VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education 2005

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

STUDENT NUMBER

									Letter
Figures									
Words									

MATHEMATICAL METHODS

Written examination 2
(Analysis task)

Monday 7 November 2005

Reading time: 9.00 am to 9.15 am (15 minutes)
Writing time: 9.15 am to 10.45 am (1 hour 30 minutes)

QUESTION AND ANSWER BOOK

Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
4	4	55

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, up to four pages (two A4 sheets) of pre-written notes (typed or handwritten), one approved graphics calculator (memory DOES NOT need to be cleared) and, if desired, one scientific calculator.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied
- Question and answer book of 13 pages, with a detachable sheet of miscellaneous formulas in the centrefold.
- Working space is provided throughout the book.

Instructions
- Detach the formula sheet from the centre of this book during reading time.
- Write your **student number** in the space provided above on this page.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Instructions

Answer **all** questions in the spaces provided.

A decimal approximation will not be accepted if an **exact** answer is required to a question.

In questions where more than one mark is available, appropriate working must be shown.

Where an instruction to **use calculus** is stated for a question, you must show an appropriate derivative or antiderivative.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

Let $f: [0, \infty) \rightarrow R,\ f(t) = 2e^{-t}$.

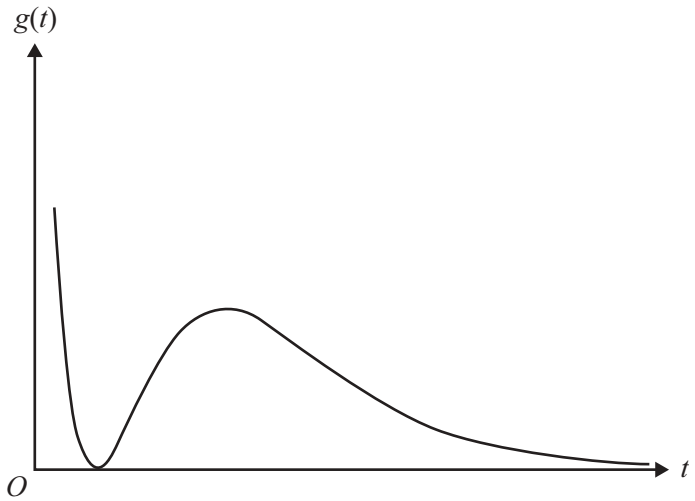
a. **i.** State the range of f.

ii. Find the rule for the inverse of f and state its domain.

1 + 2 = 3 marks

b. Let $g: [0, \infty) \rightarrow R,\ g(t) = (t - 1)^2 e^{-t}$.

Part of the graph of g is shown.



i. The rule for the derivative of g may be expressed in the form $g'(t) = (-t^2 + bt + c)e^{-t}$.
Find the exact values of b and c.

ii. The graph of $y = g(t)$ has stationary points $(1, p)$ and (m, n).
Find the exact values of p, m and n.

iii. For the function $q: [0, \infty) \rightarrow R$, $q(t) = 2g(t) - 5$, state the exact coordinates of the stationary points of the graph of $y = q(t)$.

3 + 2 + 2 = 7 marks

c. The function $h: R \rightarrow R$, $h(t) = (t^2 + at + 10)e^{-t}$, where a is a real constant, has derivative $h'(t) = (-t^2 + (2 - a)t + (a - 10))e^{-t}$.

Find the values of a such that

i. the graph of $y = h(t)$ has exactly one stationary point

ii. $h'(t) < 0$ for all $t \in R$.

3 + 2 = 5 marks

Total 15 marks

Working space

Question 2

Tasmania Jones is training to throw the javelin for the next Olympic Games.

The 'A Standard' throwing distance, to be thrown in an authorised competition, is 81.80 metres.

The current Olympic Record for the men's javelin throw is 90.17 metres.

To be selected for the Olympic Games, Tasmania needs to throw the A Standard.

Tasmania knows that the distance in metres he can throw the javelin from the marked throwing line follows a normal distribution with a mean of 80.80 and a standard deviation of 4.50.

a. Complete the following table. Give probabilities correct to three decimal places.

Distance thrown (metres)	**Probability**
greater than the A Standard	
greater than the A Standard but less than the Olympic Record	
greater than the Olympic Record	

3 marks

b. 90% of Tasmania's throws travel at least M metres. Find the value of M, correct to two decimal places.

1 mark

c. Tasmania throws a javelin that does not reach the Olympic Record. What is the probability, correct to three decimal places, that it reaches the A Standard?

2 marks

Tasmania's sponsor offers him an incentive to perform his best in competition. The cash rewards for each throw are shown in the table below.

Length of throw	**Amount paid ($)**
under his personal mean	0
between his personal mean and A Standard	1 000
between A Standard and Olympic Record	2 000
over the Olympic Record	10 000

d. Calculate the expected reward, correct to the nearest 10 dollars, for Tasmania for each throw he completes in competition.

2 marks

e. In a particular competition, Tasmania completes five throws.
Find

i. the total reward he would expect to receive, correct to the nearest 10 dollars

ii. the probability, correct to three decimal places, that at least three of the throws will be over the A Standard

iii. the expected number of times his throw will be over the A Standard, correct to two decimal places

iv. the probability, correct to three decimal places, that Tasmania earns a reward of at least $10 000.

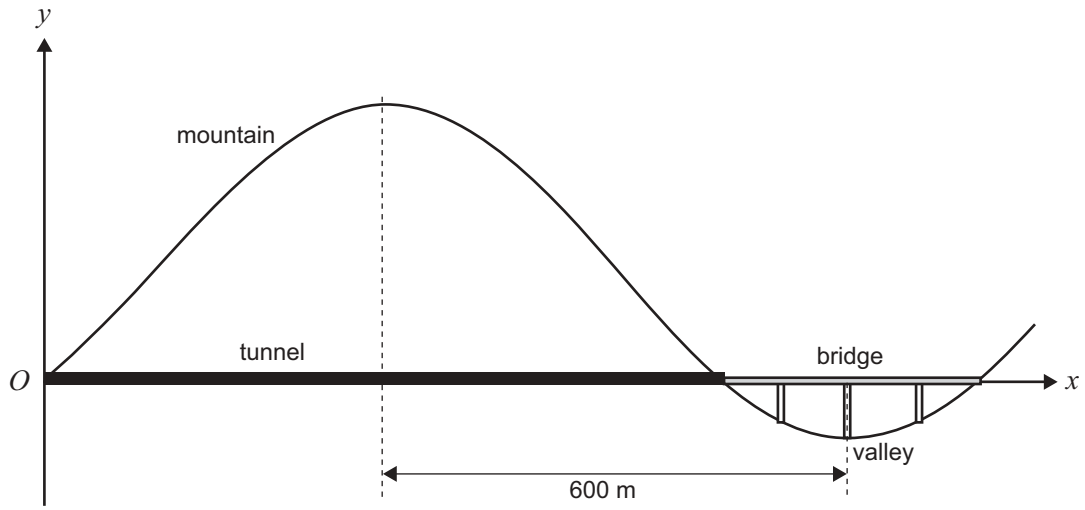
1 + 2 + 1 + 3 = 7 marks

Total 15 marks

TURN OVER

Question 3

A hydroelectric authority is proposing to build a horizontal pipeline which will pass through a new tunnel and over a bridge. The diagram below shows a cross-section of the proposed route with a tunnel through the mountain and a bridge over the valley to carry the pipeline.



The boundary of the cross-section can be modelled by a function of the form

$$y = 100\cos\left[\frac{\pi(x-400)}{600}\right] + 50, \ 0 \le x \le 1600$$

where y is the height, in metres, above the proposed bridge and x is the distance, in metres, from a point O where the tunnel will start.

a. What is the height (in metres) of the top of the mountain above the bridge?

1 mark

b. How many metres below the bridge is the bottom of the valley?

1 mark

c. What is the exact length of

i. the tunnel

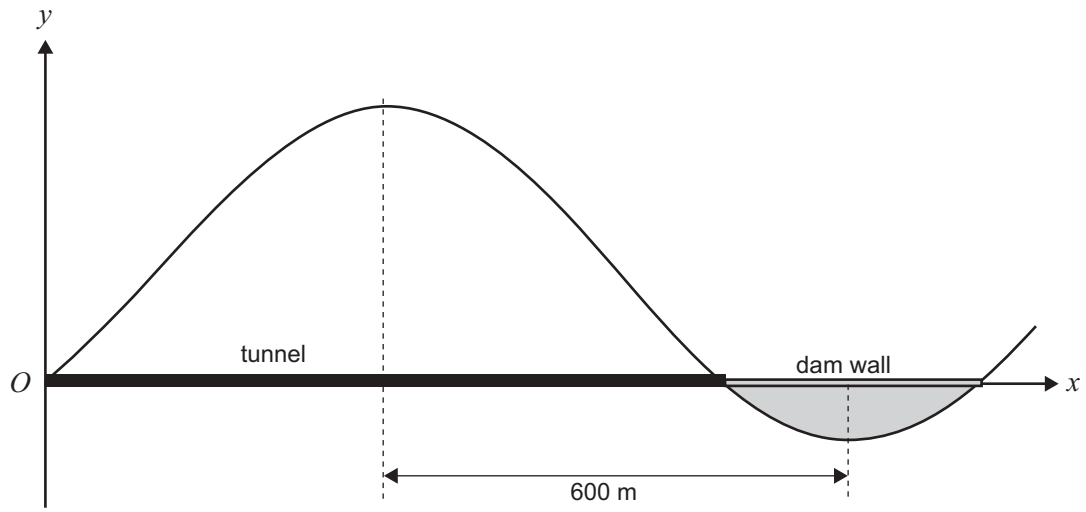
ii. the bridge?

1 + 1 = 2 marks

d. What would be the length (correct to the nearest metre) of the **tunnel** if it were built 20 m higher up the mountain?

2 marks

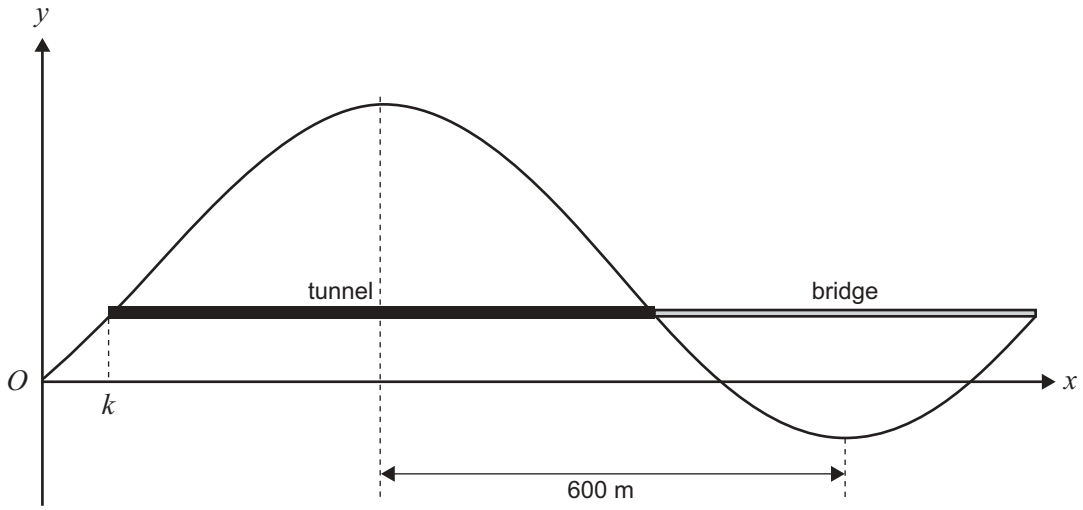
A second proposal is to build a solid concrete dam instead of a bridge. The shaded area in the diagram below shows a cross-section of the dam wall.



e. Use calculus to find the area of the cross-section of the dam wall, correct to the nearest square metre.

3 marks

A third proposal is to build the tunnel and bridge above the original proposed position.



f. Suppose the tunnel is built at a height such that it starts at a point on the mountain when $x = k$, $0 < k < 400$.

i. Find the length of the tunnel in terms of k.

ii. Find the length of the bridge in terms of k.

iii. The estimated total cost, C thousand dollars, of building the tunnel and bridge for this third proposal is equal to the sum of the square of the length (in metres) of the tunnel and the square of the length (in metres) of the bridge.

Write down an expression for the estimated total cost of building the tunnel and the bridge if the tunnel starts when $x = k$, in terms of k.

iv. Hence use calculus to find the exact value of k for which the estimated cost of the proposal is minimum.

1 + 1 + 1 + 2 = 5 marks

Total 14 marks

Question 4

The pollution level, y units, along a straight road between factories A and B, which are 10 km apart, is given by

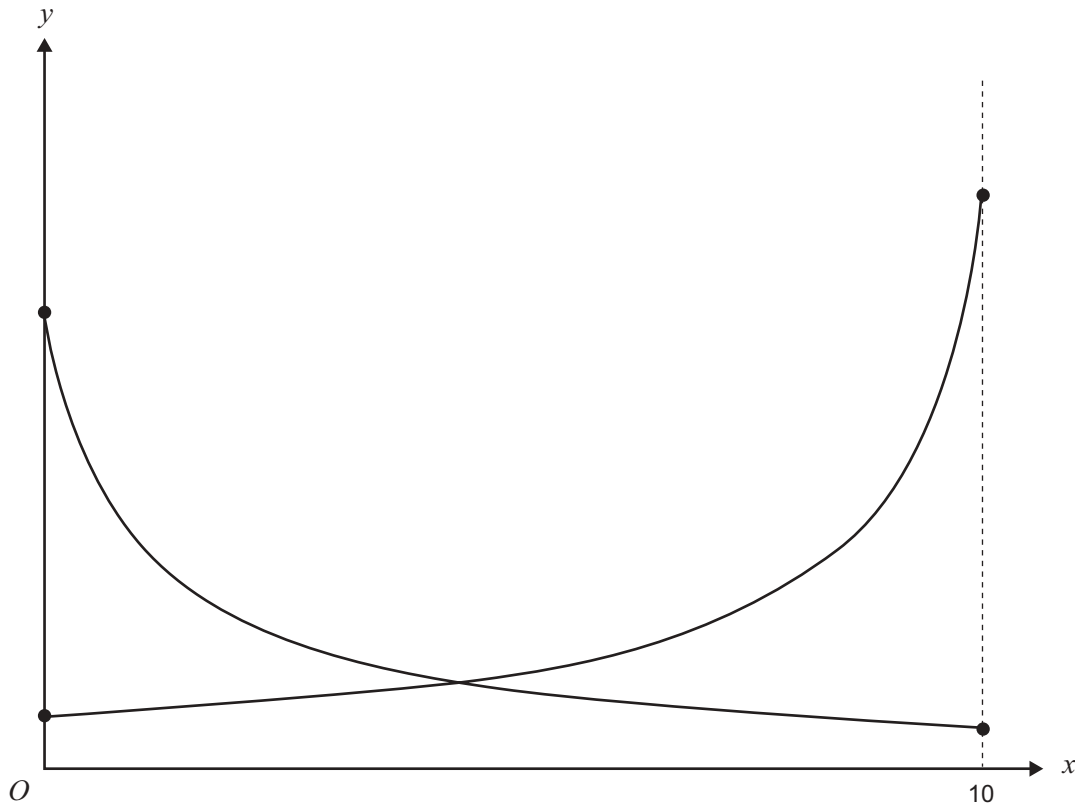
$$y = \frac{p}{x+1} + \frac{q}{11-x}, \text{ where } 0 \leq x \leq 10$$

where x km is the distance from Factory A, and p and q are positive constants.

a. Find the value of y, expressed in terms of p and q, at the point between the two factories where $x = 3$.

1 mark

b. On a particular day, the values of p and q are such that sections of the graphs of $y_1 = \frac{p}{x+1}$ and of $y_2 = \frac{q}{11-x}$ are as shown below. On this set of axes, sketch the graph of y.



2 marks

A week later, the values of p and q are measured to be $p = 9$ and $q = 4$.

c. Use calculus to find an equation in x, a solution of which is the value of x at which the pollution level is a minimum.

2 marks

d. **i.** Use your equation from **part c.** to find the value of x at which the pollution level y is a minimum and the value of this minimum, correct to three decimal places.

ii. Jack travels from Factory A to Factory B along the road. For what length of his journey (in kilometres correct to three decimal places) is the pollution level less than 5?

3 + 1 = 4 marks

e. The total pollution along the road is given by $\int_0^{10} y\,dx$. Use calculus to find the total pollution (correct to two decimal places) along the road between the factories when $p = 9$ and $q = 4$.

2 marks

Total 11 marks

END OF QUESTION AND ANSWER BOOK

MATHEMATICAL METHODS

Written examinations 1 and 2

FORMULA SHEET

Directions to students

Detach this formula sheet during reading time.

This formula sheet is provided for your reference.

Mathematical Methods Formulas

Mensuration

area of a trapezium: $\frac{1}{2}(a+b)h$

volume of a pyramid: $\frac{1}{3}Ah$

curved surface area of a cylinder: $2\pi rh$

volume of a sphere: $\frac{4}{3}\pi r^3$

volume of a cylinder: $\pi r^2 h$

area of a triangle: $\frac{1}{2}bc\sin A$

volume of a cone: $\frac{1}{3}\pi r^2 h$

Calculus

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1} \qquad \int x^n dx = \frac{1}{n+1}x^{n+1} + c, n \neq -1$$

$$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax} \qquad \int e^{ax} dx = \frac{1}{a}e^{ax} + c$$

$$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x} \qquad \int \frac{1}{x} dx = \log_e(x) + c, \text{for } x > 0$$

$$\frac{d}{dx}\left(\sin(ax)\right) = a\ \cos(ax) \qquad \int \sin(ax)dx = -\frac{1}{a}\cos(ax) + c$$

$$\frac{d}{dx}\left(\cos(ax)\right) = -a\ \sin(ax) \qquad \int \cos(ax)dx = \frac{1}{a}\sin(ax) + c$$

$$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\ \sec^2(ax)$$

product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

approximation: $f(x+h) \approx f(x) + hf'(x)$

Statistics and Probability

$\Pr(A) = 1 - \Pr(A')$

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$

mean: $\mu = \mathrm{E}(X)$

variance: $\mathrm{var}(X) = \sigma^2 = \mathrm{E}((X-\mu)^2) = \mathrm{E}(X^2) - \mu^2$

Discrete distributions			
	$\Pr(X = x)$	mean	variance
general	$p(x)$	$\mu = \sum x\,p(x)$	$\sigma^2 = \sum (x-\mu)^2 p(x)$ $= \sum x^2 p(x) - \mu^2$
binomial	${}^nC_x\, p^x (1-p)^{n-x}$	np	$np(1-p)$
hypergeometric	$\frac{{}^DC_x\ {}^{N-D}C_{n-x}}{{}^NC_n}$	$n\frac{D}{N}$	$n\frac{D}{N}\left(1-\frac{D}{N}\right)\left(\frac{N-n}{N-1}\right)$
Continuous distributions			
normal	If X is distributed N(μ, σ^2) and $Z = \frac{X-\mu}{\sigma}$, then Z is distributed N(0, 1).		

Table 1 Normal distribution – cdf

x	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359	4	8	12	16	20	24	28	32	36
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753	4	8	12	16	20	24	28	32	35
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141	4	8	12	15	19	23	27	31	35
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517	4	8	11	15	19	23	26	30	34
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879	4	7	11	14	18	22	25	29	32
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224	3	7	10	14	17	21	24	27	31
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549	3	6	10	13	16	19	23	26	29
0.7	.7580	.7611	.7642	.7673	.7703	.7734	.7764	.7793	.7823	.7852	3	6	9	12	15	18	21	24	27
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133	3	6	8	11	14	17	19	22	25
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389	3	5	8	10	13	15	18	20	23
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621	2	5	7	9	12	14	16	18	21
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830	2	4	6	8	10	12	14	16	19
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015	2	4	6	7	9	11	13	15	16
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177	2	3	5	6	8	10	11	13	14
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319	1	3	4	6	7	8	10	11	13
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441	1	2	4	5	6	7	8	10	11
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545	1	2	3	4	5	6	7	8	9
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633	1	2	3	3	4	5	6	7	8
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706	1	1	2	3	4	4	5	6	6
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767	1	1	2	2	3	4	4	5	5
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817	0	1	1	2	2	3	3	4	4
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857	0	1	1	2	2	2	3	3	4
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890	0	1	1	1	2	2	2	3	3
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916	0	1	1	1	1	2	2	2	2
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936	0	0	1	1	1	1	1	2	2
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952	0	0	0	1	1	1	1	1	1
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964	0	0	0	0	1	1	1	1	1
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974	0	0	0	0	0	1	1	1	1
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981	0	0	0	0	0	0	0	1	1
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986	0	0	0	0	0	0	0	0	0
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990	0	0	0	0	0	0	0	0	0
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993	0	0	0	0	0	0	0	0	0
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995	0	0	0	0	0	0	0	0	0
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997	0	0	0	0	0	0	0	0	0
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998	0	0	0	0	0	0	0	0	0
3.5	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	0	0	0	0	0	0	0	0	0
3.6	.9998	.9998	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	0	0	0	0	0	0	0	0	0
3.7	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	0	0	0	0	0	0	0	0	0
3.8	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	0	0	0	0	0	0	0	0	0
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0	0	0	0	0	0	0	0	0

END OF FORMULA SHEET