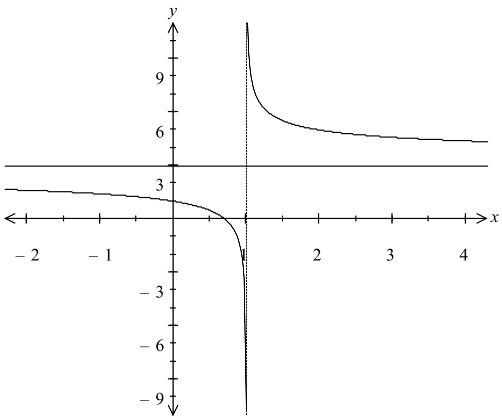
Mathematical Methods Exam 2: SOLUTIONS

Section 1: Multiple Choice

Question 1 — Answer B

$y = \dfrac{2}{\sqrt[3]{(x-1)}} + 3$



$x - 1 \neq 0$

$x = 1$ is a vertical asymptote

$y = 3$ is a horizontal asymptote

Question 2 — Answer C

$(x-a)^3(x+b)^2(x^2-c) = 0$

Using the Null Factor Law

$x - a = 0$

$x = a$

or

$x + b = 0$

$x = -b$

or

$x^2 - c = 0$

$x^2 = c$

no real solution as $c < 0$
There are two distinct solutions.

Question 3 — Answer E

The graph could be of the form
$y = A(x-1)^9 + 3$ where A is a positive real constant.

The y-intercept is negative

$-A + 3 < 0$

$A > 3$

Hence $y = 4(x-1)^9 + 3$

Alternatively,

The y-intercept is negative. Substituting $x = 0$ into each of the equations gives:

A. $-1 + 3 = 2$ No!

B. $+1 + 3 = 4$ No!

C. $-(-1) + 3 = 4$ No!

D. $-4 + 3$ Possibility

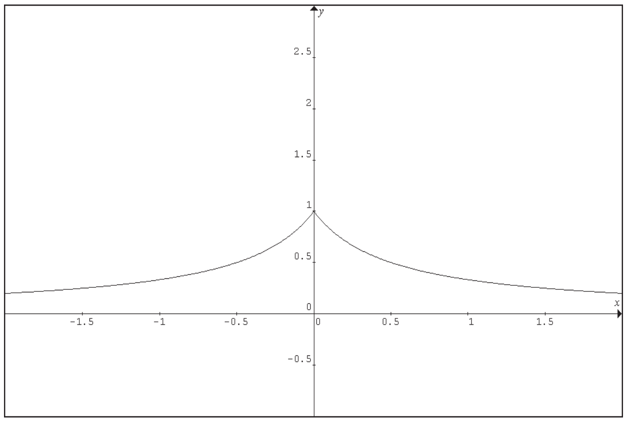
E. $-4 + 3$ Possibility

Try the graphs of these last two on the graphical calculator.

Hence $y = 4(x-1)^9 + 3$

Question 4 — Answer A

$h(k(x)) = \dfrac{1}{|2x| + 1}$ with domain R.



The range is $(0, 1]$.

Question 5 — Answer C

$f(g(x)) = 2e^{-x+1} = 2e^{-(x-1)}$
reflection in the y-axis
translation 1 unit parallel to x-axis

Question 6 — Answer E

$f(x) = 3e^{-2x} + 1$ with domain R and range $(1, \infty)$

Let $y = 3e^{-2x} + 1$

Inverse: swap x and y and solve for y

$$x = 3e^{-2y} + 1$$

$$\frac{x-1}{3} = e^{-2y}$$

$$y = -\frac{1}{2}\log_e\left(\frac{x-1}{3}\right) \quad \text{with domain } (1, \infty)$$

$$= -\frac{1}{2}\log_e\left(\frac{x-1}{3}\right)^{-\frac{1}{2}}$$

$$= \log_e\left(\sqrt{\frac{3}{x-1}}\right)$$

Therefore, $f^{-1}(x) = \log_e\left(\sqrt{\frac{3}{x-1}}\right)$

Question 7 — Answer B

$$2^{2x} - 9 \times 2^x + 8 = 0$$

$$(2^x - 8)(2^x - 1) = 0$$

$$2^x = 8 \text{ or } 2^x = 1$$

$$\therefore x = 3 \text{ or } x = 0$$

Question 8 — Answer C

$$2\log_e|x-1| + \log_e(9) = \log_e(a^2)$$

$$\log_e|x-1|^2 + \log_e(9) = \log_e(a^2)$$

$$\log_e(9|x-1|^2) = \log_e(a^2)$$

$$9|x-1|^2 = a^2$$

$$|x-1|^2 = \frac{a^2}{9}$$

$$|x-1| = \frac{a}{3}, a > 0$$

$$x - 1 = \frac{a}{3} \text{ and } -x + 1 = \frac{a}{3}$$

$$x = 1 + \frac{a}{3} \text{ and } x = 1 - \frac{a}{3}$$

Question 9 — Answer B

$$\sqrt{3}\tan(2\theta) + 1 = 0$$

$$\tan(2\theta) = -\frac{1}{\sqrt{3}}$$

$$2\theta = \frac{5\pi}{6}, \frac{11\pi}{6}, \frac{5\pi}{6} + 2\pi, \frac{11\pi}{6} + 2\pi$$

$$= \frac{5\pi}{6}, \frac{11\pi}{6}, \frac{17\pi}{6}, \frac{23\pi}{6}$$

$$\theta = \frac{5\pi}{12}, \frac{11\pi}{12}, \frac{17\pi}{12}, \frac{23\pi}{12}$$

Sum of the solutions

$$\frac{5\pi}{12} + \frac{11\pi}{12} + \frac{17\pi}{12} + \frac{23\pi}{12} = \frac{56\pi}{12} = \frac{14\pi}{3}$$

Question 10 — Answer E

$$\text{Period} = \frac{2\pi}{\pi/12} = 24$$

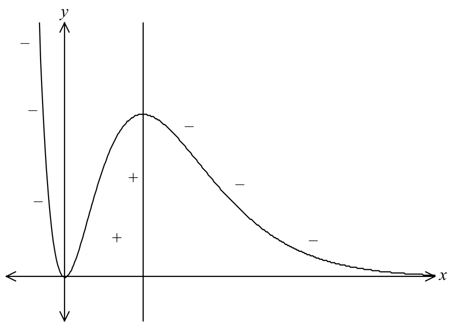
Maximum value of f is $1 + 5 = 6$

Minimum value of f is $1 - 5 = -4$

Range $= [-4, 6]$

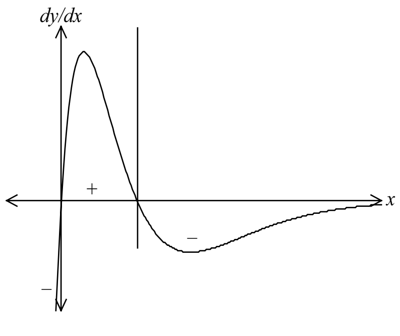
Question 11 — Answer B

Consider the sign of the gradient



x intercepts of $\frac{dy}{dx}$ are the stationary points of y

Gradient → 0 (from the negative, i.e. from below) as $x \to \infty$



Question 12 Answer C

$f:[0,\pi] \to R,\ f(x) = 4\cos\left(\frac{x}{2}\right)$

$$f'(x) = -1$$
$$-2\sin\left(\frac{x}{2}\right) = -1$$
$$\sin\left(\frac{x}{2}\right) = \frac{1}{2}$$
$$\frac{x}{2} = \frac{\pi}{6}$$
$$x = \frac{\pi}{3}$$

Question 13 Answer D

$$f(x) = \frac{1}{\sqrt[3]{x}},\ h = 0.1,\ x = 1$$
$$f(1) = 1$$
$$f'(x) = -\frac{1}{3x^{\frac{4}{3}}},\ f'(1) = -\frac{1}{3}$$
$$f(x+h) \approx f(x) + hf'(x)$$
$$\frac{1}{\sqrt[3]{1.1}} \approx 1 + 0.1 \times -\frac{1}{3}$$
$$\approx \frac{29}{30}$$

Question 14 Answer C

The domain of f is $(-\infty, 2]$ and the domain of g is $[-3, \infty)$.

Hence the domain of h is $[-3,2]$.

The domain of the derivative is $(-3,2)$.

Question 15 Answer A

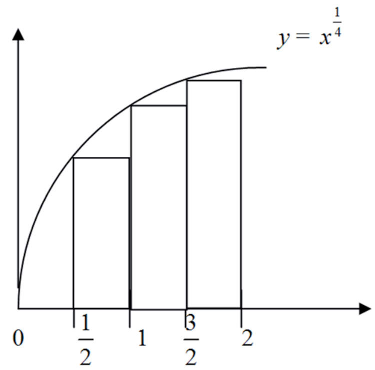
$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$
$$y = \left(\frac{x}{2} - 3\right)^8$$
$$\frac{dy}{dx} = 4\left(\frac{x}{2} - 3\right)^7 \text{ and } \frac{dx}{dt} = 3$$
$$\frac{dy}{dt} = 4\left(\frac{x}{2} - 3\right)^7 \times 3$$
$$= 12\left(\frac{x}{2} - 3\right)^7$$

Question 16 Answer A

$$\int \frac{1}{1-2x}dx = -\int \frac{1}{2x-1}dx$$
$$= -\frac{1}{2}\int \frac{2}{2x-1}dx$$
$$= -\frac{1}{2}\log_e(|2x-1|) + c,$$
for $R\backslash\left\{\frac{1}{2}\right\}$

c omitted, anti-derivative only required

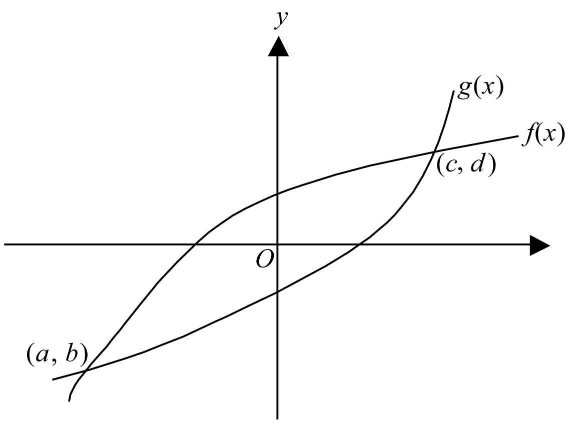
Question 17 Answer A



The area of the rectangles

$$= \frac{1}{2}\left(f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right)\right)$$
$$= \frac{1}{2}\left(\left(\frac{1}{2}\right)^{\frac{1}{4}} + 1 + \left(\frac{3}{2}\right)^{\frac{1}{4}}\right) \text{ square units}$$

Question 18 Answer D



The area between the curves is

$$\int_a^c (\text{top curve} - \text{bottom curve})dx$$
$$= \int_a^c (f(x) - g(x))dx$$

Question 19 — Answer A

$2t^2 + t = 1$

$2t^2 + t - 1 = 0$

$(2t-1)(t+1) = 0$

$2t = 1$ (reject $t = -1$ because $0 \le p(x) \le 1$)

$t = \frac{1}{2}$

Question 20 — Answer D

$k\int_{-\infty}^{\infty} e^{-(x^2/2)}dx = 1$

$k \times \sqrt{2\pi} = 1$

$k = \frac{1}{\sqrt{2\pi}}$

$E(X) = \mu$

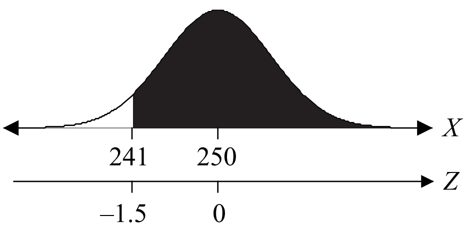
$= k\int_{-\infty}^{\infty} xe^{-(x^2/2)}dx$

$= k \times 0$

$= 0$

Hence, $k = \frac{1}{\sqrt{2\pi}}$ and $E(X) = 0$

Question 21 — Answer C



$250 - 241 = 9$

$1.5\sigma = 9$

$\sigma = \frac{9}{1.5}$

$\sigma = 6$

Question 22 — Answer B

$X \sim Bi\left(3, \frac{1}{3}\right)$

$\Pr(X = 2 \mid X \ge 1) = \frac{\Pr(X=2)}{1 - \Pr(X=0)}$

$= \frac{3 \times \left(\frac{1}{3}\right)^2 \times \frac{2}{3}}{1 - \left(\frac{2}{3}\right)^3}$

$= \frac{\frac{2}{9}}{\frac{19}{27}}$

$= \frac{6}{19}$

Mathematical Methods Exam 2: SOLUTIONS

Section 2: Extended answers

Question 1

a. i. $a = \frac{6}{3} = 2$ **1A**

$b = -6$ **1A**

ii. $x^2 + y^2 = 9$

$y^2 = 9 - x^2$

$y = \sqrt{9 - x^2}$

$c = 9$ **1A**

b. i. $-2\int_0^3 (2x - 6)dx$ **1M**

$= -2[x^2 - 6x]_0^3$ **1M**

$= -2(9 - 18 - 0 + 0)$

$= 18 \text{ cm}^2$

ii. $2\int_0^3 \sqrt{9 - x^2}\, dx$ **1M**

$= 2\left[\frac{9\sin^{-1}\left(\frac{x}{\sqrt{9}}\right)}{2} + \frac{x\sqrt{9 - x^2}}{2}\right]_0^3$ **1M**

$= 2\left(\frac{9\sin^{-1}(1)}{2} + \frac{3\sqrt{9-9}}{2} - \left(\frac{9\sin^{-1}(0)}{2} + \frac{0\sqrt{9-0}}{2}\right)\right)$

$= \frac{9\pi}{2} \text{ cm}^2$ **1M**

c. i. $\frac{dr}{dt} = \frac{dv}{dt} \times \frac{dr}{dv}$

$v = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2 2r = \frac{2}{3}\pi r^3$ **1M**

$\frac{dv}{dr} = 2\pi r^2, \frac{dr}{dv} = \frac{1}{2\pi r^2}$ **1M**

$\frac{dr}{dt} = -\pi \times \frac{1}{2\pi r^2} = -\frac{1}{2r^2}$ **1M**

At $r = 2$ cm

$\frac{dr}{dt} = -\frac{1}{8}$ cm/s

Decreasing at $\frac{1}{8}$ cm/s **1A**

ii. $v = 18\pi \text{ cm}^3, \frac{dv}{dt} = -\pi \text{ cm}^3/\text{s}$ **1M**

$t = \frac{18\pi}{\pi} = 18$ s

$t \geq 18$ s **1A**

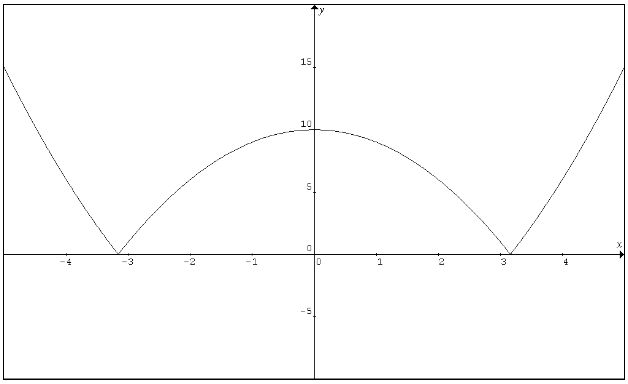
Question 2

a. i. $g(f(x)) = |x^2 - 10|$ **1A**

ii. Domain of $g(f(x))$ = Domain of $f(x)$

$= [-5, 5]$ **1A**

b.



Correct endpoints **1A**

Correct cusps and local maximum **1A**

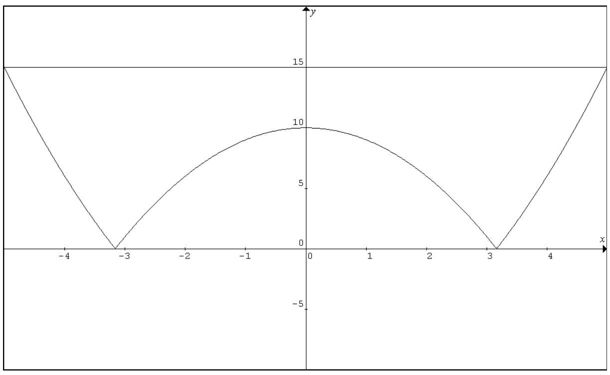
Correct x intercepts **1A**

c. i. $0 < k < 10$ **1A**

ii. $k = 0$ or **1A**

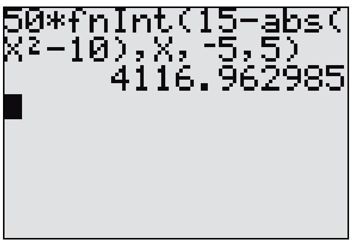
$10 < k \leq 15$ **1A**

d.



Cost $= 50 \times \int_{-5}^{5} (15 - |x^2 - 10|)dx$ **1M**

$\approx \$4117$ **1A**



e. i. Substitute $(0,0)$ into the equation.

$0 = 5e^0 + C$

$C = -5$ **1M**

ii. Substitute $(14,100)$ into the equation.

$100 = 5e^{14B} - 5$ **1M**

$e^{14B} = \frac{105}{5}$ **1M**

$= 21$

$B = \frac{\log_e(21)}{14}$

f. Solve $200 = 5e^{\frac{\log_e(21)}{14}t} - 5$

$t \approx 17.1$

During the 18th day. **1A**

g. Let $y = 5e^{\frac{\log_e(21)}{14}t} - 5$

Inverse: swap t and y.

$t = 5e^{\frac{\log_e(21)}{14}y} - 5$ **1M**

$\frac{t+5}{5} = e^{\frac{\log_e(21)}{14}y}$

$y = \frac{14}{\log_e(21)}\log_e\left(\frac{t+5}{5}\right)$

$r^{-1}(t) = \frac{14}{\log_e(21)}\log_e\left(\frac{t+5}{5}\right), t \geq 0$ **1A**

h. $r^{-1}(t) = \frac{14}{\log_e(21)}\log_e\left(\frac{20+5}{5}\right)$

≈ 7.4

7 foxes **1A**

Question 3

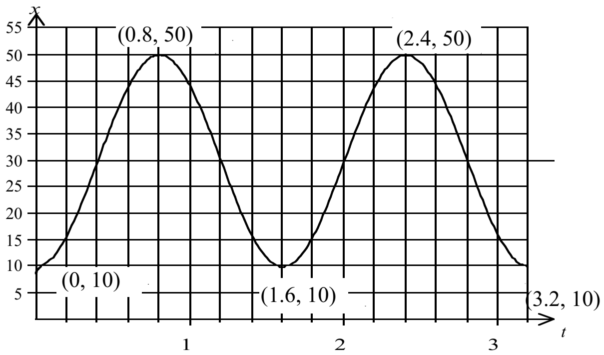
a. $x_{max} = 30 + 20 = 50$ cm **1A**

$x_{min} = 30 - 20 = 10$ cm **1A**

b. Period $= \frac{2\pi}{5\pi/4} = \frac{2\pi}{1} \times \frac{4}{5\pi} = 1.6$

The period is 1.6 seconds. **1A**

c.



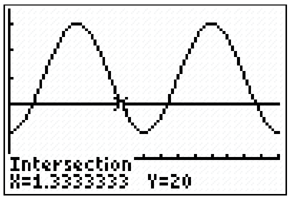
Shape **1 mark**

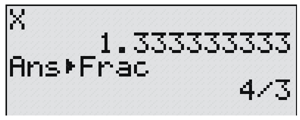
Endpoints **1 mark**

Correct coordinates for maximum and minimum values **1 mark**

d. Time $= \frac{4}{3} - \frac{4}{15}$ **1M**

$= \frac{16}{15}$ seconds **1A**





e. $m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

From the graph in part c,

$m = \frac{50-10}{0.8-0} = 50$ **1M**

The average rate of change is 50 cm/s. **1A**

f. Product rule

$\frac{d}{dt}\left(30 - 20e^{-t/10}\cos\left(\frac{5\pi}{4}t\right)\right)$ **1M**

$= \frac{-20}{-10}e^{-t/10}\cos\left(\frac{5\pi}{4}t\right) - \frac{-20 \times 5\pi}{4}e^{-t/10}\sin\left(\frac{5\pi}{4}t\right)$

$= 2e^{-t/10}\cos\left(\frac{5\pi}{4}t\right) + 25\pi e^{-t/10}\sin\left(\frac{5\pi}{4}t\right)$ **1A**

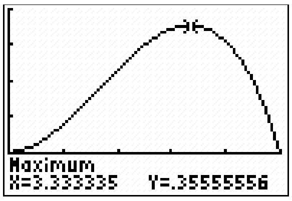
Question 4

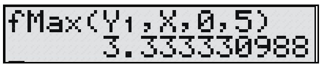
a. i. $-k\int_0^5 (t^2(t-5))dt = 1$ **1M**

$-k \times -\dfrac{625}{12} = 1$ **1M**

$k = \dfrac{12}{625} = 0.0192$

ii. The mode occurs at the maximum value of the function, which in this case, is a turning point **1M**



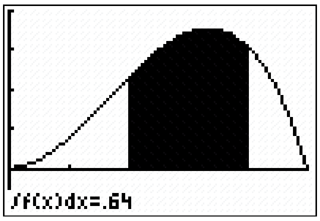


Mode = 3.33 **1A**

iii. $\Pr(2 < T < 4) =$

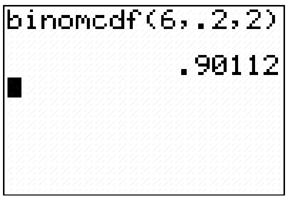
$\int_2^4 (0.0192x^2(x-5))dx$ **1M**

$\Pr(2 < T < 4) = 0.64$ **1A**



iv. Let X be the number of people from area *A*.

$X \sim Bi(6, 0.2)$ **1M**



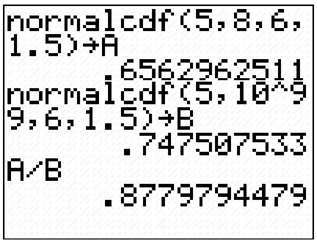
$\Pr(X \le 2) = \sum_{x=0}^{2} {}^6C_x (0.2)^x (0.8)^{6-x}$ **1A**

$= 0.9011$

b. $X \sim N(6, 1.5^2)$

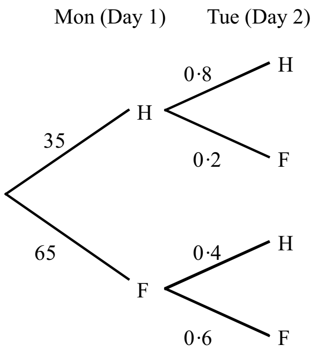
Conditional probability $\Pr(X < 8 \mid X > 5)$ **1M**

$\Pr(X < 8 \mid X > 5) = \dfrac{\Pr(5 < X < 8)}{\Pr(X > 5)}$ **1M**



$\Pr(X < 8 \mid X > 5) = 0.8780$ **1A**

c. Let H denote choosing from *Healthy* menu and F denote choosing from *Fast* menut.



$\Pr(H_2 \mid H_1) = 35 \times 0.8 = 28$

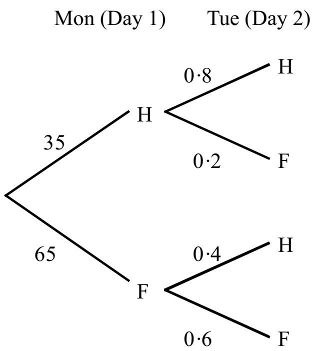
$\Pr(F_2 \mid H_1) = 35 \times 0.2 = 7$

$\Pr(H_2 \mid F_1) = 65 \times 0.4 = 26$

$\Pr(F_2 \mid F_1) = 65 \times 0.6 = 39$ **1M**

$\Pr(H_2) = 28 + 26 = 54$ and

$\Pr(F_2) = 7 + 39 = 46$ **1A**



$\Pr(H_2 \mid H_1) = 35 \times 0.8 = 28$

$\Pr(F_2 \mid H_1) = 35 \times 0.2 = 7$

$\Pr(H_2 \mid F_1) = 65 \times 0.4 = 26$

$\Pr(F_2 \mid F_1) = 65 \times 0.6 = 39$ **1M**

$\Pr(H_3) = 43.2 + 18.4 = 61.6$

On Wednesday, 62 people will choose from the *Healthy* menu. **1A**