Insight Publications

INSIGHT

Trial Exam Paper

2006

MATHEMATICAL METHODS

Written examination 1

Worked solutions

This book presents:

- correct answers
- worked solutions, giving you a series of points to show you how to work through the questions and achieve top marks
- mark allocation details.

Question 1

1a. Use the factor theorem to show that $(x+2)$ is a factor of $9x^3+12x^2-11x+2$.

Answer

$p(-2)=9\times-8+12\times4+22+2$

$=-72+48+24=0 \therefore x+2$ is a factor

Marks

- 1 mark for using $x=-2$
- 1 mark for showing equals to zero.

1b. The equation $y=9x^3+12x^2-11x+2$ can be written in the form $y=(x+2)(ax-b)^2$ where $\{a, b\} > 0$. State the values of *a* and *b.*

Answer

By inspection, $a^2=9 \Rightarrow a=3$. Equating the coefficients gives $2\times b^2=2 \Rightarrow b=1$

Marks

- 1 mark for $a = 3$
- 1 mark for $b = 1$

2 + 2 = 4 marks

Question 2

2a. The graph of a function *g* is obtained from the graph of the function *f* which has the rule $f(x)=2(x-2)^5$ by a performing a translation of −4 units parallel to the *x*-axis. Write down the rule for *g.*

Answer

Replace *x* with *x*+ 4 to give $g(x)=2(x+2)^5$

2b. The graph of a function *h* is obtained from the graph of *g* by a reflection in the *y*-axis. Write down the rule for *h.*

Answer

Replace *x* with –*x* to give $h(x)=2(2-x)^5$

2c. The graph of a function *k* is obtained from the graph of *h* by a dilation by a scale factor of $\frac{1}{2}$ along the *y*-axis. Write down the rule for *k.*

Answer

Replace *x* with 2*x* to give $k(x)=2(2-2x)^5$

Marks

- 1 mark for each correct equation

1 + 1 + 1 = 3 marks

Question 3

Solve the equation $\sqrt{3}\sin(2x)+\cos(2x)=0$ for $x \in [0,2\pi]$, giving exact values in terms of π.

Answer

$\sqrt{3}\sin 2x = -\cos 2x$

$\tan 2x = \frac{-1}{\sqrt{3}}$

$2x = \frac{5\pi}{6}, \frac{11\pi}{6}, \frac{17\pi}{6}, \frac{23\pi}{6}$

$x = \frac{5\pi}{12}, \frac{11\pi}{12}, \frac{17\pi}{12}, \frac{23\pi}{12}$

Marks

- 1 mark for rearranging to get tan $(2x)$
- 1 mark for finding $\frac{5\pi}{6}, \frac{11\pi}{6}$
- 1 mark for all four correct angles

3 marks

Question 4

Let $f(x)=x^2$ and $g(x)=3x-5$.

4a. Write down the rule of $f(g(x))$.

Answer

$f(g(x))=(3x-5)^2$

4b. Find the derivative of $f(g(x))$.

Answer

Let $y=f(g(x))$,

$\frac{dy}{dx}=2(3x-5)\times 3=6(3x-5)$ using the chain rule.

4c. Hence, find the coordinates of the point P on the curve with the equation $y = f(g(x))$ at which the tangent is parallel to the line $2y - 12x = 7$.

Answer

'Parallel' means a line has the same gradient as another.

The gradient of the line $2y - 12x = 7$ is 6 (after rearranging to $y = mx+c$ form), so:

$6(3x - 5) = 6$

$(3x - 5) = 1$

$x = 2, \ y = 1$

Marks

- 1 mark for the rule in **a.**
- 1 mark for the derivative in **b.**
- 1 mark for equating the derivative to equal 6
- 1 mark for the correct coordinate

1 + 1 + 2 = 4 marks

Question 5

For the function $f : [-\pi, \pi] \rightarrow R, \ f(x) = -2\cos(2t + \frac{\pi}{2})$

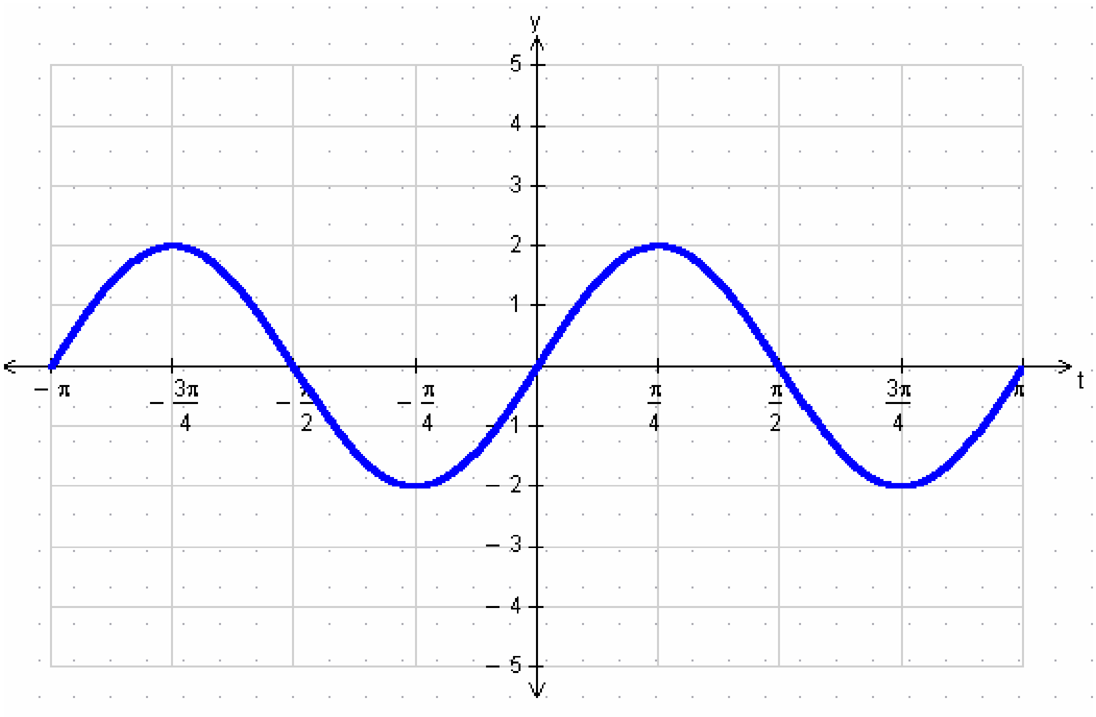
5a. Write down the period of the function.

Answer

$$p = \frac{2\pi}{n} = \frac{2\pi}{2} = \pi$$

5b. On the set of axes below, sketch the graph of the function f.

Answer



5c. State the number of solutions to the equation $\cos(2t + \frac{\pi}{2}) = \frac{1}{2}$, where $-\pi \le x \le \pi$.

Answer

Look at the intersection of the graph with the line y = −1. There are four points of intersection, and therefore four solutions to the equation.

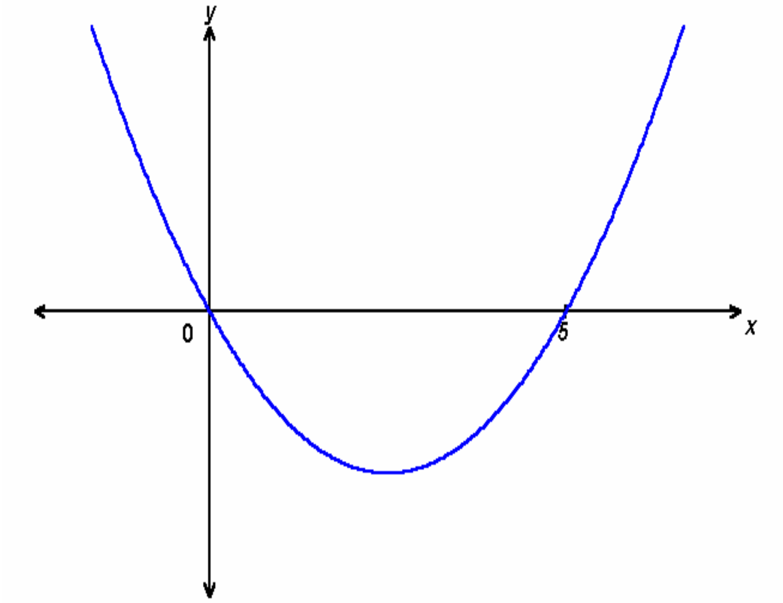
Marks

- 1 mark for correct period
- 1 mark for sketching two cycles of graph
- 1 mark for correct shape and amplitude of graph
- 1 mark for stating '4'

1 + 2 + 1 = 4 marks

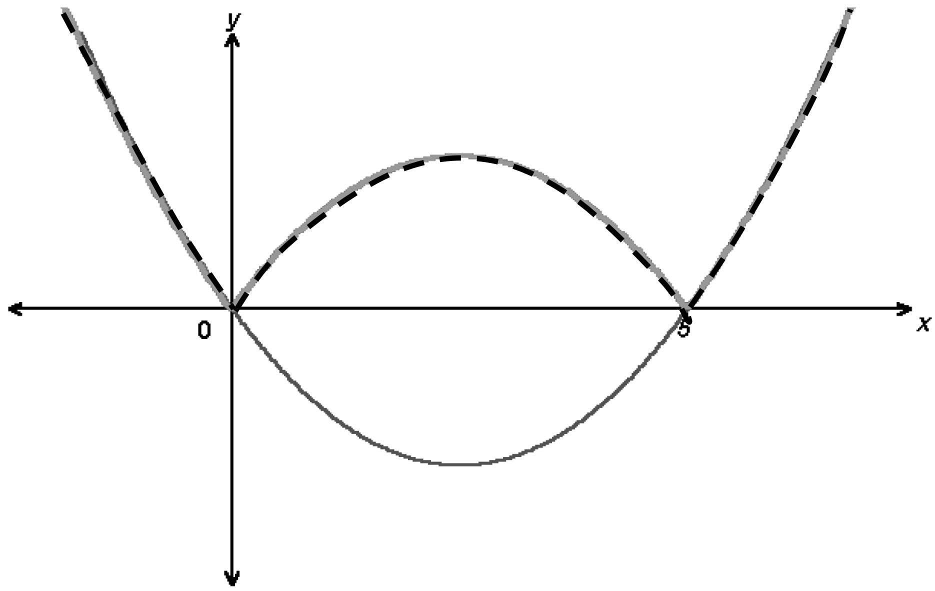
Question 6

Part of the graph of $y = x^2 - 5x$ is shown below.



6a. On the same set of axes sketch the graph of $y = |x^2 - 5x|$.

Answer



6b. Find the set of values of x for which $|x^2 - 5x| \geq 6$.

Answer

Solve

$x^2 - 5x \geq 6$ and $-x^2 + 5x \geq 6$

to give $2 \leq x \leq 3$

and $x \geq 6$ *or* $x \leq -1$

Marks

- 1 mark for the correct sketch
- 1 mark for setting up two equations to solve
- 1 mark for the correct intervals

1 + 2 = 3 marks

Question 7

For the function $f(x) = 2e^{1-x}$,

7a. find the rule of the inverse function f^{-1}.

Answer

Swap x and y to give $x = 2e^{1-y}$, then rearrange to make y the subject, as follows:

$$\frac{x}{2} = e^{1-y}$$

$$\log_e \frac{x}{2} = 1 - y \Rightarrow y = 1 - \log_e \frac{x}{2}$$

7b. find the domain of the inverse function f^{-1}.

Answer

Range of the original function = domain of the inverse function, so the domain of the inverse function $= R^+$.

Marks

- 1 mark for knowing to swap x and y and an attempt to find y
- 1 mark for the correct equation
- 1 mark for the correct domain

2 + 1 = 3 marks

Question 8

The random variable X has the following probability distribution.

x	−1	0	1	2
Pr(X = x)	a	$2a$	$3a$	0.4

8a. Find the value of *a*.

Answer

Since this is a probability distribution, the probabilities must total 1. Therefore,

$$6a + 0.4 = 1$$
$$6a = 0.6$$
$$a = 0.1$$

8b. If $\Pr(X \le k) > 0.5$, find the minimum value of *k*.

Answer

$\Pr(X \le 0) = 0.3$

$\Pr(X \le 1) = 0.6$ so $k = 1$

Marks

- 1 mark for the correct value of *a*
- 1 mark for the correct value of *k*

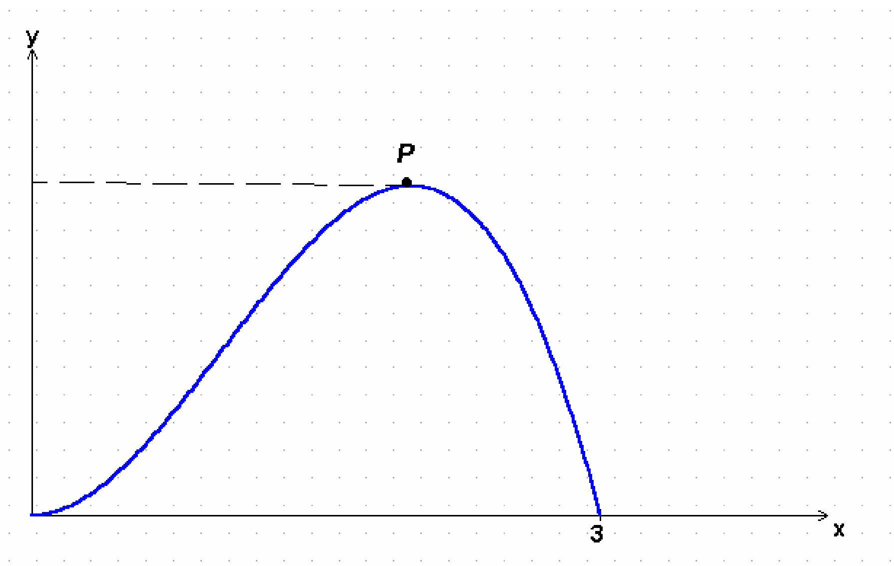
1 + 1 = 2 marks

Question 9

A continuous random variable *X* has the probability density function given by

$$f(x) = \frac{4}{27}(3x^2 - x^3), \quad 0 \le x \le 3$$

The graph of *f*, as shown below, has a maximum point at *P*.



9a. Find the value of the *x*-coordinate of *P*.

Answer

The maximum occurs when the derivative equals zero.

$$f'(x) = \frac{4}{27}(6x - 3x^2) = 0$$
$$\Rightarrow 3x(2 - x) = 0$$
$$\Rightarrow x = 0 \text{ or } x = 2 \text{ so } x = 2$$

Marks

- 1 mark for the correct derivative
- 1 mark for equating derivative to zero
- 1 mark for the correct answer

9b. Find the $\Pr(0 < X < 2)$.

Answer

$$\Pr(0 < X < 2) = \int_0^2 \frac{4}{27}(3x^2 - x^3)dx$$
$$= \frac{4}{27}[x^3 - \frac{x^4}{4}]_0^2$$
$$= \frac{4}{27}(8 - 4) = \frac{16}{27}$$

Marks

- 1 mark for setting up integral from 0 to 2
- 1 mark for the correct antiderivative
- 1 mark for the correct answer

9c. Find the mean value of *X*.

Answer

$$E(X) = \int_0^3 xf(x)dx = \frac{4}{27}\int_0^3 (3x^3 - x^4)dx$$
$$= \frac{4}{27}[\frac{3x^4}{4} - \frac{x^5}{5}]_0^3 = \frac{4}{27}(\frac{243}{4} - \frac{243}{5}) = \frac{9}{5}$$

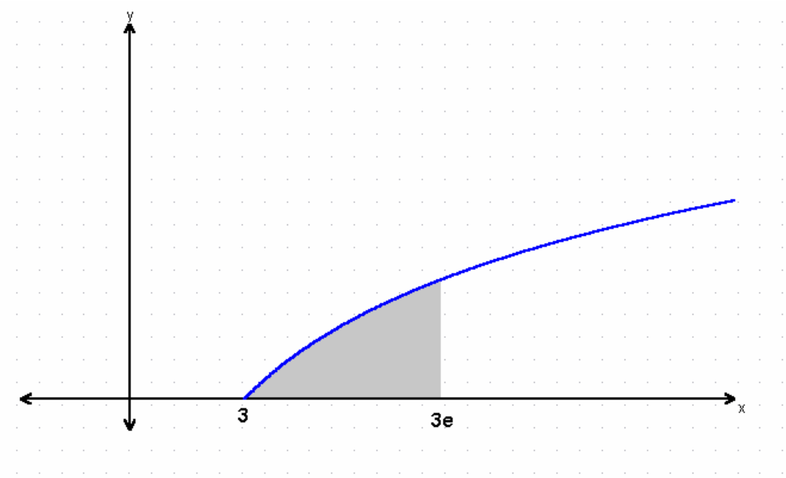
Marks

- 1 mark for setting up integral of $xf(x)$ from 0 to 3
- 1 mark for the correct antiderivative
- 1 mark for the correct answer

3 + 3 + 3 = 9 marks

Question 10

The graph of the function $f:[3,\infty) \to R,\ f(x) = \log_e \frac{x}{3}$ is shown below.



10a. If $y = x\log_e \frac{x}{3} - x$, find $\frac{dy}{dx}$.

Answer

$$\frac{dy}{dx} = x\frac{3}{x}\frac{1}{3} + \log_e \frac{x}{3} - 1$$

$$= 1 + \log_e \frac{x}{3} - 1$$

$$= \log_e \frac{x}{3}$$

Marks

- 1 mark for using product rule to differentiate
- 1 mark for the correct answer

10b. Hence, find the exact area of the shaded region.

Answer

$$\int_3^{3e} \log_e \frac{x}{3}\,dx = \left[x\log_e \frac{x}{3} - x \right]_e^{3e} \quad \text{from part a}$$

$$= 3e\log_e e - 3e - 3\log_e 1 + 3$$

$$= 3e - 3e - 0 + 3 = 3 \text{ sq units.}$$

Marks

- 1 mark for using result of part **a.**
- 1 mark for evaluating with $3e$ and e
- 1 mark for the correct answer

2 + 3 = 5 marks

END OF SOLUTIONS