VCE
2005 Mathematical Methods
Trial Examination 2

Suggested Solutions

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Question 1

a.

$$f(x) = 2\left[x^2 - 6x + \frac{23}{2}\right]$$

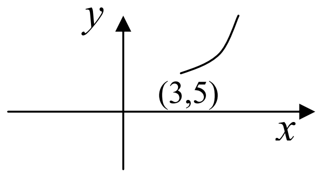
$$f(x) = 2\left[x^2 - 6x + 9 + \frac{23}{2} - 9\right] \quad \text{(1 mark)}$$

$$f(x) = 2\left[(x-3)^2 + \frac{5}{2}\right]$$

$$f(x) = 2(x-3)^2 + 5$$

$A = 2, B = 3, C = 5$ (1 mark)

b.



$f(x)$ must have one - one correspondence for $f^{-1}(x)$ to exist.

$a = 3$ (1 mark)

c.

$$x = 2(y-3)^2 + 5$$

$$x - 5 = 2(y-3)^2$$

$$\frac{x-5}{2} = (y-3)^2$$

$$y - 3 = \pm\sqrt{\frac{x-5}{2}}$$

$$y = 3 \pm \sqrt{\frac{x-5}{2}} \quad \text{(1 mark)}$$

But $y \geq 3$

$$\therefore y = 3 + \sqrt{\frac{x-5}{2}}$$

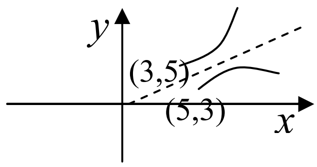
$$\therefore f^{-1}(x) = 3 + \sqrt{\frac{x-5}{2}} \quad x \geq 5 \quad \text{(1 mark)}$$

Domain $[5,\infty)$ (1 mark)

Range $[3,\infty)$ (1 mark)

d.

1 mark for each shape with its end point.



e.(i)

$f'(x) = 4x - 12$

When $x = 4, f'(x) = 4$ (1 mark)

e.(ii)

$f^{-1}(x) = 4 = 3 + \sqrt{\dfrac{x-5}{2}}$

$\sqrt{\dfrac{x-5}{2}} = 1$

$\dfrac{x-5}{2} = 1$

$x - 5 = 2$

$x = 7$ (1 mark)

$f'^{-1}(x) = \dfrac{1}{2}\left(\dfrac{x-5}{2}\right)^{-\frac{1}{2}} \times \dfrac{1}{2} = \dfrac{1}{4}\left(\dfrac{x-5}{2}\right)^{-\frac{1}{2}}$ (1 mark)

When $x = 7$,

$f'^{-1}(x) = \dfrac{1}{4}$ (1 mark)

f.

Let point of intersection be (a, b)

For $f^{-1}(x)$ when $x = 7, y = 4$

$\therefore$ for $f(x)$ when $x = 4, y = 7$

Gradient of $f(x) = 4$

$\therefore \dfrac{7-b}{4-a} = 4$

$\therefore 7 - b = 16 - 4a$

$4a - b = 9$ (1) (1 mark)

But gradient of $f^{-1}(x) = \dfrac{1}{4}$

$\therefore \dfrac{4-b}{7-a} = \dfrac{1}{4}$

$\therefore 7 - a = 16 - 4b$

$-a + 4b = 9$

$-4a + 16b = 36$ (2)

$(1) + (2)$

$15b = 45$

$b = 3$

Sub $b = 3$ in (1)

$a = 3$

$(3,3)$ (1 mark)

g.

All the points will be of the form (a, a),

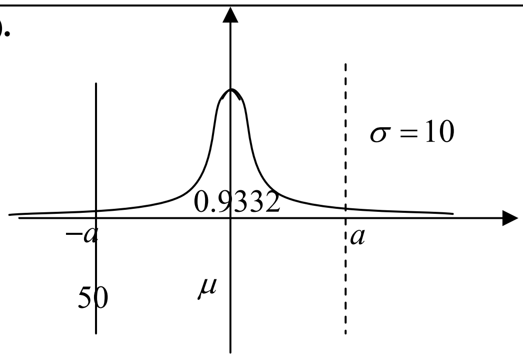
so the equation of the line will be $y = x$ (1 mark)

Question 2

a. $\frac{35}{100} \times 18{,}000 = 6{,}300$ (1 mark)	**b.** $4k^2 + k + 6k^2 + 4k - 14k^2 = 1$ $-4k^2 + 5k - 1 = 0$ $4k^2 - 5k + 1 = 0$ $(4k-1)(k-1) = 0$ (1 mark) $k = \frac{1}{4}$ or 1 But $0 < k < 1$ $\therefore k = \frac{1}{4}$ (1 mark)
c. $\Pr = 4k^2 + k + 6k^2 = 10k^2 + k = \frac{7}{8}$ (1 mark)	
d.(i) $\Pr(X = 1) = \binom{10}{1}\left(\frac{3}{10}\right)^1\left(\frac{7}{10}\right)^9 = 0.121$ (1 mark)	**d(ii).** $\Pr(X \geq 2) = 1 - [\Pr(X = 0) + \Pr(X = 1)]$ (1 mark) $= 1 - \left[\binom{10}{0}\left(\frac{3}{10}\right)^0\left(\frac{7}{10}\right)^{10} + \binom{10}{1}\left(\frac{3}{10}\right)^1\left(\frac{7}{10}\right)^9\right]$ = 0.851 to 3 dec. places. (1 mark)

Question 2(continued)

e(i).



$\Pr(X > 50) = 0.9332$
$\Pr\left(Z > -a\right) = 0.9332$
$\Pr\left(Z < a\right) = 0.9332$
$a = 1.5$
$-a = -1.5$ (1 mark)
$Z = \dfrac{x-\mu}{\sigma}$
$-1.5 = \dfrac{50-\mu}{10}$
$\mu = 65$ (1 mark)

e(ii).

$\Pr(X > 82 / X > 50)$
$= \dfrac{\Pr(X > 82 \cap X > 50)}{\Pr(X > 50)}$
$= \dfrac{\Pr(X > 82)}{0.9332}$ (1 mark)
$Z = \dfrac{x-\mu}{\sigma} = \dfrac{82-65}{10} = 1.7$
$\Pr(X > 82 / X > 50) = \dfrac{\Pr(Z > 1.7)}{0.9332}$
$= \dfrac{1-\Pr(Z < 1.7)}{0.9332} = \dfrac{1-0.9554}{0.9332} = 0.05$ (1 mark)

f.

$\Pr(X = 4) = \dfrac{\binom{20}{7}\binom{10}{4}}{\binom{30}{11}}$ (1 mark)

$= 0.298$ (1 mark)

Question 3

a.
Maximum $h = 10 + 3 = 13$ hours (1 mark)

b.
Minimum $h = 10 - 3 = 7$ hours (1 mark)

c.
Period $= \dfrac{2\pi}{n}$ where $n = \dfrac{2\pi}{365}$

Period $= 2\pi \div \dfrac{2\pi}{365} = 365$ days (1 mark)

d.
Minimum occurs when

$\cos\dfrac{2\pi(t-100.5)}{365} = -1$ (1 mark)

$\dfrac{2\pi(t-100.5)}{365} = -\pi, \pi, 3\pi, 5\pi....$ (1 mark)

$\dfrac{2(t-100.5)}{365} = -1, 1, 3, 5....$

$2(t-100.5) = -365, 365....$

$t - 100.5 = -182.5, 182.5...$

$t = 283 \quad t > 0$

Minimum daylight hours occur on the 283rd day of the year. (1 mark)

e.
$10 + 3\cos\dfrac{2\pi(t-100.5)}{365} = 12$

$\cos\dfrac{2\pi(t-100.5)}{365} = \dfrac{2}{3}$

$\dfrac{2\pi(t-100.5)}{365} = -0.8411,\ 0.8411,\ 5.442$ (1 mark)

$2\pi(t-100.5) = -306.99,\ 307.0015,\ 1986.37$

$t = 51.64,\ 149.36$

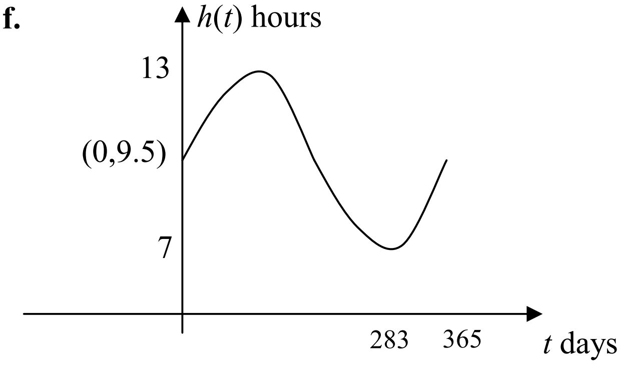
$t = 52,\ 149$ (1 mark)

52nd day of the year is February 21

149th day of the year is May 29 (1 mark)

Question 3 (continued)

f.



When $t = 0, h = 10 + 3\cos\left[\frac{2\pi(-100.5)}{365}\right] = 9.5$

(1 mark)

g.
There are 12 hours of sunlight on day 52 and day 149.
Number of days from day 52 to day 149 = 97
More than 12 hours of sunlight =96 days because day 52 and day 149 should not be included.

(1 mark)

h.

$$\frac{dh}{dt} = -3\sin\left[\frac{2\pi(t-100.5)}{365}\right] \times \frac{2\pi}{365} \quad \text{(1 mark)}$$

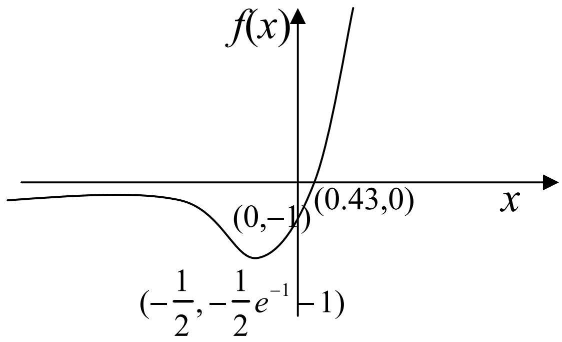
When $t = 30$,

$$\frac{dh}{dt} = -3\sin\left[\frac{2\pi(-70.5)}{365}\right] \times \frac{2\pi}{365}$$

$$\frac{dh}{dt} = 0.048 \text{ hours / day} = 2.89 \text{ min / day}$$

$$\frac{dh}{dt} = 3 \text{ min/ day} \qquad \text{(1 mark)}$$

Question 4

a. $f(0) = 0 \times 1 - 1 = -1$ (1 mark)	**b.** $f'(x) = xke^{kx} + e^{kx}$ (1 mark)
c. Turning Point occurs when $f'(x) = 0$ $e^{kx}(kx+1) = 0$ (1 mark) $e^{kx} \neq 0$ $\therefore kx + 1 = 0$ $kx = -1$ $x = -\frac{1}{k} = -\frac{1}{2}$ $\therefore k = 2$ (1 mark)	**d.**  Asymptote: $y = 0$ When $x = -\frac{1}{2}, y = -\frac{1}{2}e^{-1} - 1$ When $x = 0, y = -1$ 1 mark for shape and x, y intercepts 1 mark for equation of asymptote 1 mark for exact value of minimum turning point.

Question 4 (continued)

e.

$\int xe^{5x}$

$f(x) = xe^{kx} - 1$

$f'(x) = xke^{kx} + e^{kx}$

$\int (xke^{kx} + e^{kx})dx = xe^{kx} - 1$ (1 mark)

If $k = 5$

$\int 5xe^{5x} dx + \int e^{5x}dx = xe^{5x} - 1$

$5\int xe^{5x}dx + \frac{1}{5}e^{5x} = xe^{5x} - 1$

$5\int xe^{5x}dx = xe^{5x} - 1 - \frac{1}{5}e^{5x}$

$\int xe^{5x}dx = \frac{1}{5}(xe^{5x} - 1 - \frac{1}{5}e^{5x}) + c$ (1 mark)

where c is a constant.

So $\int xe^{5x} dx = \frac{1}{5}\left(xe^{5x} - \frac{1}{5}e^{5x}\right) + c_1$

where c_1 is a constant.

$\int xe^{5x} dx = \frac{1}{25}(5x-1)e^{5x} + c_1$

f.

$A = \left|\int_0^{0.43} f(x)dx\right| + \int_{0.43}^{1} f(x)dx$

$\int f(x)dx = \int (xe^{2x} - 1)dx$

$A = \left|\left[\frac{1}{4}(2x-1)e^{2x} - x\right]_0^{0.43}\right| + \frac{1}{4}(2x-1)e^{2x} - x\Big]_{0.43}^{0}$

(1 mark)

$A = |-0.2627| + 1.36$ (1 mark)

$A = 0.2627 + 1.36$

$A = 1.62$ to 2 decimal places.

(1 mark)

g.

$g(x)$ and $g^{-1}(x)$ intersect on the line $y = x$

$\therefore x = xe^{5x}$

$x - xe^{5x} = 0$ (1 mark)

$x(1 - e^{5x}) = 0$

$\Rightarrow x = 0$ or $e^{5x} = 1$

$\Rightarrow x = 0$ or $5x = 0$

$\Rightarrow x = 0$

When $x = 0, y = 0$

$\therefore$ point is $(0, 0)$ (1 mark)

END OF SUGGESTED SOLUTIONS
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