VCAA 2005 Mathematical Methods
Written Examination 2

Suggested Answers & Solutions

Question 1

a. i $(0, 2]$

ii Interchange t for y and y for t: $t = 2e^{-y}$ and so $\frac{t}{2} = e^{-y}$

Taking logarithm of both sides (to base e): $\log_e(\frac{t}{2}) = \log_e(e^{-y}) = -y$

Hence $f^{-1}(t) = -\log_e(\frac{t}{2})$ and the domain is $(0, 2]$

b. i Using the Product Rule $g'(t) = 2(t-1)e^{-t} - (t-1)^2 e^{-t}$

$= [2t - 2 - (t^2 - 2t + 1)]e^{-t}$

$= (-t^2 + 4t - 3)e^{-t}$

Answers: $b = 4$, $c = -3$

ii $g'(t) = -(t-1)(t-3)e^{-t}$

$= 0$ for stationary values.

This occurs at $t = 1$ and $t = 3$. It should be noted that e^{-t} is never zero.

When $t = 1$, $g(1) = 0$ and so one stationary value is $(1, 0)$.

When $t = 3$, $g(3) = (3-1)^2 e^{-3} = 4e^{-3}$ and so the other stationary value is $(3, 4e^{-3})$.

Answers: $p = 0$, $m = 3$ and $n = 4e^{-3}$

iii $q(t) = 2(t-1)^2 e^{-t} - 5$

Stationary points occur when q'(t)=0.

ie. 2g'(t)=0

g'(t)=0 , so at t=1 and 3

This has stationary points at $(1, -5)$ and $(3, 8e^{-3} - 5)$

c i If this has one stationary value only then the discriminant of the quadratic factor is zero.

$\Delta = (2-a)^2 + 4(a-10)$

Therefore $4 - 4a + a^2 + 4a - 40 = 0$

$a^2 - 36 = 0$ and so $a = \pm 6$

ii If $h'(t) < 0$ for all t then the same discriminant must be negative as $e^{-t} > 0$ for all t.

Therefore $a^2 - 36 < 0$ and so $-6 < a < 6$.

Question 2

a. Greater than the A Standard:

Probability = normalcdf (81.8, 10^{10}, 80.8, 4.5) = 0.412

Greater than A but less than Olympic:

Probability = normalcdf (81.8, 90.17, 80.8, 4.5) = 0.393

Greater than Olympic:

Probability = normalcdf (90.17, 10^{10}, 80.8, 4.5) = 0.019

b. Invnorm (0.1, 80.8, 4.5) = 75.03 and so M = 75.03

c. Pr (X > A / X < Olympic) = $\frac{0.4121 - 0.01866}{1 - 0.01866}$
= 0.401

d. The following table assists in finding this reward:

Length of throw	**Amount paid $**	**Probability**
Under personal best (PB)	0	0.5
Between PB and A	1000	0.088
Between A and Olympic	2000	0.393
Over Olympic	10000	0.019

Expected reward $= 0 \times 0.5 + 1000 \times 0.088 + 2000 \times 0.393 + 10000 \times 0.019$
$= 0 + 88 + 786 + 190$
$= 1064$

Answer: $1060 (to nearest $10)

e. i $5 \times 1064 = 5320$
Answer: $5320

ii Binomial distribution: n=5 and p = 0.393+0.019=0.412
i.e. Bi(5, 0.412)
Probability of at least three throws = Pr (X=3) + Pr (X=4) + Pr (X=5)
Pr (3) + Pr (4) + Pr (5) $= {}^5C_3\,(.412)^3(.588)^2 + {}^5C_4\,(.412)^4(.588)^1 + (.412)^5$
$= 0.2418 + 0.0847 + 0.01187$
$= 0.3384$
Answer: 0.338

iii Expected number = $n\,p = 5 \times 0.412$
Answer: 2.06

iv Pr (Throws at least one Olympic Record) + Pr(Throws all five greater than Standard but less than Olympic)
$= (1 - \text{Pr (throwing no Olympic)}) + 0.393^5$
$= (1 - 0.981^5) + 0.393^5$
$= (1 - 0.9085) + 0.0094$
$= 0.101$

Question 3

a. 150 m

b. 50 m

c. i 800 m
ii 1200 − 800 = 400 m

d. Solve $y_1 = 100\cos\left(\frac{\pi(x-400)}{600}\right) + 50$ and $y_2 = 20$ to find intersection points on the graphics calculator. The two values of x found are 41.81 and 758.19.
The length of the tunnel = 758.19 − 41.81
= 716 m (to the nearest metre).

e. $\int_{1200}^{800} (100\cos\left(\frac{\pi(x-400)}{600}\right)+50)dx$

$= [\ \frac{100\times 600}{\pi}\sin\left(\frac{\pi(x-400}{600}\right)+50x\ ]_{1200}^{800}$

$=\ (\ \frac{60000}{\pi}\sin\left(\frac{400\pi}{600}\right)+40000\ ) - (\ \frac{60000}{\pi}\sin\left(\frac{800}{600}\right)+60000)$

= 13080 m^2 (to the nearest square metre).

f. i $800 - 2k$

ii $400 + 2k$

iii $C = (800 - 2k)^2 + (400 + 2k)^2$

iv $\frac{dC}{dk} = 2 \times -2\ (800 - 2k) + \ 2 \times 2\ (400 + 2k)$

$= -3200 + 8k + 1600 + 8k$

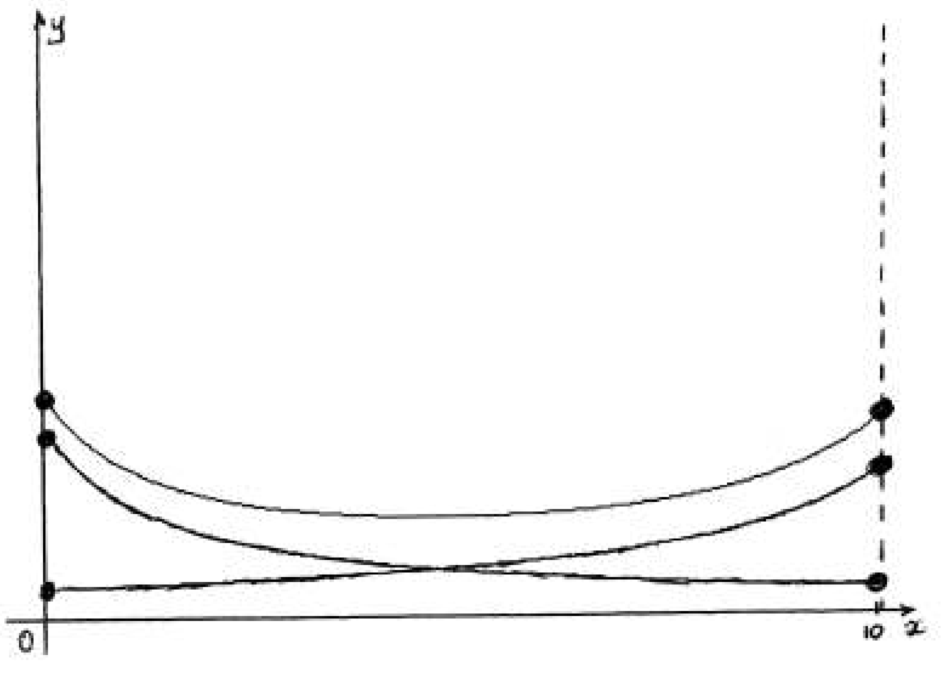
$= 0$ for a minimum

$0 = 16k - 1600$ and so $k = 100$.

Question 4

a. $y = \frac{p}{4} + \frac{q}{8}$

b.



Two features need to be shown here:

- the end-points need to be closed and filled in
- the addition graph is to have a minimum *above* the intersection of the two original graphs.

c. $y = \dfrac{9}{x+1} + \dfrac{4}{11-x}$

$\dfrac{dy}{dx} = \dfrac{-9}{(x+1)^2} + \dfrac{4}{(11-x)^2}$

$\dfrac{-9}{(x+1)^2} + \dfrac{4}{(11-x)^2} = 0$ for a minimum value.

d. i Now adding these two fractions together gives:

$\dfrac{-9(11-x)^2 + 4(x+1)^2}{(x+1)^2(11-x)^2} = 0$

If this fraction is zero then the numerator is zero:

$4(x + 1)^2 - 9(11 - x)^2 = 0$

This can be factorised using the difference of two squares:

$[2(x + 1) - 3(11 - x)][2(x + 1) + 3(11 - x)] = 0$

$[2x + 2 - 33 + 3x][2x + 2 + 33 - 3x] = 0$

$(5x - 31)(-x + 35) = 0$

Hence $x = 6.2$ is the only answer within the domain.

When $x = 6.2$ then the minimum is 2.083 (correct to three decimal places).

ii Solve $\dfrac{9}{x+1} + \dfrac{4}{11-x} = 5$ using the calculator to give $x = 0.956$.

The pollution level will be less than 5 for a journey from $x = 0.956$ to $x = 10$, which is a distance of 9.044 km.

e. $\displaystyle\int_0^{10} \left(\frac{9}{x+1} + \frac{4}{11-x}\right)dx$

$= [\, 9\log_e(x + 1) - 4\log_e(11 - x)\,]_0^{10}$

$= (\, 9\log_e(11) - 4\log_e(1)\,) - (9\log_e(1) - 4\log_e(11)\,)$

$= 13\log_e(11)$

Answer: 31.17 correct to two decimal places.